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Geometric Auslander criterion for openness of an algebraic morphism

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Abstract

We give a topological analogue for openness of a criterion for flatness that originates with Auslander. Over a normal base of dimension n, failure of openness is detected by a *vertical* component in the nth fibred power of the morphism.

1. *Introduction*

The purpose of this note is to give a topological analogue for openness of a criterion for flatness that originates in a classical paper of Auslander [**3**]. Consider a morphism $\varphi : X \to Y$ of schemes of finite type over a field, where φ is locally of finite type. We show that, if Y is normal and of dimension n, then failure of openness of φ is equivalent to a severe discontinuity of the fibres (the existence of an irreducible component of the source whose image is nowhere dense in Y), after passage to the n-fold fibred power of the mapping (Theorem 1.1). The criterion is effective (see Remark 1.3).

In the analogous criterion for (non-)flatness, one replaces the 'existence of an irreducible component of the source' with 'existence of an associated component of the source', where the associated component can be isolated or embedded. Auslander's criterion is for flatness of finitely generated modules over a ring. The flatness criterion was recently extended to algebraic or analytic morphisms over the complex numbers by the authors [**2**], and then to algebraic morphisms over arbitrary fields by Avramov and Iyengar [**4**] (see Section 2).

Let Y be a normal scheme of finite type over a field $\underline{k}$. (*Normal* means that every local ring of Y is a normal ring; that is, an integrally closed domain.) Let $\varphi : X \to Y$ be a morphism which is locally of finite type. We say that an associated component (isolated or embedded) of X is *vertical* if its image lies in a proper subvariety of Y (equivalently, is nowhere-dense in Y). If $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$, where R is a normal algebra of finite type over $\underline{k}$ and A is an R-algebra of finite type, and φ is the induced morphism, then X has a vertical associated component (resp. vertical irreducible component) over Y if and only if there is an associated prime (resp. minimal associated prime) $\mathfrak{p}$ in $\operatorname{Spec} A$ with $\mathfrak{p} \cap R \neq (0)$. Equivalently, by the prime avoidance lemma [**6**, Lemma 3.3], A (resp. A_{red}) has a zero-divisor $r \in R$. The following is our main result. (See [**1**, Theorem 2.2] for the complex analytic case.)

Theorem 1.1. *Let Y be a scheme of finite type over a field $\underline{k}$, and let $\varphi : X \to Y$ denote a morphism which is locally of finite type. Assume that Y is normal, of dimension n. Then φ is open if and only if there are no vertical irreducible components in the n-fold fibred power $\varphi^{\{n\}} : X^{\{n\}} \to Y$.*

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Openness is a local property; that is, a mapping $\varphi : X \to Y$ is open if and only if it is open in a neighbourhood of ξ, for every $\xi \in X$. Theorem 1.1 therefore reduces to the following affine criterion:

Theorem 1.2. *Let R be an algebra of finite type over a field $\underline{k}$, and let A be an R-algebra of finite type. Assume that R is normal, of dimension n. Then the induced morphism of spectra* $\operatorname{Spec} A \to \operatorname{Spec} R$ *is open if and only if the reduced n-fold tensor power $(A^{\otimes_R^n})_{\mathrm{red}}$ is a torsion-free R-module.* (*Equivalently, $\mathfrak{p} \cap R = (0)$ for every minimal prime $\mathfrak{p}$ in $A^{\otimes_R^n}$.*)

Remark 1.3. By Theorem 1.2, to verify non-openness of the morphism $\operatorname{Spec} A \to \operatorname{Spec} R$, it suffices to find a minimal associated prime in $A^{\otimes_R^n}$ which contains a non-zero element $r \in R$. Similarly, by Theorem 2.1, in order to verify that a finite A-module F is not R-flat, it is enough to find an associated prime of $F^{\otimes_R^n}$ in $A^{\otimes_R^n}$ which contains a non-zero element $r \in R$. Thus, Theorems 1.2 and 2.1 together with Gröbner-basis algorithms for primary decomposition (see, for example, [**11**]) provide tools for checking openness and flatness by effective computation.

2. *Criterion for flatness*

Theorem 1.2 is a topological analogue of the criterion for flatness mentioned above. In contrast to the general openness criterion, however, the flatness results are known for a smooth target only. On the other hand, flatness makes sense also for modules over a morphism, and the corresponding flatness criterion holds also in this context.

