On the Meyniel condition for hamiltonicity in bipartite digraphs

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We prove a sharp Meyniel-type criterion for hamiltonicity of a balanced bipartite digraph: For $a \geq 2$, a strongly connected balanced bipartite digraph D on $2a$ vertices is hamiltonian if $d(u) + d(v) \geq 3a$ whenever $uv \notin A(D)$ and $vu \notin A(D)$. As a consequence, we obtain a sharp sufficient condition for hamiltonicity in terms of the minimal degree: a strongly connected balanced bipartite digraph D on $2a$ vertices is hamiltonian if $\delta(D) \geq 3a/2$.

Keywords: digraph, bipartite digraph, cycle, cycle factor, hamiltonicity, degree condition

1 Introduction

1.1 Results

The main goal of this article is to prove a Meyniel-type sufficient condition for hamiltonicity of a balanced bipartite digraph. We consider digraphs in the sense of [3], and use standard graph theoretical terminology and notation (see Section 1.2 for details).

Our object of study in the present article is the class of bipartite digraphs satisfying the following Meyniel-type condition (cf. Thm. 1.6).

Definition 1.1. Consider a balanced bipartite digraph D on $2a$ vertices. For $k \geq 0$, we will say that D satisfies *condition* $(\mathcal{M}_k)$ when

$$d(u) + d(v) \geq 3a + k$$

for every pair of distinct vertices $u, v \in V(D)$ such that $uv \notin A(D)$ and $vu \notin A(D)$.

Our main result is the following:

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Theorem 1.2. *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. Then D is hamiltonian provided one of the following holds:*

(a) D satisfies condition $(\mathcal{M}_1)$, or

(b) D is strongly connected and satisfies condition $(\mathcal{M}_0)$.

(A digraph is called *strongly connected* when, for every (ordered) pair of distinct vertices u and v, D contains a directed path originating in u and terminating in v.)

There are numerous sufficient conditions for existence of hamiltonian cycles in digraphs (see [3, 5]). In this article, we will be concerned with the degree conditions. For general digraphs, let us recall the following four classical results.

Theorem 1.3 (Ghouila-Houri, 1960, [7])**.** *Let D be a strongly connected digraph on n vertices, where $n \geq 3$. If $\delta(D) \geq n$, then D is hamiltonian.*

Theorem 1.4 (Nash-Williams, 1969, [10])**.** *Let D be a digraph on n vertices, where $n \geq 3$. If $\delta^+(D) \geq n/2$ and $\delta^-(D) \geq n/2$, then D is hamiltonian.*

Theorem 1.5 (Woodall, 1972, [11])**.** *Let D be a digraph on n vertices, where $n \geq 3$. If $d^+(u) + d^-(v) \geq n$ for every pair of distinct vertices $u, v \in V(D)$ satisfying $uv \notin A(D)$, then D is hamiltonian.*

Theorem 1.6 (Meyniel, 1973, [9])**.** *Let D be a strongly connected digraph on n vertices, where $n \geq 3$. If $d(u) + d(v) \geq 2n - 1$ for any two vertices u and v such that $uv \notin A(D)$ and $vu \notin A(D)$, then D is hamiltonian.*

All the above criteria are sharp (see [5]). Note also that Theorems 1.3, 1.4 and 1.5 follow from Theorem 1.6. A beautiful short proof of the latter can be found in [6].

Naturally, for bipartite digraphs one might expect bounds for degrees of order $|D|/2$ rather than $|D|$. This is the case, indeed, for analogues of the Nash-Williams and Woodall theorems. As for the analogues of the Ghouila-Houri and Meyniel theorems, however, this expectation is quite far from reality (cf. Remark 1.10). For minimal half-degrees we have the following result.

Theorem 1.7 (Amar & Manoussakis, 1990, [2])**.** *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. If $\delta^+(D) \geq (a + 2)/2$ and $\delta^-(D) \geq (a + 2)/2$, then D is hamiltonian.*

The above criterion is sharp. Moreover, it is shown in [2] that the only non-hamiltonian digraph D satisfying $\delta^+(D) \geq (a + 1)/2$ and $\delta^-(D) \geq (a + 1)/2$ is the digraph on 6 vertices depicted in Fig. 1.

