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Tameness of Complex Dimension in a Real Analytic Set

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Abstract. Given a real analytic set X in a complex manifold and a positive integer d, denote by $\mathcal{A}^d$ the set of points p in X at which there exists a germ of a complex analytic set of dimension d contained in X. It is proved that $\mathcal{A}^d$ is a closed semianalytic subset of X.

1 Introduction and Main Results

The existence or non-existence of complex analytic germs in a given real hypersurface X of a complex manifold plays an important role in the theory of holomorphic mappings. A particularly interesting case is when X is real analytic. For example, in [7] Diederich and Fornæss showed that a compact real analytic set X in $\mathbb{C}^n$ does not contain germs of complex analytic sets of positive dimension. If X is not compact, then the set $\mathcal{A}^1$ of points p in X such that there exists a positive-dimensional complex analytic germ Y_p with $Y_p \subset X_p$ is non-empty in general. It is a natural problem to describe the structure of the set $\mathcal{A}^1$. D'Angelo [6] and Diederich and Mazzilli [8], using different methods, proved that $\mathcal{A}^1$ is closed in X. In [8] the authors also asked whether $\mathcal{A}^1$ is a real analytic subset of X. Our main theorem answers this question.

Theorem 1.1 *Let X be a closed real analytic subset of an open set in $\mathbb{C}^n$. Let $\mathcal{A}^d$ denote the set of points p in X such that X_p, the germ of the set X at p, contains a complex analytic germ of dimension d. Then $\mathcal{A}^d$ is a closed semianalytic subset of X, for every $d \in \mathbb{N}$. Moreover, if X is real algebraic, then $\mathcal{A}^d$ is semialgebraic in X.*

The proof of closedness of $\mathcal{A}^d$, given in Proposition 3.2, is similar in spirit to [8] (where it is done for $\mathcal{A}^1$), but we do not use volume estimates or Bishop's theorem. Instead, our proof relies purely on the properties of Segre varieties. The following example, which is due to Meylan, Mir, and Zaitsev [12], shows that the set $\mathcal{A}^d$ is not in general real analytic. Consider

$$X = \left\{ (z_1, \dots, z_4) \in \mathbb{C}^4 : x_1^2 - x_2^2 + x_3^2 = x_4^3 \right\},$$

where $z_j = x_j + i y_j$, $j = 1, \dots, 4$. Near $(1, 1, 0, 0)$ the set X is a smooth real algebraic manifold. For every point z in X with $x_4 \geq 0$ there is a complex line passing through z and contained in X. But if $x_4 < 0$, then X can be expressed (locally near z) as a graph

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of a strictly convex function, and therefore there cannot be any germs of positive-dimensional complex analytic sets in X. Thus $\mathcal{A}^1$ coincides with $X \cap \{x_4 \geq 0\}$, which is semianalytic (even semialgebraic) but not analytic.

Remark 1.2 Another (in a sense, dual) question that can be asked about a germ X_p of a real analytic set is: what is the smallest dimension of a complex analytic germ at p *containing* X_p, and what can be said about the structure of the subset of X along which this minimal dimension is realized? It is shown in [2, Thm. 1.5] that for an irreducible real analytic subset X of $\mathbb{C}^n$ of pure dimension $d > 0$ this so-called *holomorphic closure dimension* attains its minimum h outside a closed semianalytic subset $S \subset X$ of dimension less than d. In fact, $X \setminus S$ is a CR manifold of CR dimension $d - h$. Interestingly, X does not in general admit semianalytic (not even subanalytic, see [2, Ex. 6.3]) stratification by holomorphic closure dimension beyond S. (See also [1] for the semialgebraic context.) By comparison, Theorem 1.1 implies a semianalytic filtration of X, $X = \mathcal{A}^0 \supset \mathcal{A}^1 \supset \cdots \supset \mathcal{A}^{n-1}$.

