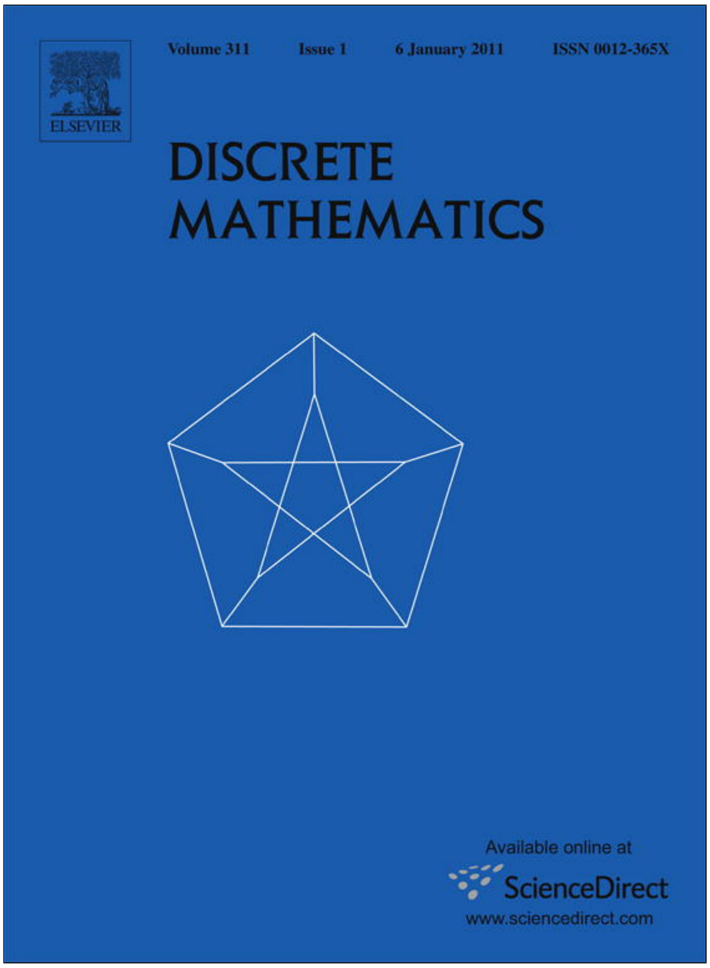
Volume 311
Issue 1
6 January 2011
ISSN 0012-365X
ELSEVIER
DISCRETE
MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 312 (2012) 1117–1122

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A degree condition for cycles of maximum length in bipartite digraphs

Janusz Adamus [a,b,*], Lech Adamus [c]

[a] Department of Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7 Canada
[b] Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland
[c] AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Krakow, Poland

ARTICLE INFO

Article history:
Received 1 February 2011
Received in revised form 27 November 2011
Accepted 28 November 2011

Keywords:
Digraph
Bipartite digraph
Cycle
Hamiltonicity

ABSTRACT

We prove a sharp Ore-type criterion for hamiltonicity of balanced bipartite digraphs: for $a \geq 2$, a bipartite digraph D with colour classes of cardinalities a is hamiltonian if $d^+(u) + d^-(v) \geq a + 2$ whenever u and v lie in opposite colour classes and $uv \notin A(D)$.

1. Introduction

The main purpose of this note is to give a sharp Ore-type sufficient condition for hamiltonicity of balanced bipartite digraphs. A *digraph* D is a pair $(V(D), A(D))$, where $V(D)$ is a finite set (of *vertices*) and $A(D)$ is a set of ordered pairs of elements of $V(D)$, called *arcs*. For vertices u and v from $V(D)$, we write $uv \in A(D)$ to say that $A(D)$ contains the ordered pair (u, v). For a vertex $v \in V(D)$, we denote by $d^+(v)$ (resp. $d^-(v)$) the number of vertices $u \in V(D)$ such that $vu \in A(D)$ (resp. $uv \in A(D)$). We call $d^+(v)$ and $d^-(v)$ the *positive* and *negative half-degree* of v, respectively. Further, $\delta^+(D)$ (resp. $\delta^-(D)$) denotes the minimum of $d^+(v)$ (resp. $d^-(v)$) as v runs over all vertices of D. A digraph D is *bipartite* when $V(D)$ is a disjoint union of sets X and Y (the *colour classes*) such that $A(D) \cap (X \times X) = \varnothing$ and $A(D) \cap (Y \times Y) = \varnothing$. It is called *balanced* if $|X| = |Y|$. See Section 1.1 for details on notation and terminology.