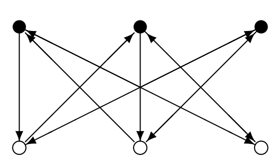

Fig. 1

An analogue of Woodall's theorem was given by Manoussakis and Millis in [8], and recently considerably strengthened by J. Adamus and L. Adamus.

Theorem 1.8 (Adamus & Adamus, 2012, [1])**.** *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. If $d^+(u)+d^-(v) \geq a+2$ for every pair of vertices u and v from the opposite colour classes such that $uv \notin A(D)$, then D is hamiltonian.*

In the present paper, we give bipartite analogues of the Ghouila-Houri and Meyniel theorems. These are Theorems 1.9 (below) and 1.2, respectively. Quite surprisingly, the bounds on degrees are much bigger than one might expect from Theorems 1.7 and 1.8 above. It is worth remarking that the methods of [6] do not carry over to the bipartite case.

Theorem 1.9. *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. Then D is hamiltonian provided one of the following holds:*

(a) $\delta(D) \geq (3a+1)/2$, or

(b) D is strongly connected and $\delta(D) \geq 3a/2$.

Of course, Theorem 1.9 is an immediate corollary of Theorem 1.2.

Remark 1.10. The bounds in Theorems 1.2 and 1.9 are sharp, as can be seen in the following two examples. Example 1.11 provides, for every even $a \geq 2$, a non-hamiltonian balanced bipartite digraph $D(a)$ on $2a$ vertices in which every vertex has degree $3a/2$. The second example (due to Amar and Manoussakis [2]) shows that, for every $a \geq 3$ and every $1 \leq l < a/2$, there is a non-hamiltonian strongly connected balanced bipartite digraph $D(a,l)$ on $2a$ vertices with $\delta(D(a,l)) = a+l$.

Example 1.11. Let a be a positive even integer, and let $D(a)$ be a bipartite digraph with colour classes V_1 and V_2 such that V_1 (resp. V_2) is a disjoint union of sets R, S (resp. U, W) of cardinality $a/2$ each, and $A(D(a))$ consists of the following arcs:
(a) ry, for all $r \in R$ and $y \in V_2$,
(b) ux, for all $u \in U$ and $x \in V_1$, and
(c) sw and ws, for all $s \in S$ and $w \in W$.
Then every vertex of $D(a)$ is of degree $3a/2$, but $D(a)$ contains no hamiltonian cycle. Notice that $D(a)$ is not strongly connected.

Example 1.12. For $a \geq 3$ and $1 \leq l < a/2$, let $D(a,l)$ be a bipartite digraph with colour classes V_1 and V_2 such that V_1 (resp. V_2) is a disjoint union of sets R, S (resp. U, W) with $|R| = |U| = l$, $|S| = |W| = a-l$, and $A(D(a,l))$ consists of the following arcs:
(a) ry and yr, for all $r \in R$ and $y \in V_2$,
(b) ux and xu, for all $u \in U$ and $x \in V_1$, and
(c) sw, for all $s \in S$ and $w \in W$.
Then $\delta(D(a,l)) = a+l$. In particular, for odd a, $\delta(D(a,(a-1)/2)) = (3a-1)/2$. Notice that $D(a,l)$ is strongly connected, but not hamiltonian.

Remark 1.13. It is also interesting to observe that under the assumptions (a) in Theorems 1.2 and 1.9, the strong-connectedness is redundant. In fact, condition $(\mathcal{M}_1)$ implies a much stronger property: a bipartite digraph D satisfying condition $(\mathcal{M}_1)$ contains a perfect matching M (cf. Lemma 2.1), and, for every pair of distinct vertices u, v, D contains a directed path from u to v which is compatible with M (i.e., a path whose every other arc belongs to M; cf. Lemma 2.2).

Finally, notice that conditions $(\mathcal{M}_k)$ cannot be weakened to apply only to pairs of vertices from the opposite colour classes (à la Theorem 1.8). This follows from the fact that there exist strongly connected non-hamiltonian balanced bipartite tournaments (Example 1.14 below). Recall that a *bipartite tournament* is a bipartite digraph D in which, for every pair of vertices x, y from the opposite colour classes, precisely one of the arcs xy, yx belongs to $A(D)$.