Semianalyticity is a consequence of the description of the set $\mathcal{A}^d$ given in Theorem 1.4 below. We first need to introduce some notation. Let $\varrho(z, \bar{z})$ be a real analytic function on some open polydisc $V \Subset \mathbb{C}^n$ given by a power series convergent in a neighbourhood of $\overline{V}$ such that

$$X \cap V = \{z \in V : \varrho(z, \bar{z}) = 0\}. \tag{1.1}$$

As in the smooth case (see, *e.g.*, [16]), for a point $w \in V$, we define the *Segre variety* of w as

$$S_w = \{z \in V : \varrho(z, \overline{w}) = 0\}. \tag{1.2}$$

For more about Segre varieties, see Section 2. Geometric properties of these varieties will play a crucial role in the proof of Theorem 1.4.

Let κ be a positive integer, and let $n \geq 1$ be the complex dimension of the ambient space of X with variables $z = (z_1, \ldots, z_n)$. For $1 \leq d \leq n$, let

$$\Lambda(d, n) := \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{N}^d : 1 \leq \lambda_1 < \cdots < \lambda_d \leq n\}.$$

Given $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda(d, n)$, we will denote by z_λ the sub-collection of variables $(z_{\lambda_1}, \ldots, z_{\lambda_d})$.

Definition 1.3 For any $1 \leq d \leq n$ and $\lambda \in \Lambda(d, n)$, we define a *$\kappa$-grid with d-dimensional base z_λ*, denoted $\mathcal{G}^\kappa_\lambda$, as follows. Let $\mathcal{G}^\kappa_\lambda$ be a collection of $(\kappa + 1)^d$ distinct points $p_\nu \in V$, where $\nu = (\nu_1, \ldots, \nu_d) \in \{1, \ldots, \kappa + 1\}^d$, such that

(i) for each pair $(p_\nu, p_{\nu'})$ of elements of $\mathcal{G}^\kappa_\lambda$, we have $\varrho(p_\nu, \overline{p_{\nu'}}) = 0$, and
(ii) for p_ν and $p_{\nu'}$ in $\mathcal{G}^\kappa_\lambda$, we have $\nu_j = \nu'_j$ if and only if p_ν and $p_{\nu'}$ have the same λ_j-th coordinate (as vectors in $\mathbb{C}^n$).

We denote by $\mathbb{B}(p, \varepsilon)$ the standard open Euclidean ball of radius ε centred at p.

Theorem 1.4 *Let X be a closed real analytic subset of an open set in $\mathbb{C}^n$, and let V and ϱ be such that (1.1) holds. Let $1 \le d < n$, and let $\mathcal{A}^d$ be the set of points p in X such that X_p contains a complex analytic germ of dimension d. Then there exists a positive integer κ such that the following two statements are equivalent:*

(i) $p \in \mathcal{A}^d \cap V$;

(ii) *for any $\varepsilon > 0$, there exists a κ-grid $\mathcal{G}^\kappa_\lambda$ with a d-dimensional base z_λ for some $\lambda \in \Lambda(d, n)$ such that $\mathcal{G}^\kappa_\lambda \subset \mathbb{B}(p, \varepsilon)$.*

In general, the number κ in Theorem 1.4 depends on the defining function ϱ. However, if X is a smooth real analytic hypersurface, then Segre varieties do not depend on the choice of ϱ provided that the differential of ϱ does not vanish on X, and in fact, S_w are local biholomorphic invariants of X. Thus, in this case κ is also a local biholomorphic invariant of X (*cf.* Section 4).

Another question raised in [8] is whether the set of points on X of infinite D'Angelo type is exactly $\mathcal{A}^1$. The proof of this fact is given in D'Angelo [6, Sec. 3.3.3, Thm. 4]; however, in [8], the validity of this proof is questioned. We address this issue in the last section. Our goal is to clarify the definition of type for real analytic sets, and to give a concise but self-contained proof of the fact that the subset of X of points of infinite type indeed coincides with the set $\mathcal{A}^1$. Combining this with Theorem 1.1 immediately gives the following result.