Definition 1.1. Consider a balanced bipartite digraph D with colour classes X and Y of cardinalities a. For $k \geq 0$, we will say that D satisfies condition A_k^* when

$$d^+(u) + d^-(v) \geq a + k$$

for all u and v from opposite colour classes such that $uv \notin A(D)$.

Our main result is the following:

Theorem 1.2. *Let D be a balanced bipartite digraph with colour classes X and Y of cardinalities a, where $a \geq 2$. If D satisfies condition A_2^*, then D contains an oriented cycle of length $2a$.*

* Correspondence to: Department of Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7 Canada.
E-mail addresses: jadamus@uwo.ca (J. Adamus), adamus@agh.edu.pl (L. Adamus).

0012-365X/$ – see front matter Crown Copyright © 2011 Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.11.032

There are numerous sufficient conditions for existence of cycles in digraphs (see [3]). In this note, we will be concerned with the degree conditions. For general digraphs, the Dirac- and Ore-type conditions are due, respectively, to Nash-Williams and Woodall.

Theorem 1.3 (*Nash-Williams [7]*). *Let D be a digraph on n vertices, where $n \geq 3$. If $\delta^+(D) \geq n/2$ and $\delta^-(D) \geq n/2$, then D contains an oriented cycle of length n.*

Theorem 1.4 (*Woodall [8]*). *Let D be a digraph on n vertices, where $n \geq 3$. If $d^+(x) + d^-(y) \geq n$ for every pair of distinct vertices $x, y \in V(D)$ satisfying $xy \notin A(D)$, then D contains an oriented cycle of length n.*

In terms of the total degrees, we have the following result of Meyniel (see [4] for a short proof). Here $d(x) = d^+(x)+d^-(x)$.

Theorem 1.5 (*Meyniel [6]*). *Let D be a digraph on n vertices ($n \geq 3$) in which, for any two distinct vertices x and y, there is an oriented path from x to y and from y to x. If $d(x)+d(y) \geq 2n-1$ for any two vertices x and y such that $xy \notin A(D)$ and $yx \notin A(D)$, then D contains an oriented cycle of length n.*

Naturally, for bipartite digraphs one can expect degree bounds of roughly $|D|/2$ rather than $|D|$.

Theorem 1.6 (*Amar and Manoussakis [1]*). *Let D be a bipartite digraph having colour classes X and Y such that $|X| = a \leq b = |Y|$. If $\delta^+(D) \geq (a+2)/2$ and $\delta^-(D) \geq (a+2)/2$, then D contains an oriented cycle of length 2a.*

In case $a = b$, the above theorem gives a Dirac-type condition for hamiltonicity of a balanced bipartite digraph. In [1], one also finds a characterization of all the bipartite digraphs that do not contain an oriented cycle of length $2a$, but satisfy $\delta^+(D) \geq (a+1)/2$ and $\delta^-(D) \geq (a+1)/2$.

As far as the Ore-type conditions for bipartite digraphs go, relatively little is known. The following result of [5] was the main motivation for the present work. A bipartite digraph D, with colour classes X and Y such that $|X| = a \leq b = |Y|$, is said to satisfy condition A_k ($k \geq 0$) when $d^+(u) + d^-(v) \geq a + k$ for all u and v such that $uv \notin A(D)$.

Theorem 1.7 (*Manoussakis and Milis [5]*). *Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \leq b = |Y|$. If D satisfies A_2, then D contains an oriented cycle of length 2a.*

The problem with the above result is that condition A_2 concerns *all* pairs of non-neighbouring vertices of D. In particular, it concerns the pairs of vertices from the same colour class, which puts a very restrictive assumption on D. To make condition A_2 more meaningful, one thus needs to require that only the pairs of vertices from opposite colour classes be considered (as in Definition 1.1 above).

