A DEGREE SUM CONDITION FOR HAMILTONICITY IN BALANCED BIPARTITE DIGRAPHS

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Abstract. We prove that a strongly connected balanced bipartite digraph D of order $2a$ is hamiltonian, provided $a \geq 3$ and $d(x)+d(y) \geq 3a$ for every pair of vertices x, y with a common in-neighbour or a common out-neighbour in D.

1. Introduction

In [5], Bang-Jensen et al. conjectured the following strengthening of a classical Meyniel theorem: If D is a strongly connected digraph on n vertices in which $d(u)+d(v) \geq 2n-1$ for every pair of non-adjacent vertices u, v with a common out-neighbour or a common in-neighbour, then D is hamiltonian. (An *in-neighbour* (resp. *out-neighbour*) of a vertex u is any vertex v such that $vu \in A(D)$ (resp. $uv \in A(D)$).)

The conjecture has been partially verified under additional assumptions in [3], but has remained in its full generality a difficult open problem. The goal of the present note is to prove a bipartite analogue of the conjecture (Theorem 1.2 below).

We consider digraphs in the sense of [4], and use standard graph theoretical terminology and notation (see Section 2 for details).

Definition 1.1. Consider a balanced bipartite digraph D with partite sets of cardinalities a. We will say that D satisfies *condition* $(\mathcal{A})$ when

$$d(x)+d(y) \geq 3a$$

for every pair of vertices x, y with a common in-neighbour or a common out-neighbour.

Theorem 1.2. *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities a, where $a \geq 3$. If D satisfies condition $(\mathcal{A})$, then D is hamiltonian.*
Moreover, the only non-hamiltonian strongly connected balanced bipartite digraph on 4 vertices which satisfies condition $(\mathcal{A})$ is the one obtained from the complete bipartite digraph $\overset{\leftrightarrow}{K}_{2,2}$ by removing one 2-cycle.

Remark 1.3. Although in light of the above mentioned conjecture one might expect something of order $2a$, it is worth noting that the bound of $3a$ in Theorem 1.2 is sharp. Indeed, this follows from Example 1.4 below (due to Amar and Manoussakis [2]).

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Example 1.4. For $a \geq 3$ and $1 \leq l < a/2$, let $D(a,l)$ be a bipartite digraph with partite sets V_1 and V_2 such that V_1 (resp. V_2) is a disjoint union of sets R, S (resp. U, W) with $|R| = |U| = l$, $|S| = |W| = a - l$, and $A(D(a,l))$ consists of the following arcs:
(a) ry and yr, for all $r \in R$ and $y \in V_2$,
(b) ux and xu, for all $u \in U$ and $x \in V_1$, and
(c) sw, for all $s \in S$ and $w \in W$.
Then $d(r) = d(u) = 2a$ for all $r \in R$ and $u \in U$, and $d(s) = d(w) = a + l$ for all $s \in S$ and $w \in W$. In particular, for odd a, in $D(a, (a-1)/2)$ we have $d(x) + d(y) \geq 3a - 1$ for every pair of non-adjacent vertices x, y. Notice that $D(a,l)$ is strongly connected, but not hamiltonian.

A weaker version of Theorem 1.2 was recently proved in [1]. There, it is assumed that the inequality $d(x)+d(y) \geq 3a$ is satisfied by *every* pair of non-adjacent vertices x and y. It is thus a bipartite analogue of the original Meyniel's hamiltonicity criterion for ordinary digraphs. The author is happy to acknowledge the influence of [1] on the present work. In fact, Lemma 3.1 and the first part of the proof of Theorem 1.2 are direct adaptations of the ideas from [1], developed together with Lech Adamus and Anders Yeo.

2. Notation and terminology

A *digraph* D is a pair $(V(D), A(D))$, where $V(D)$ is a finite set (of *vertices*) and $A(D)$ is a set of ordered pairs of distinct elements of $V(D)$, called *arcs* (i.e., D has no loops or multiple arcs). The number of vertices $|V(D)|$ is the *order* of D (also denoted by $|D|$). For vertices u and v from $V(D)$, we write $uv \in A(D)$ to say that $A(D)$ contains the ordered pair (u, v).