Example 1.14. For $a \geq 3$ and $1 \leq l < a/2$, let $T(a,l)$ be a bipartite digraph with colour classes V_1 and V_2 such that V_1 (resp. V_2) is a disjoint union of sets R, S (resp. U, W) with $|R| = |U| = l$, $|S| = |W| = a - l$, and $A(T(a,l))$ consists of the following arcs:
(a) ru, for all $r \in R$ and $u \in U$,
(b) us, for all $u \in U$ and $s \in S$,
(c) sw, for all $s \in S$ and $w \in W$, and
(d) wr, for all $w \in W$ and $r \in R$.
Then $T(a,l)$ is strongly connected and vacuously satisfies condition $(\mathcal{M}_1)$ (hence also condition $(\mathcal{M}_0)$) for every pair of vertices from the opposite colour classes, but $T(a,l)$ contains no hamiltonian cycle.

1.2 Notation and terminology

A *digraph* D is a pair $(V(D), A(D))$, where $V(D)$ is a finite set (of *vertices*) and $A(D)$ is a set of ordered pairs of distinct elements of $V(D)$, called *arcs* (i.e., D has no loops or multiple arcs). The number of vertices $|V(D)|$ is the *order* of D (also denoted by $|D|$). For vertices u and v from $V(D)$, we write $uv \in A(D)$ to say that $A(D)$ contains the ordered pair (u, v).

For a vertex set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ *dominated* by the vertices of S; i.e.,

$$N^+(S) = \{u \in V(D) : vu \in A(D) \text{ for some } v \in S\}.$$

Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ *dominating* vertices of S; i.e,

$$N^-(S) = \{u \in V(D) : uv \in A(D) \text{ for some } v \in S\}.$$

If $S = \{v\}$ is a single vertex, the cardinality of $N^+(\{v\})$ (resp. $N^-(\{v\})$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the *outdegree* (resp. *indegree*) of v in D. The *degree* of v is $d(v) = d^+(v) + d^-(v)$. Further, by $\delta^+(D)$ and $\delta^-(D)$ we will denote respectively the minimal outdegree and the minimal indegree of D; i.e., $\delta^+(D) = \min\{d^+(v) : v \in V(D)\}$ and $\delta^-(D) = \min\{d^-(v) : v \in V(D)\}$. The minimal degree of D will be denoted by $\delta(D)$.

For vertex sets $S, T \subset V(D)$, we denote by $A[S,T]$ the set of all arcs of $A(D)$ from a vertex in S to a vertex in T. Let $\overset{\leftrightarrow}{a}(S,T) = |A[S,T]| + |A[T,S]|$. Note that $\overset{\leftrightarrow}{a}(\{v\}, V(D) \setminus \{v\}) = d(v)$. A set of vertices $\{v_1, \dots, v_k\}$ such that $\overset{\leftrightarrow}{a}(\{v_i\}, \{v_j\}) = 0$, for all $i \neq j$, is called *independent*.

A directed cycle (resp. directed path) on vertices $v_1, \dots, v_m$ in D is denoted by $[v_1, \dots, v_m]$ (resp. $(v_1, \dots, v_m)$). We will refer to them as simply *cycles* and *paths* (skipping the term "directed"), since their non-directed counterparts are not considered in this article at all.

A cycle passing through all the vertices of D is called *hamiltonian*. A digraph containing a hamiltonian cycle is called a *hamiltonian digraph*. A *cycle factor* in D is a collection of vertex-disjoint cycles $C_1, \dots, C_l$ such that $V(C_1) \cup \dots \cup V(C_l) = V(D)$.

A digraph D is *bipartite* when $V(D)$ is a disjoint union of independent sets V_1 and V_2 (the *colour classes*). It is called *balanced* if $|V_1| = |V_2|$. One says that a bipartite digraph D is *complete* when $d(x) = 2|V_2|$ for all $x \in V_1$.

A *matching from* V_1 *to* V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (u_1u_2 and v_1v_2 are *independent arcs* when $u_1 \neq v_1$ and $u_2 \neq v_2$). If D is balanced, one says that such a matching is *perfect* if it consists of precisely $|V_1|$ arcs. After [2], we call a path *compatible* with a matching M from V_1 to V_2 (or, M*-compatible*, for short) if its arcs are alternately in M and in $A(D) \setminus M$.