We conjecture the following (and prove it for $a = b$ in the next section).

Conjecture 1.8. *Let D be a bipartite digraph with colour classes X and Y such that $|X| = a \leq b = |Y|$. If*

$$d^+(u) + d^-(v) > \frac{a+b+2}{2} \tag{1.1}$$

whenever u and v lie in opposite colour classes and $uv \notin A(D)$, then D contains an oriented cycle of length 2a.

Remark 1.9. We suspect that condition (1.1) is sharp, but we do not know how to generalise the following example of [1] (Fig. 1) for arbitrarily large a. Here $a = b = 3$, and all the vertices have both positive and negative half-degree equal to 2. Therefore, the sum of half-degrees of any pair of vertices is 4; i.e., equal to $(a + b + 2)/2$. However, no oriented cycle of length 6 is contained in this digraph.

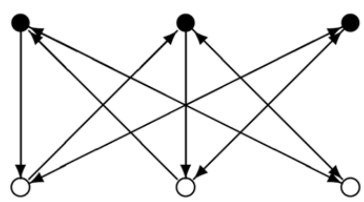

Fig. 1.

Remark 1.10. Note also that the bound $(a + b + 2)/2$ in (1.1) cannot be replaced, in general, by a bound of the type $a + k$, for any $k \in \mathbb{N}$. Indeed, for $k \in \mathbb{N}$ and any $b \geq a+2k+2$, let D be the disjoint union of digraphs $K^*_{1,k+2}$ and $K^*_{a-1,b-k-2}$ (Fig. 2), where $K^*_{k,l}$ denotes the complete bipartite digraph with colour classes of cardinalities k and l. Clearly D does not contain an oriented cycle of length $2a$, but the sum of half-degrees of non-neighbouring vertices from opposite colour classes is either $(a-1)+(k+2) = a+k+1$ or $1+(b-k-2) = b-k-1$, so in any case it is greater than or equal to $a+k+1$.

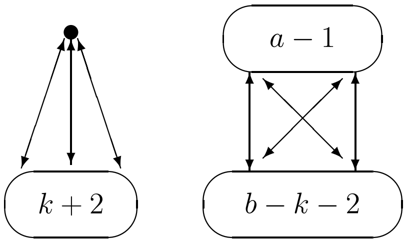

Fig. 2.

1.1. Notation and terminology

This paper is concerned with digraphs, in the sense of [3]. That is, the set $A(D)$ of arcs of D consists only of ordered pairs of vertices of D (i.e., D has no loops or multiple arcs). Given a digraph D, we denote by $V(D)$ the set of its vertices, and the number of vertices $|V(D)|$ is the *order* of D. We write $xy \in A(D)$ to say that an arc from a vertex x to a vertex y is contained in D. If $xy \in A(D)$, then x and y are called *neighbours*. For a set $S \subset V(D)$, we denote by $N^+(S)$ the set of *vertices dominated* by the vertices of S; i.e.,

$$N^+(S) = \{v \in V(D) : uv \in A(D) \text{ for some } u \in S\}.$$

Similarly, $N^-(S)$ denotes the set of *vertices dominating* the vertices of S; i.e.,

$$N^-(S) = \{v \in V(D) : vu \in A(D) \text{ for some } u \in S\}.$$

For $S = \{u\}$, we set $d^+(u) = |N^+(u)|$ and $d^-(u) = |N^-(u)|$, which we call the *positive* and *negative half-degree* of u, respectively.[1] Further, $\delta^+(D)$ and $\delta^-(D)$ denote respectively the least positive and the least negative half-degrees of D. A digraph obtained from D by removing the vertices of S and their incident arcs is denoted by $D \setminus V(S)$.

For $u \in V(D)$ and $S \subset V(D)$, we set $N_S^+(u)$ (resp. $N_S^-(u)$) to be the set of vertices of S dominated by (resp. dominating) u, and denote its cardinality by $d_S^+(u)$ (resp. $d_S^-(u)$).