For a vertex set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ *dominated* by the vertices of S; i.e.,

$$N^+(S) = \{u \in V(D) : vu \in A(D) \text{ for some } v \in S\}\,.$$

Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ *dominating* vertices of S; i.e,

$$N^-(S) = \{u \in V(D) : uv \in A(D) \text{ for some } v \in S\}\,.$$

If $S = \{v\}$ is a single vertex, the cardinality of $N^+(\{v\})$ (resp. $N^-(\{v\})$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the *outdegree* (resp. *indegree*) of v in D. The *degree* of v is $d(v) = d^+(v) + d^-(v)$.

For vertex sets $S, T \subset V(D)$, we denote by $A[S,T]$ the set of all arcs of $A(D)$ from a vertex in S to a vertex in T. Let $\overset{\leftrightarrow}{a}(S,T) = |A[S,T]| + |A[T,S]|$. Note that $\overset{\leftrightarrow}{a}(\{v\}, V(D) \setminus \{v\}) = d(v)$. A set of vertices $\{v_1, \dots, v_k\}$ such that $\overset{\leftrightarrow}{a}(\{v_i\}, \{v_j\}) = 0$, for all $i \neq j$, is called *independent*.

A directed cycle (resp. directed path) on vertices $v_1, \dots, v_m$ in D is denoted by $[v_1, \dots, v_m]$ (resp. $(v_1, \dots, v_m)$). We will refer to them as simply *cycles* and *paths* (skipping the term "directed"), since their non-directed counterparts are not considered in this article at all.

A cycle passing through all the vertices of D is called *hamiltonian*. A digraph containing a hamiltonian cycle is called a *hamiltonian digraph*. A *cycle factor* in D is a collection of vertex-disjoint cycles $C_1, \dots, C_l$ such that $V(C_1) \cup \dots \cup V(C_l) = V(D)$.

A digraph D is *strongly connected* when, for every pair of vertices $u, v \in V(D)$, D contains a path originating in u and terminating in v and a path originating in v and terminating in u.

A digraph D is *bipartite* when $V(D)$ is a disjoint union of independent sets V_1 and V_2 (the *partite sets*). It is called *balanced* if $|V_1| = |V_2|$. One says that a bipartite digraph D is *complete* when $d(x) = 2|V_2|$ for all $x \in V_1$.

A *matching from* V_1 *to* V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (u_1u_2 and v_1v_2 are *independent arcs* when $u_1 \neq v_1$ and $u_2 \neq v_2$). If D is balanced, one says that such a matching is *perfect* if it consists of precisely $|V_1|$ arcs.

3. Lemmas

The proof of Theorem 1.2 will be based on the following three lemmas.

Lemma 3.1. *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities $a \geq 2$. If D satisfies condition $(\mathcal{A})$, then D contains a cycle factor.*

Proof. Let V_1 and V_2 denote the two partite sets of D. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1. Therefore, by the König-Hall theorem (see, e.g., [6]), it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

For a proof by contradiction, suppose that a non-empty set $S \subset V_1$ is such that $|N^+(S)| < |S|$. Then $V_2 \setminus N^+(S) \neq \varnothing$ and, for every $y \in V_2 \setminus N^+(S)$, we have $d^-(y) \leq a - |S|$. Hence

$$d(y) \leq 2a - |S| \quad \text{for every } y \in V_2 \setminus N^+(S). \tag{3.1}$$

If $|S| = 1$ then $|N^+(S)| = 0$, and so the only vertex of S has out-degree zero, which is impossible in a strongly connected D. If, in turn, $|S| = a$, then every vertex from $V_2 \setminus N^+(S)$ has in-degree zero, which again contradicts strong connectedness of D. Therefore, $2 \leq |S| \leq a - 1$. We now consider the following two cases.

Case 1. $\frac{a}{2} < |S| \leq a - 1$.
Since D is strongly connected, we have $d^-(y) \geq 1$ for every $y \in V_2 \setminus N^+(S)$. Note that $|V_2 \setminus N^+(S)| \geq |V_1 \setminus S| + 1 \geq 2$. On the other hand, the vertices of $V_2 \setminus N^+(S)$ are dominated only by those of $V_1 \setminus S$. It follows that $V_2 \setminus N^+(S)$ contains at least one pair of vertices, say y_1 and y_2, with a common in-neighbour. Condition $(\mathcal{A})$ together with (3.1) thus imply that

$$3a \leq d(y_1) + d(y_2) \leq 2(2a - |S|) = 4a - 2|S| < 4a - a;$$

a contradiction.