The flatness criterion (Theorem 2.1 following) has its origin in Auslander [**3**]. It was recently proved by the authors for complex analytic morphisms and, as a consequence, for morphisms of schemes of finite type over $\mathbb{C}$, using transcendental methods [**2**] (see also [**7**]), and more recently for schemes of finite type over an arbitrary field, by Avramov and Iyengar [**4**] using homological methods (which require a smooth base ring). The flatness criterion over any field of characteristic zero can, in fact, be deduced from the complex case, using the Tarski–Lefschetz principle. In other words, the following is a corollary of [**2**, Theorem 1.3].

Theorem 2.1. *Let R be an algebra of finite type over a field $\underline{k}$ of characteristic zero. Let A denote an R-algebra essentially of finite type, and let F denote a finitely generated A-module. Assume that R is regular, of dimension n. Then F is R-flat if and only if the n-fold tensor power $F^{\otimes_R^n}$ is a torsion-free R-module.* (*Equivalently, $\mathfrak{p} \cap R = (0)$ for every associated prime $\mathfrak{p}$ of $F^{\otimes_R^n}$ in $A^{\otimes_R^n}$.*)

An R-algebra *essentially of finite type* means a localization of an R-algebra of finite type. In the case that $\underline{k} = \mathbb{C}$, the above result is [**2**, Theorem 1.3].

Proof of Theorem 2.1. The 'only if' direction in Theorem 2.1 is immediate, since any tensor power of a flat R-module is R-flat, and hence a torsion-free R-module, by the characterization of flatness in terms of relations ([**6**, Corollary 6.5]). To deduce the 'if' direction of Theorem 2.1 from [**2**, Theorem 1.3], we proceed in two steps.

First, assume that $\underline{k}$ is algebraically closed. Then our result follows from [**2**, Theorem 1.3], by the Tarski–Lefschetz principle (see, for example, [**10**]), as flatness can be expressed in terms of a finite number of relations [**6**, Corollary 6.5].

Next, suppose that $\underline{k}$ is an arbitrary field of characteristic zero, and let $\mathbb{K}$ denote an algebraic closure of $\underline{k}$. Let A be an R-algebra essentially of finite type, and let F be a finitely generated A-module, which is not flat over R. Put $R' := R \otimes_{\underline{k}} \mathbb{K}$. Then R' is a faithfully flat R-module. Indeed, R' is R-flat, because flatness is preserved by any base change and $\mathbb{K}$ is trivially $\underline{k}$-flat, as a $\underline{k}$-vector space. To prove faithful flatness, by [5, Chapter I, § 3.5, Proposition 9], it suffices to show that for every maximal ideal $\mathfrak{m}$ of R there exists a maximal ideal $\mathfrak{n}$ of R' such that $\mathfrak{n} \cap R = \mathfrak{m}$. Let $\mathfrak{m}$ be an arbitrary maximal ideal in R. Then $\mathfrak{m}$ induces a proper ideal in R'. Let $\mathfrak{n}$ be a maximal ideal of R' which contains the ideal induced by $\mathfrak{m}$ in R'. Since R' is a homomorphic image of a polynomial algebra $\mathbb{K}[y_1, \dots, y_n]$, with $\mathbb{K}$ algebraically closed, it follows that $\mathfrak{n}$ is (the equivalence class of) an ideal of the form $(y_1 - a_1, \dots, y_n - a_n)$ for some $(a_1, \dots, a_n) \in \mathbb{K}^n$. Then clearly $\mathfrak{n} \cap R \subsetneq R$, and hence $\mathfrak{n} \cap R = \mathfrak{m}$, since $\mathfrak{n} \cap R \supset \mathfrak{m}$ and $\mathfrak{m}$ is maximal.

Put $A' := A \otimes_R R'$, and $F' := F \otimes_R R'$. Since F is not R-flat, F' is not R'-flat, by faithful flatness of R' over R. Hence, $F'^{\otimes^n_{R'}}$ has a zero-divisor in R', by the first part of the proof. Therefore, $F'^{\otimes^n_{R'}}$ has a non-zero associated prime $\mathfrak{q}$ in R'. But $F'^{\otimes^n_{R'}} \cong (F^{\otimes^n_R}) \otimes_R R'$, and for any R-module M we have

$$\mathrm{Ass}_{R'}(M \otimes_R R') = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathrm{Ass}_{R'}(R'/\mathfrak{p}R'),$$

by [5, Chapter IV, § 2.6, Theorem 2]. Thus, $F^{\otimes^n_R}$ has a non-zero associated prime in R; that is, $F^{\otimes^n_R}$ has a zero-divisor in R. □

3. *Zero-divisors in tensor powers and variation of fibre dimension*

Let $\varphi : X \to Y$ denote a morphism of schemes of finite type over an arbitrary field $\underline{k}$. We show that non-trivial variation of the dimension of the fibres of φ implies the existence of a vertical component in a sufficiently high fibred power of φ (that is, the existence of an irreducible component of a fibred power of φ which is mapped into a proper closed subscheme of Y). We will use the following two classical results.