Corollary 1.5 *Given a real analytic set X, the set of points of D'Angelo infinite type is a closed semianalytic subset of X.*

2 Segre Varieties

Given a closed real analytic set X in an open set in $\mathbb{C}^n$ of arbitrary positive dimension, for any point $p \in X$ there exists a neighbourhood $V \subset \mathbb{C}^n$ of p such that $X \cap V$ is precisely the zero set of a convergent power series

$$\varrho(z, \bar{z}) = \sum_{|\alpha|+|\beta|\ge 1} c_{\alpha\beta}(z-p)^\alpha(\overline{z-p})^\beta,$$

where, for a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, w^β denotes the monomial $w_1^{\beta_1}\cdots w_n^{\beta_n}$, and $|\beta| = \beta_1 + \cdots + \beta_n$. (Indeed, if X is defined near p by the vanishing of real analytic functions $h_1, \dots, h_t$, one can put $\varrho = h_1^2 + \cdots + h_t^2$.) For simplicity, assume that $p = 0$. By shrinking V if needed, we may further assume that the series $\varrho(z, \overline{w}) = \sum c_{\alpha\beta} z^\alpha \overline{w}^\beta$ is also convergent in a neighbourhood of the closure of $V \times V$. For a given $w \in V$ define the *Segre variety* S_w of w to be the complex analytic subset of V defined by (1.2).

The set

$$X^c = \{(z, \overline{w}) \in V \times V : \varrho(z, \overline{w}) = 0\}$$

is a non-empty complex analytic set defined by a single holomorphic function, and hence it is of (pure) dimension $2n-1$. It follows that a fibre $\{z \in V : (z, \overline{w}) \in X^c\}$ over a point $\overline{w}$, if nonempty, has dimension $n-1$ or n. For every point $z \in X$, we have $\varrho(z, \bar{z}) = 0$, and hence S_z is not empty. Therefore, by the analytic dependence of S_w on $\overline{w}$, there exist polydisc neighbourhoods $U_1 \Subset U_2 \Subset V$ of p such that for any $w \in U_1$, the set $S_w \cap U_2$ is a non-empty complex analytic subset of U_2 of (pure) dimension either $n-1$ or n. To simplify notation, we will write S_w for $S_w \cap U_2$, whenever $w \in U_1$. From the definition (1.2), and the fact that $\varrho(z,\overline{z})$ is real-valued, it follows that for $z, w \in U_1$,

$$z \in S_w \Longleftrightarrow w \in S_z, \tag{2.1}$$

$$z \in S_z \Longleftrightarrow z \in X. \tag{2.2}$$

Let E be the set of points z in U_1 such that $\dim S_z = n$; *i.e.*, $S_z = U_2$. Then $z \in E$ implies $z \in S_z$, and therefore $E \subset X$. Furthermore, $E \neq X$ unless X is itself complex analytic.

Remark 2.1 Apart from properties (2.1) and (2.2), the results of the following sections rely on a few basic properties of complex analytic sets, which we list here for the reader's convenience (for details, see [4] or [11]). Let Y denote a complex analytic subset of an open set in $\mathbb{C}^n$.

(i) The family of irreducible components of Y is locally finite, and each irreducible component is precisely the set-theoretic closure in Y of a connected component of the regular locus of Y.

(ii) The set Y is irreducible if and only if its regular locus Y^{reg} is a connected manifold. In this case, Y is of pure dimension. Moreover, a proper analytic subset of an irreducible set Y is of dimension at most $\dim Y - 1$.

(iii) A point $z^0 \in Y$ is regular (*i.e.*, $z^0 \in Y^{\text{reg}}$) if and only if there are a natural number d, an open polydisc U centered at z_0, and a sub-collection of variables $(z_{j_1}, \dots, z_{j_d})$, such that the projection π onto (the linear subspace of $\mathbb{C}^n$ spanned by) these variables restricted to $Y \cap U$ is a bijection between $Y \cap U$ and $\pi(U)$.