Case 2. $2 \leq |S| \leq \frac{a}{2}$.
Since D is strongly connected, we have $d^+(x) \geq 1$ for every $x \in S$. On the other hand, $|N^+(S)| \leq |S| - 1$. It follows that S contains at least one pair of vertices, say x_1 and x_2, with a common out-neighbour. Condition $(\mathcal{A})$ thus implies that

$$3a \leq d(x_1) + d(x_2) = d^-(x_1) + d^+(x_1) + d^-(x_2) + d^+(x_2) \leq \\ a + (|S| - 1) + a + (|S| - 1) \leq 3a - 2\,;$$

a contradiction.

This completes the proof of existence of a perfect matching from V_1 to V_2. The proof for a matching in the opposite direction is analogous. $\square$

Lemma 3.2. *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities $a \geq 2$, which satisfies condition $(\mathcal{A})$. Suppose that D is non-hamiltonian. Then, for every $u \in V(D)$, there exists $v \in V(D) \setminus \{u\}$ such that u and v have a common in-neigbour or out-neighbour in D.*

Proof. For a proof by contradiction, suppose that $x' \in V(D)$ has no common in-neighbour or out-neighbour with any other vertex in D. By Lemma 3.1, D has a cycle factor, say, $\mathcal{F} = \{C_1, \dots, C_l\}$, with $l \geq 2$ (as D is non-hamiltonian). Without loss of generality, we may assume that $x' \in V_1 \cap V(C_1)$.

Let x'^+ denote the successor of x' on C_1. We have $d^-(x'^+) = 1$. Indeed, for if $d^-(x'^+) \geq 2$ then x'^+ would be a common out-neighbour of x' and some other vertex from V_1. It follows that

$$d(x'^+) = d^+(x'^+) + d^-(x'^+) \leq a + 1. \tag{3.2}$$

We claim that x'^+ has no common in-neighbour or out-neighbour with any other vertex in V_2. Suppose otherwise, and let $y' \in V_2$ be a vertex which shares an in-neighbour or an out-neighbour with x'^+. Then, by condition $(\mathcal{A})$ and (3.2), we have

$$3a \leq d(y') + d(x'^+) \leq d(y') + a + 1,$$

hence $d(y') \geq 2a - 1$. It follows that $xy' \in A(D)$ for all $x \in V_1$ or else $y'x \in A(D)$ for all $x \in V_1$. In the first case, y' is a common out-neighbour of x' and every other vertex in V_1, and in the second case y' is a common in-neighbour of x' and every other vertex in V_1. This contradicts the choice of x'. Consequently, there is no such y', that is, x'^+ has no common in-neighbour or out-neighbour with any vertex in $V(D)$.

By repeating the above argument, one can now show that x'^{++}, the successor of x'^+ on C_1 has no common in-neighbour or out-neighbour with any vertex in $V(D)$, and, inductively, that no vertex of C_1 has a common in-neighbour or out-neighbour with any other vertex. In particular, this means that there are no arcs in or out of C_1, which is not possible in a strongly connected non-hamiltonian digraph. This contradiction completes the proof of the lemma. $\square$

Lemma 3.3. *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities $a \geq 2$, which satisfies condition $(\mathcal{A})$. If D is non-hamiltonian, then $d(u) \geq a$ for all $u \in V(D)$.*

Proof. This follows immediately from Lemma 3.2, condition $(\mathcal{A})$, and the fact that the degree of every vertex in D is bounded above by $2a$. $\square$

4. Proof of the main result

Proof of Theorem 1.2. Let D be a balanced bipartite digraph on $2a$ vertices, and let V_1 and V_2 denote its partite sets. Suppose first that $a = 2$. By Lemma 3.1, D contains a cycle factor. If D is not hamiltonian, this factor must consist of two 2-cycles, say C_1 with vertices $x_1 \in V_1$ and $y_1 \in V_2$, and C_2 with vertices $x_2 \in V_1$ and $y_2 \in V_2$. By strong connectedness of D there must also exist at least one arc from C_1 to C_2 and one arc from C_2 to C_1. The only configuration in which D is not hamiltonian is when there is precisely one such arc in each direction and they both join the same pair of vertices, say x_1 with y_2. D is thus obtained from $\overleftrightarrow{K}_{2,2}$ by removing the 2-cycle $[x_2, y_1]$.

