ON ARC-ANALYTIC FUNCTIONS AND ARC-SYMMETRIC SETS

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In this note, we will mostly deal with semialgebraic geometry, that is, the study of real solutions of systems of polynomial equations and inequalities. A *semialgebraic set* E in $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_s(x) > 0\},$$

where $s \in \mathbb{N}$ and $f, g_1, \ldots, g_s$ are polynomials in real variables $x = (x_1, \ldots, x_n)$. A function $f : E \to \mathbb{R}$ is called semialgebraic if its graph Γ_f is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}$. Given an open semialgebraic $U \subset \mathbb{R}^n$, a real analytic semialgebraic function $f : U \to \mathbb{R}$ is called *Nash*.

Our main object of interest here are the so called *arc-analytic* functions. A function $f : S \to \mathbb{R}$ on a set $S \subset \mathbb{R}^n$ is said to be arc-analytic when $f \circ \gamma$ is analytic for every real analytic arc $\gamma : (-\varepsilon, \varepsilon) \to S$.

Arc-analytic functions, although relatively unknown among non-specialists, play an important role in modern real algebraic and analytic geometry (see, e.g., [10] and the references therein). Indeed, Bierstone and Milman [3] proved that arc-analytic semialgebraic functions on a Nash manifold are precisely those that can be made Nash after composition with a finite sequence of blowings-up with smooth algebraic nowhere dense centres. In fact, this criterion is often the quickest way to determine arc-analyticity of a given function. Many classical examples in calculus are arc-analytic but not analytic.

Example 1. (a) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x^3/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is arc-analytic but not differentiable at the origin. Observe that f is made Nash after composition with a single blowing-up of the origin; for instance, $f(x, xy) = x/(1 + y^2)$. Note also that the graph Γ_f of f is not real analytic. In fact, the smallest real analytic subset of $\mathbb{R}^3$ containing Γ_f is the *Cartan umbrella* $\{(x, y, z) \in \mathbb{R}^3 : z(x^2 + y^2) = x^3\}$ (cf. [9, Ex. 1.2(1)]).

(b) The function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as $g(x, y) = \sqrt{x^4 + y^4}$ is arc-analytic but not $\mathcal{C}^2$. The graph Γ_g of g is not real analytic. Indeed, the Zariski closure $\{(x, y, z) \in \mathbb{R}^3 : z^2 = x^4 + y^4\}$ of Γ_g has two $\mathcal{C}^1$ sheets $z = \pm\sqrt{x^4 + y^4}$, but it is irreducible at the origin as a real analytic set (cf. [3, Ex. 1.2(3)]).

In general, the behaviour of arc-analytic functions may be surprising, if not pathological. For example, in [4] the authors construct an arc-analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not even continuous. However, in the semialgebraic setting, arc-analytic functions form a very nice family.

Arc-analytic functions were first considered by Kurdyka [9] on arc-symmetric semialgebraic sets. A set E in $\mathbb{R}^n$ is called *arc-symmetric* when, for every analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma((-1, 0)) \subset E$, one has $\gamma((-1, 1)) \subset E$. By a fundamental theorem [9, Thm. 1.4], the arc-symmetric semialgebraic sets are precisely the closed sets of a certain noetherian topology on $\mathbb{R}^n$. (A topology is called *noetherian* when every descending sequence of its closed sets is stationary.) Following [9], we will call it the *$\mathscr{AR}$ topology*, and the arc-symmetric semialgebraic sets will henceforth be called *$\mathscr{AR}$-closed sets*.

Given an $\mathscr{AR}$-closed set X in $\mathbb{R}^n$, we will denote by $\mathscr{A}_a(X)$ the ring of arc-analytic semialgebraic functions on X. By [9, Prop. 5.1], the zero locus of every $f \in \mathscr{A}_a(X)$ is $\mathscr{AR}$-closed. Interestingly, despite noetherianity of the $\mathscr{AR}$ topology, the ring $\mathscr{A}_a(\mathbb{R}^n)$ is not noetherian (see [9, Ex. 6.11]).

The usefulness of $\mathscr{AR}$ topology comes from the fact that it contains and is strictly finer than the Zariski topology on $\mathbb{R}^n$. Moreover, it follows from the semialgebraic Curve Selection Lemma that $\mathscr{AR}$-closed sets are closed in the Euclidean topology in $\mathbb{R}^n$.

Noetherianity of the $\mathscr{AR}$ topology allows one to make sense of the notions of irreducibility and components of a semialgebraic set much like in the algebraic case: An $\mathscr{AR}$-closed set X is called *$\mathscr{AR}$-irreducible* if it cannot be written as a union of two proper $\mathscr{AR}$-closed subsets. Every $\mathscr{AR}$-closed set admits a unique decomposition $X = X_1 \cup \dots \cup X_r$ into $\mathscr{AR}$-irreducible sets satisfying $X_i \not\subset \bigcup_{j\neq i} X_j$ for each $i = 1, \dots, r$. The sets $X_1, \dots, X_r$ are called the *$\mathscr{AR}$-components* of X. The decomposition into $\mathscr{AR}$-components is finer than that into algebraic or Nash components and encodes more algebro-differential information (see [11]). In particular, by a beautiful characterisation of Kurdyka, there is a one-to-one correspondence between the $\mathscr{AR}$-components of X of maximal dimension and the connected components of a desingularization of the Zariski closure of X.

Desingularization arguments play a very important role in the study of arc-symmetry and arc-analyticity. Together with H. Seyedinejad [1], we used them recently to prove that every $\mathscr{AR}$-closed set X in $\mathbb{R}^n$ is precisely the zero locus of a certain arc-analytic function $f \in \mathscr{A}_a(\mathbb{R}^n)$. It thus follows that the $\mathscr{AR}$ topology coincides with the one defined by the vanishing of semialgebraic arc-analytic functions, which is not at all apparent from the intrinsic definition above.

Extending the techniques of [1], most recently we also proved in [2] an arc-analytic analogue of Efroymson's extension theorem [5]: Every arc-analytic semialgebraic function $f : X \to \mathbb{R}$ on an $\mathscr{AR}$-closed set $X \subset \mathbb{R}^n$ is, in fact, a restriction of an arc-analytic function $F \in \mathscr{A}_a(\mathbb{R}^n)$. Moreover, the function F may be chosen real analytic outside the Zariski closure of X. This result is particularly interesting in the context of the so-called continuous rational functions, which form one of the most active research areas in contemporary real algebraic geometry (see, e.g., [7] and the references therein). A continuous function f is called *continuous rational* if it is generically of the form $\frac{p}{q}$, with p and q polynomial. Continuous rational functions on an $\mathscr{AR}$-closed set X form a subring of $\mathscr{A}_a(X)$, and the following example of Kollar-Nowak [8] shows that not every continuous rational function on an $\mathscr{AR}$-closed set admits an extension to the ambient space as a continuous rational function. Nonetheless, by [2], it does admit an extension as an arc-analytic one.

Example 2. The function $f(x, y, z) = \sqrt[3]{1 + z^2}$ is continuous rational on the real algebraic surface $S = \{(x, y, z) \in \mathbb{R}^3 : x^3 = (1 + z^2)y^3\}$, since $f|_S$ coincides with $\frac{x}{y}|_S$, but it has no continuous rational extension to $\mathbb{R}^3$ (see [8, Ex. 2]). Note that f is Nash, and hence arc-analytic, on $\mathbb{R}^3$.

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