(iv) If Y is irreducible, of dimension $k > 0$, and $0 \in Y$, then after a (generic) linear change of coordinates in $\mathbb{C}^n$, there is a neighbourhood $\Omega \times \Sigma$ of 0, where $\Omega = \{(z_1, \dots, z_k) \in \mathbb{C}^k : |z_j| < \delta\}$, $\Sigma = \{(z_{k+1}, \dots, z_n) \in \mathbb{C}^{n-k} : |z_j| < \varepsilon\}$ for some $\delta, \varepsilon > 0$, and a proper analytic subset Z of Ω such that the restriction to Y, $\pi\colon Y \cap (\Omega \times \Sigma) \to \Omega$, of the canonical projection $\Omega \times \Sigma \to \Omega$ is proper, surjective, and locally biholomorphic at every p in $(Y \cap (\Omega \times \Sigma)) \setminus (Z \times \Sigma)$, which is an open dense subset of $Y \cap (\Omega \times \Sigma)$.

(v) If π is a proper projection from Y to a linear subspace of $\mathbb{C}^n$, then $\dim \pi(Y) = \dim Y$.

By a *holomorphic disc* through a point p we mean an irreducible one-dimensional complex analytic set Y in a neighbourhood U of p such that $p \in Y$ and Y is the image of a non-constant holomorphic map γ from a disc $\{\zeta \in \mathbb{C} \colon |\zeta| < \delta\}$ to U. We say that the disc is centred at p when $\gamma(0) = p$. The following result is essentially a restatement of [7, Claim on p. 383]. It generalizes [8, Lem. 2.5], which states that a holomorphic disc Y through a point z is contained in S_z, provided $Y \subset X$.

Lemma 2.2 *Let X, p, ϱ, V, U_1 and U_2 be as above. Suppose that Y is an irreducible complex analytic subset of an open set in U_2, of positive dimension k, and such that $Y \subset X$. Then $z \in Y$ implies $Y \subset S_z$.*

Proof Fix a point $z_0 \in Y$. We shall show that $Y \subset S_{z_0}$. For simplicity of notation, assume that $z_0 = 0$. By Remark 2.1(iv), we may choose a neighbourhood $\Omega \times \Sigma$ of z_0 such that Ω is a k-dimensional polydisc, and the projection $\pi\colon Y \cap (\Omega \times \Sigma) \to \Omega$ is proper and surjective. Let $z' = (z'_1, \dots, z'_k, z'_{k+1}, \dots, z'_n)$ be an arbitrary point in $Y \cap (\Omega \times \Sigma)$, and let $L_{z'} \subset \Omega$ be the complex line segment through $(z'_1, \dots, z'_k)$ and 0 in Ω. Then $Y_{z'} := \pi^{-1}(L_{z'})$ is an analytic subset of $Y \cap (\Omega \times \Sigma)$, with a proper projection onto $L_{z'}$, and hence of dimension one, by Remark 2.1(v). We may assume that $Y_{z'}$ is irreducible by keeping only one irreducible component of $Y_{z'}$ passing through z' and z_0. Then, by the Puiseux theorem (see, *e.g.*, [11, Ch. II, §6.2]), there is a neighbourhood Ω' of $0 \in \Omega$, such that $Y_{z'} \cap (\Omega' \times \Sigma)$ is a holomorphic disc centred at z_0. By [8, Lem. 2.5], $Y_{z'} \cap (\Omega' \times \Sigma) \subset S_{z_0}$. It follows that the set $Y_{z'} \cap (\Omega' \times \Sigma) \cap S_{z_0}$ contains a non-empty open subset of $Y_{z'}$, hence is of dimension $\dim Y_{z'}$, and so is not a proper subset of $Y_{z'}$, by Remark 2.1(ii). Thus $Y_{z'} \subset S_{z_0}$ and, in particular, $z' \in S_{z_0}$. Consequently, $Y \cap (\Omega \times \Sigma) \subset S_{z_0}$, because z' was arbitrary. Hence, by Remark 2.1(ii) again, $Y \subset S_{z_0}$, as required. ■