From now on, we assume that $a \geq 3$. By Lemma 3.1, D contains a cycle factor $\mathcal{F} = \{C_1, C_2, \dots, C_l\}$. Assume that l is minimum possible, and for a proof by contradiction suppose that $l \geq 2$. Recall that $|C_i|$ denotes the order of cycle C_i. Without loss of generality, assume that $|C_1| \leq |C_2| \leq \cdots \leq |C_l|$.

Claim 1: $\overleftrightarrow{a}(V(C_i), V(C_j)) \leq \frac{|C_i|\cdot|C_j|}{2}$, for all $i \neq j$.

Proof of Claim 1. Let $q \in \{1, 2\}$, $u_i \in V(C_i) \cap V_q$ and $u_j \in V(C_j) \cap V_q$ be arbitrary. Let u_i^+ be the successor of u_i in C_i and let u_j^+ be the successor of u_j in C_j. Let $\mathcal{Z}_q(u_i, u_j)$ be defined as $A(D) \cap \{u_i u_j^+,\ u_j u_i^+\}$. If $|\mathcal{Z}_q(u_i, u_j)| = 2$ for some u_i, u_j, then the cycles C_i and C_j can be merged into one cycle by deleting the arcs $u_i u_i^+$ and $u_j u_j^+$ and adding the arcs $u_i u_j^+$ and $u_j u_i^+$. This would contradict the minimality of l, so we may assume that

$$|\mathcal{Z}_q(u_i, u_j)| \leq 1 \quad \text{for all } u_i \in V(C_i) \cap V_q \text{ and } u_j \in V(C_j) \cap V_q\,. \tag{4.1}$$

Now, consider an arc $uv \in A[V(C_i), V(C_j)]$ and assume $u \in V_q$. Let v^- denote the predecessor of v in C_j. Then $uv \in \mathcal{Z}_q(u, v^-)$. Similarly, if $uv \in A[V(C_j), V(C_i)]$, $u \in V_q$, and v^- is the predecessor of v in C_i, then $uv \in \mathcal{Z}_q(v^-, u)$. Therefore

$$\overleftrightarrow{a}(V(C_i), V(C_j)) \leq \sum_{q=1}^{2} \sum_{u_i \in V(C_i) \cap V_q} \sum_{u_j \in V(C_j) \cap V_q} |\mathcal{Z}_q(u_i, u_j)|\,,$$

and hence, by (4.1),

$$\overleftrightarrow{a}(V(C_i), V(C_j)) \leq 2 \cdot \frac{|C_i|}{2} \cdot \frac{|C_j|}{2}\,,$$

which completes the proof of Claim 1.

We now return to the proof of Theorem 1.2. Repeatedly using Claim 1, we note that the following holds

$$\begin{aligned} &\overleftrightarrow{a}(V(C_1) \cap V_1, V(D) \setminus V(C_1)) + \overleftrightarrow{a}(V(C_1) \cap V_2, V(D) \setminus V(C_1)) \\ &= \overleftrightarrow{a}(V(C_1), V(D) \setminus V(C_1)) = \sum_{j=2}^{l} \overleftrightarrow{a}(V(C_1), V(C_j)) \leq \frac{|C_1|(2a - |C_1|)}{2}\,. \end{aligned} \tag{4.2}$$

Without loss of generality, we may assume that

$$\overleftrightarrow{a}(V(C_1) \cap V_1, V(D) \setminus V(C_1)) \leq \frac{|C_1|(2a - |C_1|)}{4}, \tag{4.3}$$

as otherwise

$$\overleftrightarrow{a}(V(C_1) \cap V_2, V(D) \setminus V(C_1)) \leq \frac{|C_1|(2a - |C_1|)}{4}. \tag{4.4}$$

In other words, the average number of arcs between a vertex in $V(C_1) \cap V_1$ and $V(D) \setminus V(C_1)$ is bounded above by $(2a - |C_1|)/2$ (as $|V(C_1) \cap V_1| = |C_1|/2$). We now consider the following two cases.

Case 1. $|C_1| \geq 4$.
Let $x_1, x_2 \in V(C_1) \cap V_1$ be distinct and chosen so that $\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1))$ is minimum. By the above formula we note that $\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1)) \leq 2a - |C_1|$. Since any vertex in C_1 has at most $|C_1|$ arcs to other vertices in C_1 (as there are $|C_1|/2$ vertices from V_2 in C_1) and $|C_1| \leq a$, we get that

$$d(x_1) + d(x_2) \leq 2|C_1| + 2a - |C_1| = 2a + |C_1| \leq 3a. \tag{4.5}$$

We shall now prove that every two vertices in $V_2 \cap V(C_1)$ share a common in-neighbour and that the inequality (4.4) holds. To that end, we need to consider two sub-cases depending on the properties of x_1 and x_2.