An oriented cycle (resp. oriented path) on m vertices in D is denoted by C_m (resp. P_m). If the vertices are $v_1, \dots, v_m$, we write $C_m = [v_1, \dots, v_m]$ and $P_m = (v_1, \dots, v_m)$. We will refer to them as simply *cycles* and *paths* (skipping the term "oriented"), since their non-oriented counterparts are not considered in this note at all.

Let D be a bipartite digraph, with colour classes X and Y. We say that D is *balanced* if $|X| = |Y|$. A *matching from X to Y* is an independent set of arcs with origin in X and terminus in Y. If G is balanced, one says that such a matching is *complete* if it consists of precisely $|X|$ arcs. A path or cycle is said to be *compatible* with a matching M from X to Y if its arcs are alternately in M and in $A(D) \setminus M$.

2. Proof of the main result

In this section, we prove Theorem 1.2. For the rest of the paper, D denotes a balanced bipartite digraph with colour classes X and Y, where $|X| = |Y| = a$ (hence $|V(D)| = 2a$). Recall condition A_k^* of Definition 1.1.

2.1. Lemmas

The proof of Theorem 1.2 is based on the following four simple lemmas and a remark.

Lemma 2.1. *If D satisfies condition A_0^*, then D contains a complete matching from X to Y.*

Proof. By the König–Hall theorem (see, e.g., [2]), it suffices to show that $|N^+(S)| \geq |S|$ for every set $S \subset X$. If $N^+(S) = Y$, then there is nothing to show. Otherwise, we can choose vertices $x \in S$ and $y \in Y \setminus N^+(S)$. Now $xy \notin A(D)$; therefore, by assumption,

$$a \leq d^+(x) + d^-(y) \leq |N^+(S)| + |X \setminus S| = |N^+(S)| + a - |S|.$$

Hence $|N^+(S)| \geq |S|$, as required. □

Remark 2.2. Suppose D contains a complete matching M from X to Y, and let $(p_1, \dots, p_s)$ be a path in D compatible with M, and of maximal length among paths compatible with M. (We will say "*maximal path compatible with M*" for short.) Denote this path by P. It follows from maximality of P that $p_1 \in X$ and $p_s \in Y$. Hence, in particular, s is even.

Indeed, if $p_1 \in Y$, then p_1 is dominated by a vertex $x \in X \setminus V(P)$ such that $xp_1 \in M$ (by completeness of M). If $x = p_s$, then P is, in fact, a cycle and we can renumber its vertices so that $p_1 \in X$ (and hence $p_s \in Y$). Otherwise, $(x, p_1, \dots, p_s)$ is a path compatible with M of length greater than P; a contradiction. Similarly, if $p_s \in X$ (and $p_sp_1 \notin M$) then there exists $y \in Y \setminus V(P)$ such that $p_sy \in M$, again contradicting the maximality of P.

[1] Also known in literature as the *outdegree* and *indegree*.

Lemma 2.3. *Assume that D satisfies condition A_1^*, and the order of D is at least 4 (i.e., $a \geq 2$). Choose M a complete matching from X to Y and P a maximal path compatible with M. Write $P = (p_1, \ldots, p_s)$. If $p_s p_1 \in A(D)$, then D contains an oriented cycle C_{2a} compatible with M.*

Proof. We will show that $s = 2a$. For a proof by contradiction, suppose otherwise, so $Y \setminus V(P) \neq \varnothing$.

If $yp_i \in A(D)$ for some $y \in Y \setminus V(P)$ and $p_i \in V(P)$, then

$$(y, p_i, p_{i+1}, \ldots, p_s, p_1, \ldots, p_{i-1})$$

is a path compatible with M and longer than P; a contradiction. We can thus assume that no vertex of P is dominated by a vertex from $Y \setminus V(P)$. Hence $d^-(p_i) \leq |V(P)|/2 = s/2$ for all $p_i \in V(P)$, and $d^+(y) \leq |X \setminus V(P)| = a - s/2$ for all $y \in Y \setminus V(P)$. Therefore, for any $p_i \in X \cap V(P)$ and $y \in Y \setminus V(P)$, we have

$$a + 1 \leq d^+(y) + d^-(p_i) \leq (a - s/2) + s/2 = a.$$

The contradiction proves that $Y \setminus V(P) = \varnothing$, and hence $s = 2a$. □