(1) Chevalley's theorem on upper semicontinuity of fibre dimension [8, Theorem 13.1.3]: If $\varphi : X \to Y$ is locally of finite type and $e \in \mathbb{N}$, then $F_e^{\varphi}(X) := \{x \in X : \dim_x \varphi^{-1}(\varphi(x)) \geqslant e\}$ is closed.

(2) The *altitude formula* of Nagata: Suppose that R is a Noetherian integral domain and that A is an integral domain and an R-algebra of finite type. If $\mathfrak{p}$ is a prime ideal in A and $\mathfrak{q} = \mathfrak{p} \cap R$, then

$$\dim A_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{q}} + \mathrm{tr.deg}_{K(R)} K(A) - \mathrm{tr.deg}_{\kappa(\mathfrak{q})} \kappa(\mathfrak{p}), \tag{3.1}$$

with equality if R is universally catenary. (See [9, Theorem 15.6] or [6, Theorem 13.8].)

Here, $\kappa(\mathfrak{q})$ and $\kappa(\mathfrak{p})$ denote the residue fields $R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$ and $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, respectively, $K(R)$ and $K(A)$ are the fields of fractions of R and A, respectively, and $\mathrm{tr.deg}_K L$ denotes the transcendence degree of a field extension $K \subset L$. A ring R is called *catenary* if, for every pair of prime ideals $\mathfrak{q}_1 \subset \mathfrak{q}_2$ in R, all maximal chains of primes between $\mathfrak{q}_1$ and $\mathfrak{q}_2$ have the same length. R is called *universally catenary* if every finitely generated R-algebra is catenary.

Equality in (3.1) holds, for example, under the hypotheses of Theorems 1.1 and 1.2, since every algebra of finite type over a field is universally catenary [6, Corollary 13.4].

Recall finally that if Y is a normal scheme of finite type over a field and X is irreducible, then a morphism $\varphi : X \to Y$ which is locally of finite type is open if and only if φ is dominating and the fibres of φ are equidimensional and of constant dimension (by Grothendieck [8, Proposition 5.2.1 and Corollary 14.4.6]).

Proof of Theorem 1.2. Let R be an algebra of finite type over a field $\underline{k}$, and let A be a finite type R-algebra. Assume that R is normal, of dimension n. Let $X := \mathrm{Spec}\, A$, $Y := \mathrm{Spec}\, R$, and let $\varphi : X \to Y$ be the canonical morphism induced by the R-algebra structure of A.

First suppose that φ is open. Then φ is *universally open* (see [8, Déf. 14.3.3]), by [8, Corollary 14.4.3]. Thus, $X \times_Y X \to X$ is open, so that $\varphi^{\{2\}} : X \times_Y X \to Y$ is open since it is the composite of the open mappings $X \times_Y X \to X$ and $X \to Y$. By induction, all fibred powers $\varphi^{\{k\}} : X^{\{k\}} \to Y$, $k \geqslant 1$ are open. In particular, $\varphi^{\{n\}}|_W$ is dominating, for every isolated irreducible component W of $X^{\{n\}}$; that is, there are no vertical components.

Now suppose that φ is not open; hence, not open at some point $\xi \in X$. Let $\{Z_j\}$ denote the set of isolated irreducible components of X at ξ. We can assume that each $\varphi|_{Z_j}$ is dominating. (Otherwise, already X would have an isolated vertical component at ξ. Then, for each k, the fibred power $X^{\{k\}}$ would have an isolated vertical component at the corresponding diagonal point, since X embeds into $X^{\{k\}}$ as the diagonal. More precisely, if Z_j is an isolated vertical component of X, then $Z_j^{\{k\}}$ is an isolated vertical component of $X^{\{k\}}$.)

Since φ is not open at ξ, $\varphi|_{Z_j}$ is not open at ξ, for each j. Let $m := \dim_\xi \varphi^{-1}(\varphi(\xi))$. By Chevalley's theorem, m is the maximal fibre dimension in some open neighbourhood U of ξ. We can assume that $U = X$ (since our problem is local).