Suppose first that x_1 and x_2 have a common in-neighbour or out-neighbour. Condition $(\mathcal{A})$ then implies that we have equality in (4.5). It follows that there must be equalities in all the estimates that led to (4.5) as well. In particular,

$$\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1)) = 2a - |C_1|, \quad \text{and} \tag{4.6}$$

$$\overset{\leftrightarrow}{a}(\{x_1\}, V(C_1)) = \overset{\leftrightarrow}{a}(\{x_2\}, V(C_1)) = |C_1|. \tag{4.7}$$

By the choice of x_1 and x_2, it now follows from (4.6) that we have equality in (4.3), and hence, by (4.2), the inequality (4.4) is satisfied. Moreover, by (4.7), every two vertices in $V_2 \cap V(C_1)$ have a common in-neighbour, namely x_1.

Suppose then that x_1 and x_2 have no common in-neighbour or out-neighbour. In this case, we have

$$\begin{aligned} &|N^+(x_1) \cap (V(D) \setminus V(C_1))| + |N^+(x_2) \cap (V(D) \setminus V(C_1))| \leq a - \frac{|C_1|}{2}, \\ &|N^-(x_1) \cap (V(D) \setminus V(C_1))| + |N^-(x_2) \cap (V(D) \setminus V(C_1))| \leq a - \frac{|C_1|}{2}, \end{aligned} \tag{4.8}$$

as well as

$$\begin{aligned} &|N^+(x_1) \cap V(C_1)| + |N^+(x_2) \cap V(C_1)| \leq \frac{|C_1|}{2}, \text{ and} \\ &|N^-(x_1) \cap V(C_1)| + |N^-(x_2) \cap V(C_1)| \leq \frac{|C_1|}{2}. \end{aligned}$$

Hence, $d(x_1) + d(x_2) \leq 2a$. Therefore, by Lemma 3.3, $d(x_1) = d(x_2) = a$ and, consequently, we have equalities in (4.8). By the choice of x_1 and x_2, it follows that we have equality in (4.3), and hence, by (4.2), the inequality (4.4) holds. Moreover, by Lemma 3.2, there exists $x' \in V_1 \setminus \{x_1\}$ such that x_1 and x' have a common in-neighbour or out-neighbour. Condition $(\mathcal{A})$ then implies that $d(x') = 2a$. In particular, every two vertices in $V_2 \cap V(C_1)$ have a common in-neighbour, namely x'.

Next, let $y_1, y_2 \in V(C_1) \cap V_2$ be distinct and chosen so that $\overset{\leftrightarrow}{a}(\{y_1, y_2\}, V(D) \setminus V(C_1))$ is minimum. By (4.4), we have $\overset{\leftrightarrow}{a}(\{y_1, y_2\}, V(D) \setminus V(C_1)) \leq 2a - |C_1|$. Since any vertex in C_1 has at most $|C_1|$ arcs to other vertices in C_1 (as there are $|C_1|/2$ vertices from V_2 in C_1) and $|C_1| \leq a$, we get that

$$d(y_1) + d(y_2) \leq 2|C_1| + 2a - |C_1| = 2a + |C_1| \leq 3a. \tag{4.9}$$

Since y_1 and y_2 have a common in-neighbour, condition $(\mathcal{A})$ implies that we have equality in (4.9). It follows that there must be equalities in all the estimates that led to (4.9) as well. That is,

$$\overleftrightarrow{a}(\{y_1, y_2\}, V(D) \setminus V(C_1)) = 2a - |C_1|, \tag{4.10}$$

$$\overleftrightarrow{a}(\{y_1\}, V(C_1)) = \overleftrightarrow{a}(\{y_2\}, V(C_1)) = |C_1|, \tag{4.11}$$

$$|C_1| = a. \tag{4.12}$$

By the choice of y_1 and y_2, it now follows from (4.10) and (4.4) that

$$\overleftrightarrow{a}(\{y', y''\}, V(D) \setminus V(C_1)) = 2a - |C_1|$$

for any distinct $y', y'' \in V_2 \cap V(C_1)$. Since any two such y', y'' have a common in-neighbour, we can repeat the above argument with y' and y'' in place of y_1 and y_2 and conclude that (4.11) is satisfied by all vertices in $V_2 \cap V(C_1)$. In other words, D contains a complete bipartite digraph spanned on the vertices of C_1.