Lemma 2.4. *Assume that D satisfies condition A_k^*, where $k \geq 1$, and the order of D is at least 4 (i.e., $a \geq 2$). If M is a complete matching from X to Y, then there exists l, $l \geq a + k$, such that D contains an oriented cycle C_l compatible with M.*

Proof. Let P be a maximal path compatible with M. Write $P = (p_1, \ldots, p_s)$. If $p_s p_1 \in A(D)$, then, by Lemma 2.3, D contains a cycle C_{2a} compatible with M. Suppose then that $p_s p_1 \notin A(D)$. Recall that $p_1 \in X$ and $p_s \in Y$ (Remark 2.2). By maximality of P, vertex p_1 is not dominated by any $y \in Y \setminus V(P)$, and vertex p_s does not dominate any $x \in X \setminus V(P)$. Therefore, by assumption,

$$a + k \leq d^+(p_s) + d^-(p_1) = d^+_{V(P)}(p_s) + d^-_{V(P)}(p_1),$$

and hence $d^+_{V(P)}(p_s) \geq (a + k)/2$ or else $d^-_{V(P)}(p_1) \geq (a + k)/2$.

In the first case, let $i_0 = \min\{i : p_s p_i \in A(D)\}$. Then $[p_{i_0}, p_{i_0+1}, \ldots, p_s]$ is a cycle in D compatible with M and of length at least $2d^+_{V(P)}(p_s)$, which is greater than or equal to $a+k$. In the second case, let $j_0 = \max\{j : p_j p_1 \in A(D)\}$. Then $[p_1, p_2, \ldots, p_{j_0}]$ is a required cycle of length at least $2d^-_{V(P)}(p_1)$, which is greater than or equal to $a + k$. □

Lemma 2.5. *Let M be a complete matching from X to Y in D. Let C be a maximal cycle in D compatible with M, and let $(u_1, v_1, \ldots, u_p, v_p)$ be a path in $D \setminus V(C)$, denoted by P, compatible with M, where $u_i \in X$ and $v_i \in Y$. If $d^-_{V(C)}(u_1) > 0$ and $d^+_{V(C)}(v_p) > 0$, then $d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1) \leq m - p + 1$, where m is half the length of C.*

Proof. Write $C = [x_1, y_1, \ldots, x_m, y_m]$, with $x_\nu \in X$ and $y_\nu \in Y$ $(1 \leq \nu \leq m)$. By assumption, there exist y_i and x_j on C such that $y_i u_1 \in A(D)$ and $v_p x_j \in A(D)$. Let $(x_{i+1}, y_{i+1}, \ldots, x_{j-1}, y_{j-1})$ be the path, denoted by P^{ij}, between y_i and x_j on C, traversed according to the orientation of C; of order, say, $2l$. Then $l \geq p$, because otherwise the cycle $[v_p, x_j, \ldots, y_i, u_1, v_1, \ldots, u_p]$ would be strictly longer than C.

We can choose the y_i and x_j so that u_1 is not dominated by any $y_\nu \in V(P^{ij})$, and that v_p does not dominate any $x_\nu \in V(P^{ij})$. Note that for every pair of vertices y_s, x_{s+1} from $V(C) \setminus V(P^{ij})$ at most one of the arcs $y_s u_1$ and $v_p x_{s+1}$ belongs to $A(D)$, for else D would contain a cycle

$$[v_p, x_{s+1}, y_{s+1}, \ldots, x_s, y_s, u_1, v_1, \ldots, u_p]$$

strictly longer than C. There is precisely $m - l - 1$ of such pairs. Accounting for the arcs $y_i u_1$ and $v_p x_j$, we get the required estimate

$$d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1) \leq (m - l - 1) + 2 \leq m - p + 1. \quad □$$

2.2. Proof of Theorem 1.2

Assume then that D satisfies condition A_2^*. Choose M a complete matching from X to Y, and an oriented cycle C, of length $2m$, compatible with M in such a way that C is of maximal length among all the oriented cycles in D compatible with some complete matching from X to Y. Write $C = [x_1, y_1, \ldots, x_m, y_m]$, with $x_\nu \in X$ and $y_\nu \in Y$, $1 \leq \nu \leq m$. By Lemma 2.4, $2m \geq a + 2$.