For a perfect matching M from V_1 to V_2 and a vertex $x' \in V_1$, we will denote by $M(x')$ the unique vertex $y' \in V_2$ such that $x'y' \in M$. Similarly, for $y' \in V_2$, we will denote by $M^{-1}(y')$ the unique vertex $x' \in V_1$ for which $x'y' \in M$. Finally, for a subset $S \subset V_2$, we will denote by $M^{-1}(S)$ the set $\{M^{-1}(y) : y \in S\}$.

2 Lemmas

We prove Theorem 1.2 in Section 3. Our argument is based on the following lemma.

Lemma 2.1. *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. Suppose that D satisfies condition $(\mathcal{M}_1)$, or D is strongly connected and satisfies condition $(\mathcal{M}_0)$. Then D contains a cycle factor.*

Proof: Let V_1 and V_2 denote the two colour classes of D. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1. Therefore, by the König-Hall theorem (see, e.g., [4]), it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

For a proof by contradiction, suppose that a non-empty set $S \subset V_1$ is such that $|N^+(S)| < |S|$. Then $V_2 \setminus N^+(S) \neq \varnothing$ and, for every $y \in V_2 \setminus N^+(S)$, we have $d^-(y) \leq a - |S|$, hence $d(y) \leq 2a - |S|$. We now consider the following two cases.

Case 1.

$|S| > a/2$.
Then $d(y) < 2a - a/2 = 3a/2$ for all $y \in V_2 \setminus N^+(S)$. Therefore, if $V_2 \setminus N^+(S)$ contains at least two elements, say y_1 and y_2, then $d(y_1) + d(y_2) < 3a$, which contradicts condition $(\mathcal{M}_0)$ (hence also $(\mathcal{M}_1)$). Suppose then that $V_2 \setminus N^+(S) = \{y\}$. This can only happen when $|S| = a$, that is, when $S = V_1$. In this case, however, the vertex y is not dominated by any vertex of D. Clearly, such a digraph is not strongly connected. Without the strong connectedness assumption, in turn, we have $d(y) = d^+(y) + d^-(y) \leq a + 0$, hence $d(y) + d(y') \leq a + 2a$, for any $y' \in V_2 \setminus \{y\}$, which contradicts condition $(\mathcal{M}_1)$.

Case 2.

$|S| \leq a/2$.
If this is so then, for every $x \in S$, we have $d(x) = d^-(x) + d^+(x) \leq a + (|S| - 1) \leq (3a-2)/2$. Therefore, if S contains at least two elements, say x_1 and x_2, we get $d(x_1) + d(x_2) \leq 3a - 2$, which contradicts $(\mathcal{M}_0)$ (hence also $(\mathcal{M}_1)$). Suppose then that $S = \{x\}$, and hence $N^+(S) = \varnothing$, by our hypothesis. It follows that x does not dominate any vertex in D, which leads to a contradiction if D is assumed strongly connected. Without the strong connectedness assumption, in turn, we have $d(x) = d^-(x) + d^+(x) \leq a + 0$, hence $d(x) + d(x') \leq a + 2a$, for any $x' \in V_1 \setminus \{x\}$, which contradicts condition $(\mathcal{M}_1)$.

This completes the proof of existence of a perfect matching from V_1 to V_2. The proof for a matching in the opposite direction is analogous. □

As advertised in Remark 1.13, every digraph D satisfying condition $(\mathcal{M}_1)$ is strongly connected. In fact, much more is true: By Lemma 2.1, condition $(\mathcal{M}_1)$ implies that D contains a perfect matching from V_1 to V_2. Moreover, D contains a perfect matching M such that any two vertices in D can be connected by an M-compatible path. This is clear when D is hamiltonian. The non-hamiltonian case is settled in the following lemma. (Note that the lemma is not needed for the proof of Theorem 1.2.)