Next observe that, by minimality of $|C_1|$, (4.12) implies that $l = 2$ and $|C_1| = |C_2| = a$. Consequently, we can swap C_1 and C_2 and repeat the argument of Case 1 to get that D contains also a complete bipartite digraph spanned on the vertices of C_2.

Now, we claim that
(i) $A[V(C_1) \cap V_1, V(C_2)] \neq \varnothing$ and $A[V(C_2), V(C_1) \cap V_2] \neq \varnothing$, or
(ii) $A[V(C_1) \cap V_2, V(C_2)] \neq \varnothing$ and $A[V(C_2), V(C_1) \cap V_1] \neq \varnothing$.
Indeed, condition $(\mathcal{A})$ applied to pairs of vertices from $V(C_1) \cap V_1$ implies that there exists $x \in V(C_1) \cap V_1$ with $\overleftrightarrow{a}(\{x\}, V(C_2)) > 0$. Similarly, there exists $y \in V(C_1) \cap V_2$ such that $\overleftrightarrow{a}(\{y\}, V(C_2)) > 0$. Therefore, if neither (i) nor (ii) held, then all the arcs between C_1 and C_2 would need to go in the same direction (i.e., either $A[V(C_1), V(C_2)] = \varnothing$ or $A[V(C_2), V(C_1)] = \varnothing$). But such an arrangement is impossible in a strongly connected digraph.

Thus, without loss of generality we can assume that D contains an arc from $V(C_1) \cap V_1$ to $V(C_2)$ and an arc from $V(C_2)$ to $V(C_1) \cap V_2$. Then, however, D must be hamiltonian, because it contains complete bipartite digraphs on $V(C_1)$ and on $V(C_2)$. This contradiction completes the proof of Case 1.

Case 2. $|C_1| < 4$.
In this case $|C_1| = 2$. Let $V(C_1) \cap V_1 = \{x_1\}$ and $V(C_1) \cap V_2 = \{y_1\}$. Note that, by (4.3), we have $d(x_1) \leq 2 + (2a - |C_1|)/2 = a + 1$. By Lemma 3.2, x_1 shares a common in-neighbour or out-neighbour with a vertex, say x', in $V_1 \setminus \{x_1\}$. By condition $(\mathcal{A})$, $d(x') \geq 2a - 1$, and so

$$x'y \in A(D) \text{ for all } y \in V_2 \qquad \text{or else} \qquad yx' \in A(D) \text{ for all } y \in V_2. \tag{4.13}$$

That is, y_1 has a common in-neighbour with every vertex in $V_2 \setminus \{y_1\}$ or else y_1 has a common out-neighbour with every vertex in $V_2 \setminus \{y_1\}$. The remainder of the proof of this case is divided into two sub-cases depending on the actual value of $d(x_1)$.

Case 2a. $d(x_1) = a + 1$.
Then, by (4.2), $d(y_1) \leq a + 1$. Hence, by (4.13) and condition $(\mathcal{A})$, we have

$$d(y) \geq 2a - 1 \quad \text{for all } y \in V_2 \setminus \{y_1\}. \tag{4.14}$$

It follows that, for every $y \in V_2 \setminus \{y_1\}$, at least one of the arcs x_1y, yx_1 belongs to $A(D)$. Moreover, every $x \in V_1 \setminus \{x_1\}$ shares a common in-neighbour or out-neighbour with x_1, and so

$$d(x) \geq 2a - 1 \quad \text{for all } x \in V_1 \setminus \{x_1\}. \tag{4.15}$$

We now claim that, for every $x \neq x_1$, at most one of the arcs xy_1, y_1x is contained in $A(D)$. Suppose otherwise, and let $\tilde{x} \in V_1 \setminus \{x_1\}$ be such that $\tilde{x}y_1, y_1\tilde{x} \in A(D)$. Say, $\tilde{x} \in V(C_j)$ for some $j \neq 1$. Let $\tilde{x}^+$ (resp. $\tilde{x}^-$) denote the successor (resp. predecessor) of $\tilde{x}$ on C_j. By (4.14), one of the following must hold:

(i) $x_1\tilde{x}^+ \in A(D)$, or
(ii) $\tilde{x}^-x_1 \in A(D)$, or else
(iii) $x_1\tilde{x}^+ \notin A(D)$, $\tilde{x}^-x_1 \notin A(D)$, and $\tilde{x}^+x_1, x_1\tilde{x}^- \in A(D)$.