We want to show that $m = a$. Suppose otherwise. Then we can choose a path P, of order $2p$, contained in $D \setminus V(C)$, compatible with M and of maximal length among such paths in $D \setminus V(C)$. Write $P = (u_1, v_1, \ldots, u_p, v_p)$, with $u_\nu \in X$ and $v_\nu \in Y$, $1 \leq \nu \leq p$ (cf. Remark 2.2). Let R denote the remaining vertices of D; i.e., $R = V(D) \setminus (V(C) \cup V(P))$. Write $|R| = 2r$ for some $r \geq 0$. Then

$$a = m + p + r \quad \text{and} \quad 2p + 2r = 2a - 2m \leq a - 2.$$

The remainder of the proof splits into several cases according to the properties of $d^-_{V(C)}(u_1)$ and $d^+_{V(C)}(v_p)$. Note that, by maximality of P, we have $d^-_{V(R)}(u_1) = 0$ and $d^+_{V(R)}(v_p) = 0$.

Case A: $d^-_{V(C)}(u_1) = 0$.

Subcase A.1: $d^+_{V(C)}(v_p) > 0$.

Let then $x_i \in V(C)$ be such that $v_p x_i \in A(D)$. It follows from maximality of C that $d^+_{V(P)}(y_{i-1}) = 0$. In particular, $y_{i-1}u_1 \notin A(D)$, and hence $d^+(y_{i-1}) + d^-(u_1) \geq a + 2$. Therefore

$$\begin{aligned} a + 2 \leq d^+(y_{i-1}) + d^-(u_1) &= (d^+_{V(C)}(y_{i-1}) + d^+_{V(R)}(y_{i-1})) + d^-_{V(P)}(u_1) \\ &\leq m + r + p = a; \end{aligned}$$

a contradiction.

Subcase A.2: $d^+_{V(C)}(v_p) = 0$.

If $v_p u_1 \notin A(D)$, then, by assumption,

$$a + 2 \leq d^+(v_p) + d^-(u_1) = d^+_{V(P)}(v_p) + d^-_{V(P)}(u_1) \leq 2(p - 1) < a;$$

a contradiction. Therefore $v_p u_1 \in A(D)$, and so P is, in fact, a cycle. Hence $d^-_{V(R)}(u_i) = 0$ and $d^+_{V(R)}(v_j) = 0$ for all $u_i, v_j \in V(P)$, by maximality of P.

Suppose now that $d^+_{V(C)}(v_j) = 0$ for all $v_j \in V(P)$. Then, for any such v_j and $x_i \in V(C)$, we get

$$\begin{aligned} a + 2 \leq d^+(v_j) + d^-(x_i) &= d^+_{V(P)}(v_j) + (d^-_{V(C)}(x_i) + d^-_{V(R)}(x_i)) \\ &\leq p + m + r = a; \end{aligned}$$

a contradiction. Therefore there exist $x_i \in V(C)$ and $v_j \in V(P)$ such that $v_j x_i \in A(D)$. It follows, as in Subcase A.1, that $y_{i-1}u_1 \notin A(D)$, and hence

$$\begin{aligned} a + 2 \leq d^+(y_{i-1}) + d^-(u_1) &= (d^+_{V(C)}(y_{i-1}) + d^+_{V(R)}(y_{i-1})) + d^-_{V(P)}(u_1) \\ &\leq m + r + p = a; \end{aligned}$$

a contradiction.

Case B: $d^-_{V(C)}(u_1) > 0$.

Subcase B.1: $d^+_{V(C)}(v_p) = 0$.