Lemma 2.3 (*cf.* [8, Thm. 1.2], see also [7]) *Let X, p, ϱ, V, U_1, and U_2 be as above. For a non-empty subset Y of U_2, with $Y \cap U_1 \neq \varnothing$, define*

$$Y^1 = \bigcap_{z \in Y \cap U_1} S_z \quad \text{and} \quad Y^2 = \bigcap_{w \in Y^1 \cap U_1} S_w.$$

(i) *Y^1 and Y^2 are complex analytic subsets of U_2. If $Y^1 \cap U_1 \neq \varnothing$, then $Y \cap U_1 \subset Y^2 \cap U_1$.*
(ii) *Moreover, if Y is an irreducible positive-dimensional complex analytic subset of an open set in U_2, such that $Y \subset X$, then $Y \cap U_1 \subset Y^1 \cap U_1$.*
(iii) *If $Y \cap U_1 \subset Y^1 \cap U_1$, then $Y^2 \subset Y^1$ and $Y^2 \cap U_1 \subset X$.*

Proof (i) The Segre varieties S_z are complex analytic in U_2, for $z \in U_1$, hence so are Y^1 and Y^2. By definition, $z \in Y^2$ if and only if $z \in S_w$ for all $w \in Y^1 \cap U_1$. Hence, by (2.1), $z \in Y^2 \cap U_1$ if and only if $w \in S_z$ for all $w \in Y^1 \cap U_1$. On the other hand, $z \in Y \cap U_1$ implies that $w \in S_z$ for all $w \in Y_1$, and so $z \in Y^2$.

(ii) Suppose now that Y is an irreducible positive-dimensional complex analytic subset of an open set in U_2, such that $Y \subset X$. Then, by Lemma 2.2, $Y \subset S_z$ for every $z \in Y$, and so $Y \cap U_1 \subset \big(\bigcap_{z \in Y \cap U_1} S_z\big) \cap U_1 = Y^1 \cap U_1$.

(iii) Finally, assume that $Y \cap U_1 \subset Y^1 \cap U_1$. Then $\bigcap_{z \in Y^1 \cap U_1} S_z \subset \bigcap_{z \in Y \cap U_1} S_z$; *i.e.*, $Y^2 \subset Y^1$. For the proof of the last inclusion, let $z \in Y^2 \cap U_1$ be arbitrary. Then $z \in S_w$ for every $w \in Y^1 \cap U_1$, hence, by (2.1) again, $w \in S_z$ for all $w \in Y^1 \cap U_1$. In particular, $z \in S_z$, since $z \in Y^2 \subset Y^1$. Therefore $z \in X$, by (2.2). ■

3 Topology of the Set of Points of Positive Complex Dimension

In this section we prove that $\mathcal{A}^d$ is closed in X, for any $d \geq 1$. The openness of the set of points of finite type in the hypersurface case was already established in [5, Thm. 4.11] and later extended to smooth real analytic sets of arbitrary codimension in [6]. Via the equivalence between the finiteness of the type at p and the property

$p \notin \mathcal{A}^1$, which we recall in Section A, D'Angelo proved in [6] the openness of $X \setminus \mathcal{A}^1$. The result was recently reproved in [8]. In the proof of Proposition 3.2, we use Lemma 2.3 to replace complex analytic germs by their representatives in a fixed open set (*cf.* [8]) and then show that their Hausdorff limit is contained in a complex analytic set in X that has dimension at least d.

For a non-empty set $E \subset \mathbb{C}^n$ and a point $p \in \mathbb{C}^n$, put

$$d(p, E) = \inf\{\, d(p, q) : q \in E \,\},$$

where $d(p, q)$ is the Euclidean distance between p and q. Recall that $\overline{U}_1$ being compact, the space $\mathcal{K}(\overline{U}_1)$ of closed subsets of $\overline{U}_1$ equipped with the *Hausdorff distance*

$$d_H(K_1, K_2) = \min\{r \geq 0 : d(x_1, K_2), d(x_2, K_1) \leq r \text{ for all } (x_1, x_2) \in K_1 \times K_2\}$$

is a compact metric space (see, *e.g.*, [13]).