Let $F_m^\varphi(X) := \{x \in X : \dim_x \varphi^{-1}(\varphi(x)) = m\}$. For each j, set

$$F_m^\varphi(Z_j) := \{x \in Z_j : \dim_x (\varphi|_{Z_j})^{-1}(\varphi(x)) = m\}.$$

Clearly, $F_m^\varphi(X) = \bigcup_j F_m^\varphi(Z_j)$. By Grothendieck's criterion mentioned before the proof, each $F_m^\varphi(Z_j)$ is a proper closed subset of Z_j (perhaps empty). For each j, put

$$d_j := \dim Z_j, \quad m_j := \min_{x \in Z_j} \dim_x (\varphi|_{Z_j})^{-1}(\varphi(x)).$$

Then $\dim F_m^\varphi(Z_j) \leqslant d_j - 1$, since Z_j is irreducible.

For each j, $d_j = \dim Y + m_j = n + m_j$, by equality in (3.1). If $F_m^\varphi(Z_j) \neq \emptyset$, then $\varphi|_{F_m^\varphi(Z_j)}$ has generic fibre dimension m and it follows that

$$\dim \varphi(F_m^\varphi(Z_j)) \leqslant (d_j - 1) - m = (n + m_j - 1) - m \leqslant n - 1,$$

by equality in (3.1) again. Therefore, $\varphi(F_m^\varphi(X)) = \bigcup_j \varphi(F_m^\varphi(Z_j))$ is a constructible set ([**8**, Chapter IV, Theorem 1.8.4]) of dimension at most $n - 1$, so it lies in a proper closed subset Y' of Y.

Let $\eta = \varphi(\xi)$. Let W be an irreducible component of the n-fold fibred power $X^{\{n\}}$ which contains an nm-dimensional irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(\eta)$. Let $\mathfrak{s}$ be the (minimal) prime ideal in $A^{\otimes_R^n}$ which defines W (that is, $W = \operatorname{Spec}(A^{\otimes_R^n}/\mathfrak{s})$). We claim that $\mathfrak{s} \cap R \neq (0)$ (that is, W is mapped by $\varphi^{\{n\}}$ into a proper subvariety of Y).

Suppose that $\mathfrak{s} \cap R = (0)$ (that is, suppose W is dominant). Then $A^{\otimes_R^n}/\mathfrak{s} \supset R$, and $A^{\otimes_R^n}/\mathfrak{s} \otimes_R K(R)$ is an algebra of finite type over $K(R)$. Hence,

$$\operatorname{tr.deg}_{K(R)} K(A^{\otimes_R^n}/\mathfrak{s}) = \dim A^{\otimes_R^n}/\mathfrak{s} \otimes_R K(R)\,.$$

The latter is the common length of all maximal chains of prime ideals in $A^{\otimes_R^n}/\mathfrak{s} \otimes_R K(R)$ [**6**, § 8.2, Theorem A]. On the other hand, the fibres of $\varphi^{\{n\}}|_W$ over the (non-empty) open set $Y \setminus Y'$ are of dimension at most $n(m-1)$; hence, the preceding common length of maximal chains of primes is at most $n(m-1)$.

Therefore,

$$\operatorname{tr.deg}_{K(R)} K(A^{\otimes_R^n}/\mathfrak{s}) \leqslant n(m-1). \tag{3.2}$$

By (3.1) and (3.2), for every maximal ideal $\mathfrak{a}$ in $A^{\otimes_R^n}/\mathfrak{s}$, we have

$$\begin{aligned}\dim(A^{\otimes_R^n}/\mathfrak{s})_\mathfrak{a} &\leqslant \dim R_{\mathfrak{a}\cap R} + \operatorname{tr.deg}_{K(R)} K(A^{\otimes_R^n}/\mathfrak{s}) \\ &\leqslant \dim R + n(m-1) = nm.\end{aligned}$$

Hence, $\dim W = \dim(A^{\otimes_R^n}/\mathfrak{s}) \leqslant nm$. Since $nm = \dim(\varphi^{\{n\}})^{-1}(\eta)$ and W is irreducible, it follows that the generic points of W and $(\varphi^{\{n\}})^{-1}(\eta)$ coincide. Therefore $\varphi^{\{n\}}(W) = \{\eta\}$; a contradiction. $\square$

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