Let then $y_i \in V(C)$ be such that $y_i u_1 \in A(D)$. It follows from maximality of C that $d^-_{V(P)}(x_{i+1}) = 0$. In particular, $v_p x_{i+1} \notin A(D)$, and hence

$$\begin{aligned} a + 2 \leq d^+(v_p) + d^-(x_{i+1}) &= d^+_{V(P)}(v_p) + (d^-_{V(C)}(x_{i+1}) + d^-_{V(R)}(x_{i+1})) \\ &\leq p + m + r = a; \end{aligned}$$

a contradiction.

Subcase B.2: $d^+_{V(C)}(v_p) > 0$.

By Lemma 2.5, $d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1) \leq m - p + 1$. If $v_p u_1 \notin A(D)$, then

$$\begin{aligned} a + 2 \leq d^+(v_p) + d^-(u_1) &= (d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1)) + (d^+_{V(P)}(v_p) + d^-_{V(P)}(u_1)) \\ &\leq (m - p + 1) + 2(p - 1) = m + p - 1 < a; \end{aligned}$$

a contradiction. Therefore $v_p u_1 \in A(D)$, and so P is, in fact, a cycle.

We shall show that $R = \varnothing$ in this case. Suppose otherwise, and let P' be a maximal path in R compatible with M. Write $P' = (p_1, \ldots, p_t)$. Then $p_1 \in R \cap X$ and $p_t \in R \cap Y$ (see Remark 2.2). Since P is a maximal cycle in $D \setminus V(C)$ compatible with M, then $d^-_{V(P)}(p_1) = d^+_{V(P)}(p_t) = 0$. Moreover, $d^+_{V(C)}(p_t) + d^-_{V(C)}(p_1) \leq m$, because for every pair of vertices y_i, x_{i+1} on C at most one of the arcs $y_i p_1$ and $p_t x_{i+1}$ exists (by maximality of C). Hence

$$d^+(p_t) + d^-(p_1) = (d^+_{V(C)}(p_t) + d^-_{V(C)}(p_1)) + (d^+_{V(R)}(p_t) + d^-_{V(R)}(p_1)) \leq m + 2r,$$

and so

$$\begin{aligned} 2a + 4 &\leq (d^+(p_t) + d^-(u_1)) + (d^+(v_p) + d^-(p_1)) \\ &\leq (m + 2r) + (d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1)) + (d^+_{V(P)}(v_p) + d^-_{V(P)}(u_1)) \\ &\leq (m + 2r) + (m - p + 1) + 2p = 2m + 2r + p + 1 \leq 2m + 2r + 2p = 2a; \end{aligned}$$

a contradiction. We have thus shown that $r = 0$, and hence $a = m + p$.

As in the proof of Lemma 2.5, there exist x_{j_0} and y_{i_0} on C such that $y_{i_0}u_1 \in A(D)$ and $v_p x_{j_0} \in A(D)$. Let $P^{i_0 j_0}$ be the path between y_{i_0} and x_{j_0} on C, traversed according to the orientation of C; of order, say, $2l$. Write $P^{i_0 j_0} = (x_{i_0+1}, y_{i_0+1}, \ldots, x_{j_0-1}, y_{j_0-1})$. Then $l \geq p$, because otherwise the cycle $[v_p, x_{j_0}, \ldots, y_{i_0}, u_1, v_1, \ldots, u_p]$ would be strictly longer than C. Further, we can choose the x_{j_0} and y_{i_0} so that

$$y_\nu u_1 \notin A(D) \quad \text{for all } y_\nu \in P^{i_0 j_0} \quad \text{and} \quad v_p x_\nu \notin A(D) \quad \text{for all } x_\nu \in P^{i_0 j_0}. \tag{2.1}$$

As in the proof of Lemma 2.5, it follows that $d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1) \leq m - l + 1$, and hence

$$\begin{aligned} d^+(v_p) + d^-(u_1) &= (d^+_{V(C)}(v_p) + d^-_{V(C)}(u_1)) + (d^+_{V(P)}(v_p) + d^-_{V(P)}(u_1)) \\ &\leq (m - l + 1) + 2p = a - l + p + 1. \end{aligned} \tag{2.2}$$