Lemma 2.2. *Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$, which satisfies condition $(\mathcal{M}_1)$. Let V_1 and V_2 be the colour classes of D, and let M be a perfect matching from V_1 to V_2. Suppose that D is not hamiltonian. Then, for every pair of distinct vertices $u, v \in V(D)$, D contains an M-compatible path from u to v.*

Proof: First, we claim that it suffices to show that D contains an M-compatible path from y to x for every pair of vertices such that $y \in V_2$ and $x \in V_1$. Indeed, to find an M-compatible path in D from $x' \in V_1$ to $x'' \in V_1$, it suffices to find an M-compatible path from $M(x')$ to x''. Likewise, to find an M-compatible path from $y' \in V_2$ to $y'' \in V_2$, it suffices to find an M-compatible path from y' to $M^{-1}(y'')$. Finally, to find an M-compatible path from $x' \in V_1$ to $y'' \in V_2$, it suffices to find an M-compatible path from $M(x')$ to $M^{-1}(y'')$ (unless $x'y''$ already is in M).

Next, observe that one can assume

$$d^+(v) \geq 2 \quad \text{and} \quad d^-(v) \geq 2 \quad \text{for all } v \in V(D)\,. \tag{2.1}$$

Indeed, for if $d^+(v') < 2$ for some $v' \in V(D)$, then $d(v') \leq a+1$, hence, by condition $(\mathcal{M}_1)$, $d(v) \geq 2a$ for all $v \neq v'$ from the same colour class. Since every degree is bounded above by $2a$, we would actually have $d(v) = 2a$ for all $v \neq v'$ from the colour class of v', as well as $d^+(v') = 1$ and $d^-(v') = a$. It is readily seen that then D would contain a hamiltonian cycle, contrary to our assumption. The argument for $d^-(v)$ is analogous.

Now, for a proof by contradiction, suppose that $y \in V_2$ and $x \in V_1$ are such that D contains no path from y to x compatible with M. Let V_2^y denote the set of vertices in V_2 which can be reached from y by a path compatible with M, and let V_1^{no} denote the set of vertices in V_1 which cannot be reached from y by a path compatible with M. Then $y \in V_2^y$, $x \in V_1^{no}$, and $A[V_2^y, V_1^{no}] = \varnothing$, by the definition of V_2^y and V_1^{no}.

Note that, if $y' \in V_2 \setminus V_2^y$, then $M^{-1}(y') \in V_1^{no}$, for otherwise there would be an M-compatible path from y to $M^{-1}(y')$, hence also to y'; a contradiction. This implies that $|V_1^{no}| \geq |V_2 \setminus V_2^y| = a - |V_2^y|$, and so $|V_1^{no}| + |V_2^y| \geq a$. As $A[V_2^y, V_1^{no}] = \varnothing$, it follows from (2.1) that $|V_2^y| \leq a-2$ and $|V_1^{no}| \leq a-2$, which in turn implies that $|V_2^y| \geq 2$ and $|V_1^{no}| \geq 2$ (as $|V_1^{no}| + |V_2^y| \geq a$). Let then $u_1, v_1 \in V_1^{no}$ and $u_2, v_2 \in V_2^y$ be four arbitrary vertices. By the above estimates, one gets that

$$\begin{aligned}(d(u_1) + d(v_1)) + (d(u_2) + d(v_2)) &\leq 2(2a - |V_2^y|) + 2(2a - |V_1^{no}|) \\ &= 8a - 2(|V_2^y| + |V_1^{no}|) \leq 8a - 2a\,.\end{aligned}$$

Therefore, $d(u_1) + d(v_1) \leq 3a$ or $d(u_2) + d(v_2) \leq 3a$; a contradiction. □

3 Proof of the main result

Proof of Theorem 1.2

Let D be a balanced bipartite digraph on $2a$ vertices, and let V_1 and V_2 denote its colour classes. We will proceed by induction on a. For $a = 2$ the theorem is clearly true, so suppose that $a \geq 3$ and the theorem holds for $a - 1$.

By Lemma 2.1, D contains a cycle factor $C_1, C_2, \ldots, C_l$. Assume that l is minimum possible, and for a proof by contradiction suppose that $l \geq 2$. Recall that $|C_i|$ denotes the order of cycle C_i. Without loss of generality, assume that $|C_1| \leq |C_2| \leq \cdots \leq |C_l|$.

Claim 1:

$\overset{\leftrightarrow}{a}(V(C_i), V(C_j)) \leq \frac{|C_i| \cdot |C_j|}{2}$, for all $i \neq j$.

Proof of Claim 1.