In the first case, one can merge C_1 with C_j by replacing the arc $\tilde{x}\tilde{x}^+$ on C_j with the path $(\tilde{x}, y_1, x_1, \tilde{x}^+)$. This contradicts the minimality of l. In the second case, one can merge C_1 with C_j by replacing the arc $\tilde{x}^-\tilde{x}$ on C_j with the path $(\tilde{x}^-, x_1, y_1, \tilde{x})$. This contradicts the minimality of l. In the third case, in turn, both $\tilde{x}^+$ and $\tilde{x}^-$ are joined by symmetric arcs with every vertex in $V_1 \setminus \{x_1\}$, by (4.14). One can thus merge C_1 with C_j by replacing the path $(\tilde{x}^{--}, \dots, \tilde{x}^{++})$ on C_j with the path $(\tilde{x}^{--}, \tilde{x}^+, x_1, y_1, \tilde{x}, \tilde{x}^-, \tilde{x}^{++})$, where $\tilde{x}^{++}$ (resp. $\tilde{x}^{--}$) denotes the successor of $\tilde{x}^+$ (resp. predecessor of $\tilde{x}^-$) on C_j. This again contradicts the minimality of l, which completes the proof of our claim. (Note that the above argument works whenever $|C_j| \geq 4$. If $|C_j| = 2$, however, there is nothing to prove, given that $\tilde{x}y_1, y_1\tilde{x} \in A(D)$ and one of (i)-(iii) holds.)

By (4.15), we now get that every $x \neq x_1$ is joined by symmetric arcs with all vertices in $V_2 \setminus \{y_1\}$. In other words, D contains a complete bipartite digraph spanned by the vertices $V(D) \setminus \{x_1, y_1\}$. Moreover, by (4.14) and (4.15), we have $\overleftrightarrow{a}(\{x_1\}, \{y\}) \geq 1$ and $\overleftrightarrow{a}(\{y_1\}, \{x\}) \geq 1$ for all $y \neq y_1, x \neq x_1$. Since in a strongly connected digraph it cannot happen that $A[V(C_1), V(D) \setminus V(C_1)] = \varnothing$ or $A[V(D) \setminus V(C_1), V(C_1)] = \varnothing$, it follows that there exist vertices $\tilde{x}, \tilde{y} \in V(D) \setminus V(C_1)$ such that $x_1\tilde{y}, \tilde{x}y_1 \in A(D)$ or $\tilde{y}x_1, y_1\tilde{x} \in A(D)$. One can readily see that then D contains a Hamilton cycle. This contradiction completes the proof of *Case 2a*.

Case 2b. $d(x_1) = a$.
Since $a \geq 3$, it follows that there exists $\tilde{y} \in V_2 \setminus \{y\}$ such that $x_1\tilde{y} \in A(D)$ or $\tilde{y}x_1 \in A(D)$. Say, $\tilde{y} \in V(C_j)$ for some $j \neq 1$. Let $\tilde{y}^+$ (resp. $\tilde{y}^-$) denote the successor (resp. predecessor) of $\tilde{y}$ on C_j. If $x_1\tilde{y} \in A(D)$, then $\tilde{y}$ is a common out-neighbour of x_1 and $\tilde{y}^-$, and so $d(\tilde{y}^-) = 2a$, by condition $(\mathcal{A})$. In particular, $\tilde{y}^-y_1 \in A(D)$, and hence C_1 can be merged with C_j by replacing the arc $\tilde{y}^-\tilde{y}$ on C_j with the path $(\tilde{y}^-, y_1, x_1, \tilde{y})$. This contradicts the minimality of l. If, in turn, $\tilde{y}x_1 \in A(D)$, then $\tilde{y}$ is a common in-neighbour of x_1 and $\tilde{y}^+$, and so $d(\tilde{y}^+) = 2a$, by condition $(\mathcal{A})$. In particular, $y_1\tilde{y}^+ \in A(D)$, and hence C_1 can be merged with C_j by replacing the arc $\tilde{y}\tilde{y}^+$ on C_j with the path $(\tilde{y}, x_1, y_1, \tilde{y}^+)$. This again contradicts the minimality of l, which completes the proof of the theorem. $\square$

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