By (2.1), we have $v_p x_{i_0+1} \notin A(D)$ and $y_{j_0-1}u_1 \notin A(D)$. Hence, and by (2.2),

$$2a + 4 \leq (d^+(v_p) + d^-(x_{i_0+1})) + (d^+(y_{j_0-1}) + d^-(u_1)) \leq (d^+(y_{j_0-1}) + d^-(x_{i_0+1})) + (a - l + p + 1),$$

and thus

$$d^+(y_{j_0-1}) + d^-(x_{i_0+1}) \geq a + l - p + 3 = m + l + 3. \tag{2.3}$$

Note that $d^+_{V(P)}(y_{j_0-1}) = d^-_{V(P)}(x_{i_0+1}) = 0$, which follows from the maximality of C and the fact that P is a cycle. Therefore $d^+(y_{j_0-1}) = d^+_{V(C)}(y_{j_0-1})$ and $d^-(x_{i_0+1}) = d^-_{V(C)}(x_{i_0+1})$, and so, by (2.3), we get

$$d^+_{V(C)}(y_{j_0-1}) + d^-_{V(C)}(x_{i_0+1}) \geq m + l + 3 > (m - l - 1) + 2l + 2. \tag{2.4}$$

Since y_{j_0-1} and x_{i_0+1} have together at most $2l + 2$ neighbours in $V(P^{i_0 j_0}) \cup \{y_{i_0}\} \cup \{x_{j_0}\}$, then (2.4) implies that there exists a pair of vertices y_s, x_{s+1} in $V(C) \setminus (V(P^{i_0 j_0}) \cup \{y_{i_0}\} \cup \{x_{j_0}\})$ such that $y_s x_{i_0+1} \in A(D)$ and $y_{j_0-1}x_{s+1} \in A(D)$. But then D contains a Hamiltonian cycle

$$[u_1, \ldots, v_p, x_{j_0}, \ldots, y_s, x_{i_0+1}, \ldots, y_{j_0-1}, x_{s+1}, \ldots, y_{i_0}].$$

This contradiction completes the proof of the theorem. □

Remark 2.6. Note that the proof of Theorem 1.2, in fact, goes under considerably weaker assumptions. Namely, it suffices to assume that the digraph D contains a complete matching from X to Y, and condition A_2^* is satisfied for every pair of vertices u and v such that $u \in X$, $v \in Y$ and $vu \notin A(D)$. That is, we do not need to require any degree condition on pairs of vertices u and v such that $u \in X$, $v \in Y$ and $uv \notin A(D)$. Of course, symmetrically, it suffices to assume a complete matching from Y to X and condition A_2^* satisfied for every pair of vertices u and v such that $u \in X$, $v \in Y$ and $uv \notin A(D)$.

Acknowledgments

The authors' research was partially supported by Natural Sciences and Engineering Research Council of Canada (J. Adamus) and Polish Ministry of Science and Higher Education (L. Adamus).

References

[1] D. Amar, Y. Manoussakis, Cycles and paths of many lengths in bipartite digraphs, J. Combin. Theory Ser. B 50 (1990) 254–264.
[2] C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.
[3] J.C. Bermond, C. Thomassen, Cycles in digraphs—a survey, J. Graph Theory 5 (1981) 1–43.
[4] J.A. Bondy, C. Thomassen, A short proof of Meyniel's theorem, Discrete Math. 19 (1977) 195–197.
[5] Y. Manoussakis, I. Milis, A sufficient condition for maximum cycles in bipartite digraphs, Discrete Math. 207 (1999) 161–171.
[6] M. Meyniel, Une condition suffisante d'existence d'un circuit Hamiltonien dans un graphe orienté, J. Combin. Theory Ser. B 14 (1973) 137–147.
[7] C.St.J.A. Nash-Williams, Hamilton circuits in graphs and digraphs, in: The Many Facets of Graph Theory, in: Lecture Notes, vol. 110, Springer, 1969, pp. 237–243.
[8] D.R. Woodall, Sufficient conditions for circuits in graphs, Proc. Lond. Math. Soc. 24 (1972) 739–755.