Let $q \in \{1, 2\}$, $x_i \in V(C_i) \cap V_q$ and $x_j \in V(C_j) \cap V_q$ be arbitrary. Let x_i^+ be the successor of x_i in C_i and let x_j^+ be the successor of x_j in C_j. Let $\mathcal{Z}_q(x_i, x_j)$ be defined as $A(D) \cap \{x_i x_j^+,\ x_j x_i^+\}$. If $|\mathcal{Z}_q(x_i, x_j)| = 2$ for some x_i, x_j, then the cycles C_i and C_j can be merged into one cycle by deleting the arcs $x_i x_i^+$ and $x_j x_j^+$ and adding the arcs $x_i x_j^+$ and $x_j x_i^+$. This would contradict the minimality of l, so we may assume that

$$|\mathcal{Z}_q(x_i, x_j)| \leq 1 \quad \text{for all } x_i \in V(C_i) \cap V_q \text{ and } x_j \in V(C_j) \cap V_q\,. \tag{3.1}$$

Now, consider an arc $uv \in A[V(C_i), V(C_j)]$ and assume $u \in V_q$. Let v^- denote the predecessor of v in C_j. Then $uv \in \mathcal{Z}_q(u, v^-)$. Similarly, if $uv \in A[V(C_j), V(C_i)]$, $u \in V_q$, and v^- is the predecessor of v in C_i, then $uv \in \mathcal{Z}_q(v^-, u)$. Therefore

$$\overset{\leftrightarrow}{a}(V(C_i), V(C_j)) \leq \sum_{q=1}^{2} \sum_{x_i \in V(C_i) \cap V_q} \sum_{x_j \in V(C_j) \cap V_q} |\mathcal{Z}_q(x_i, x_j)|\,,$$

and hence, by (3.1),

$$\overset{\leftrightarrow}{a}(V(C_i), V(C_j)) \leq 2 \cdot \frac{|C_i|}{2} \cdot \frac{|C_j|}{2}\,,$$

which completes the proof of Claim 1.

We now return to the proof of Theorem 1.2. Repeatedly using Claim 1, we note that the following holds

$$\begin{aligned} &\overset{\leftrightarrow}{a}(V(C_1) \cap V_1, V(D) \setminus V(C_1)) + \overset{\leftrightarrow}{a}(V(C_1) \cap V_2, V(D) \setminus V(C_1)) \\ &\qquad = \overset{\leftrightarrow}{a}(V(C_1), V(D) \setminus V(C_1)) = \sum_{j=2}^{l} \overset{\leftrightarrow}{a}(V(C_1), V(C_j)) \leq \frac{|C_1|(2a - |C_1|)}{2}\,. \end{aligned} \tag{3.2}$$

Without loss of generality, we may assume that

$$\overset{\leftrightarrow}{a}(V(C_1) \cap V_1, V(D) \setminus V(C_1)) \leq \frac{|C_1|(2a - |C_1|)}{4}, \tag{3.3}$$

as otherwise

$$\overset{\leftrightarrow}{a}(V(C_1) \cap V_2, V(D) \setminus V(C_1)) \leq \frac{|C_1|(2a - |C_1|)}{4}.$$

In other words, the average number of arcs between a vertex in $V(C_1) \cap V_1$ and $V(D) \setminus V(C_1)$ is bounded above by $(2a - |C_1|)/2$ (as $|V(C_1) \cap V_1| = |C_1|/2$). We now consider the following two cases.

Case 1.

$|C_1| \geq 4$.

Let $x_1, x_2 \in V(C_1) \cap V_1$ be distinct and chosen so that $\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1))$ is minimum. By the above formula we note that $\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1)) \leq 2a - |C_1|$. Since any vertex in C_1 has at most $|C_1|$ arcs to other vertices in C_1 (as there are $|C_1|/2$ vertices from V_2 in C_1) and $|C_1| \leq a$, we get that

$$d(x_1) + d(x_2) \leq 2|C_1| + 2a - |C_1| = 2a + |C_1| \leq 3a\,, \tag{3.4}$$

which leads to contradiction if D is assumed to satisfy condition $(\mathcal{M}_1)$.

Suppose then that D is strongly connected and satisfies condition $(\mathcal{M}_0)$, and we have equality in (3.4). It follows that there must be equalities in all the estimates that led to (3.4) as well. That is,

$$\overset{\leftrightarrow}{a}(\{x_1, x_2\}, V(D) \setminus V(C_1)) = 2a - |C_1|, \tag{3.5}$$

$$\overset{\leftrightarrow}{a}(\{x_1\}, V(C_1)) = \overset{\leftrightarrow}{a}(\{x_2\}, V(C_1)) = |C_1|, \tag{3.6}$$

$$|C_1| = a. \tag{3.7}$$

By the choice of x_1 and x_2, it now follows from (3.3) and (3.5) that $\overset{\leftrightarrow}{a}(\{x_i, x_j\}, V(D) \setminus V(C_1)) = 2a - |C_1|$ for any distinct $x_i, x_j \in V(C_1) \cap V_1$. Therefore, either we can find a pair of vertices $x_1', x_2' \in V(C_1) \cap V_1$ with $d(x_1') + d(x_2') < 3a$, or else condition (3.6) holds for all vertices from $V(C_1) \cap V_1$. The former would contradict condition $(\mathcal{M}_0)$, so we get that $\overset{\leftrightarrow}{a}(\{x\}, V(C_1)) = |C_1|$ for every $x \in V(C_1) \cap V_1$. In other words, D contains a complete bipartite digraph spanned on the vertices of C_1.

Next observe that, by minimality of $|C_1|$, (3.7) implies that $l = 2$ and $|C_1| = |C_2| = a$. Consequently, we can swap C_1 and C_2 and repeat the above argument to get that D contains also a complete bipartite digraph spanned on the vertices of C_2.

Now, we claim that

(i) $A[V(C_1) \cap V_1, V(C_2)] \neq \varnothing$ and $A[V(C_2), V(C_1) \cap V_2] \neq \varnothing$, or

(ii) $A[V(C_1) \cap V_2, V(C_2)] \neq \varnothing$ and $A[V(C_2), V(C_1) \cap V_1] \neq \varnothing$.

Indeed, condition $(\mathcal{M}_0)$ applied to pairs of vertices from $V(C_1) \cap V_1$ implies that there exists $x \in V(C_1) \cap V_1$ with $\overset{\leftrightarrow}{a}(\{x\}, V(C_2)) > 0$. Similarly, there exists $y \in V(C_1) \cap V_2$ such that $\overset{\leftrightarrow}{a}(\{y\}, V(C_2)) > 0$. Therefore, if neither (i) nor (ii) held, then all the arcs between C_1 and C_2 would need to go in the same direction (i.e., either $A[V(C_1), V(C_2)] = \varnothing$ or $A[V(C_2), V(C_1)] = \varnothing$). But such an arrangement is impossible in a strongly connected digraph.

Thus, without loss of generality we can assume that D contains an arc from $V(C_1) \cap V_1$ to $V(C_2)$ and an arc from $V(C_2)$ to $V(C_1) \cap V_2$. Then, however, D must be hamiltonian, because it contains complete bipartite digraphs on $V(C_1)$ and on $V(C_2)$. This contradiction completes the proof of Case 1.

Case 2.

$|C_1| < 4$.
In this case $|C_1| = 2$. Let $V(C_1) \cap V_1 = \{x\}$ and $V(C_1) \cap V_2 = \{y\}$. Note that, by (3.3), we have $d(x) \leq 2 + (2a - |C_1|)/2 = a + 1$. Hence, if D satisfies condition $(\mathcal{M}_1)$, then we can assume that $d(x') = 2a$ for all $x' \in V_1 \setminus \{x\}$. If $xu \in A(D)$ and $u \in C_j$ with $j > 1$, then let u^- be the predecessor of u in C_j and note that we can merge C_1 and C_j into one cycle using the arcs xu and u^-y (which exists as $d(u^-) = 2a$) instead of xy and u^-u, which contradicts the minimality of l. Therefore there are no arcs from x to $V(D) \setminus \{y\}$. Analogously, there are no arcs ux from $V(D) \setminus \{y\}$ to x (as we could then merge using ux and yu^+, where u^+ is the successor of u on its cycle). Hence $d(x) = 2$, and so $d(x) + d(x') = 2a + 2 < 3a + 1$ for every $x' \in V_1 \setminus \{x\}$, which contradicts condition $(\mathcal{M}_1)$.

Suppose then that D is strongly connected and satisfies condition $(\mathcal{M}_0)$. We can then assume that

$$d(x') \geq 2a - 1, \text{ for all } x' \in V_1 \setminus \{x\}. \tag{3.8}$$

Hence, for all such x', $A(D)$ contains at least one of the arcs $x'y$ and yx'. The remainder of the proof is divided into two subcases depending on the actual value of $d(x)$.

Case 2a.

$d(x) = a + 1$.
Then, by (3.2), $d(y) \leq a + 1$ and hence, as above,

$$d(y') \geq 2a - 1, \text{ for all } y' \in V_2 \setminus \{y\}. \tag{3.9}$$

We now want to show that $l = 2$. Let D' be the sub-digraph of D spanned on the vertices of $V(D)\setminus\{x, y\}$. Note that D' is a balanced bipartite digraph on $2a'$ vertices, where $a' = a-1$. For the sake of contradiction, suppose that $l > 2$, that is, D' is not hamiltonian. By (3.8) and (3.9), we note that $\overset{\leftrightarrow}{a}(\{u\}, V(D')) \geq 2a' - 1$ for all $u \in V(D')$, which implies that $d_{D'}(u) + d_{D'}(v) \geq 4a' - 2$ for all $u, v \in V(D')$. Therefore, if $a' \geq 3$, then $4a' - 2 \geq 3a' + 1$ and condition $(\mathcal{M}_1)$ (hence also $(\mathcal{M}_0)$) holds for D'. Then, by induction, D' is hamiltonian; a contradiction. It follows that $a' \leq 2$, which implies that $a' = 2$, $l = 3$ and $|C_1| = |C_2| = |C_3| = 2$. In this case, we have $4a' - 2 = 3a'$, so condition $(\mathcal{M}_0)$ holds for D' which, by induction, implies that D' is not strongly connected. However, some vertex $u \in V_2 \setminus V(C_1)$ must have $ux \notin A(D)$ or $xu \notin A(D)$, as otherwise $d(x) = 2a > a + 1$, contrary to the assumption of Case 2a. For such u, by (3.9), we have $\overset{\leftrightarrow}{a}(\{u\}, V(D')) = 2a'$. Therefore, there are arcs in both directions between C_2 and C_3 implying that D' is strongly connected; a contradiction. We thus proved that $l = 2$.

Pick any $u \in V_1 \setminus \{x\}$. Suppose that $uy \in A(D)$. Then $xu^+ \notin A(D)$ (for else we could merge C_1 and C_2), and hence $u^+x \in A(D)$, by (3.9). Consequently, $yu^{++} \notin A(D)$ (where u^{++} is the successor of u^+ on C_2), and hence $u^{++}y \in A(D)$, by (3.8). Repeatedly using this argument, we eventually arrive at u^-, the predecessor of u on C_2: Its predecessor u^{--} dominates y, so $xu^- \notin A(D)$. Then $u^-x \in A(D)$, by (3.9), and thus $yu \notin A(D)$. Therefore we obtain that $A[V(C_1), V(C_2)] = \varnothing$, which contradicts the strong connectedness of D.

If, in turn, $uy \notin A(D)$, then $yu \in A(D)$, by (3.8), hence $u^-x \notin A(D)$ (for else we could merge C_1 and C_2), and so $xu^- \in A(D)$, by (3.9). Consequently, $u^{--}y \notin A(D)$, hence $yu^{--} \in A(D)$, by (3.8). Repeatedly using this argument, we get that $A[V(C_2), V(C_1)] = \varnothing$, which is impossible in a strongly connected D. This contradiction completes the proof of Case 2a.

Case 2b.

$d(x) < a + 1$.

Then, by condition $(\mathcal{M}_0)$, we have $d(x') \geq 2a$ for all $x' \in V_1 \setminus \{x\}$. Hence, in fact, $d(x') = 2a$ for all such x', and $d(x) = a$, by $(\mathcal{M}_0)$ again. Now, $A(D)$ contains all the arcs between the vertices of $V_1 \setminus \{x\}$ and V_2. Also, $A(D)$ contains xy, yx, and at least one arc of the form xy' or $y'x$ for some $y' \in V_2 \setminus \{y\}$ (because $d(x) = a \geq 3$). One readily sees that such a D is necessarily hamiltonian. This contradicts the hypothesis that $l > 1$, which completes the proof of the theorem. □

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