Remark 3.1 Suppose that the sequence $(K_j)_{j=1}^{\infty} \subset \mathcal{K}(\overline{U}_1)$ converges to K in this metric, with $d_H(K_j, K) \leq 2^{-j}$. Then K is precisely the set of points p for which there is a sequence $(p_j)_{j=1}^{\infty}$ with $p_j \in K_j$ and $d(p_j, p) \leq 2^{-j}$. In particular, if $K_j \subseteq L_j$ are closed subsets of $\overline{U}_1$ with the sequence (K_j) (resp. (L_j)) converging to the set K (resp. L), then $K \subseteq L$.

Proposition 3.2 *Let X be a closed real analytic subset of an open set in $\mathbb{C}^n$, and let $\mathcal{A}^d$ be the set of points p in X such that X_p contains a complex analytic germ of dimension d. Then $\mathcal{A}^d$ is closed in X for every $d \geq 1$.*

Proof Fix $d \geq 1$, and let $p_0 \in X$ be a limit point of $\mathcal{A}^d$. Then there exists a sequence of d-dimensional complex analytic germs $(Y_j)_{p_j} \subset X_{p_j}$ at points $p_j \in X$ such that $p_0 = \lim_{j\to\infty} p_j$. We restrict our considerations to neighbourhoods U_1 and U_2 of p_0, as discussed in Section 2. Without loss of generality, we may assume that the Y_j are irreducible.

One difficulty arising here is that the $(Y_j)_{p_j}$ may not simultanously admit representatives in a fixed neighbourhood of p_0. We can, however, replace the Y_j by irreducible complex analytic subsets of U_2 by setting

$$Y_j^1 = \bigcap_{z \in Y_j \cap U_1} S_z \qquad \text{and} \qquad Y_j^2 = \bigcap_{w \in Y_j^1 \cap U_1} S_w.$$

Indeed, by Lemma 2.3, the Y_j^1 and Y_j^2 are complex analytic subsets of U_2, $Y_j \subset Y_j^2$ and $Y_j^2 \cap U_1 \subset X$. The first inclusion implies also that $\dim Y_j^2 \geq d$, for all j, since the Y_j are d-dimensional. We may also assume that the Y_j^2 are irreducible, by keeping only one irreducible component of Y_j^2 passing through p_j. To simplify the notation, from now on we denote Y_j^2 by Y_j. Since $\dim Y_j \in \{d, \dots, n-1\}$ for all j, there exists an integer $d' \geq d$ such that $\dim Y_j = d'$ for infinitely many j. Let us then replace the original sequence $(Y_j)_{j=1}^{\infty}$ by this infinite subsequence.

By compactness of $\mathcal{K}(\overline{U}_1)$, the sequence $(Y_j \cap \overline{U}_1)_{j=1}^{\infty}$ contains an infinite subsequence convergent in the Hausdorff metric to a set Y_0 closed in $\overline{U}_1$. Therefore, without loss of generality, we may assume that

$$Y_0 = \lim_H (Y_j \cap \overline{U}_1),$$

and further that

$$d_H(Y_j \cap \overline{U}_1, Y_0) \leq 2^{-j} \tag{3.1}$$

(by throwing out some terms of the sequence, if necessary). Notice that $p_0 = \lim_{j\to\infty} p_j$ belongs to Y_0, by Remark 3.1.

We will show that Y_0 is contained in a complex analytic set of dimension at least d contained in X. Set

$$Y_j^1 = \bigcap_{z \in Y_j \cap U_1} S_z, \quad Y_0^1 = \bigcap_{z \in Y_0 \cap U_1} S_z, \quad \text{and} \quad \widetilde{Y}_0^1 = \lim_H (Y_j^1 \cap \overline{U}_1),$$

where $\lim_H(Y_j^1 \cap \overline{U}_1)$ is again the limit of (an infinite convergent subsequence of) $Y_j^1 \cap \overline{U}_1$ in the sense of the Hausdorff metric on $\mathcal{K}(\overline{U}_1)$. (Notice that replacing $(Y_j \cap \overline{U}_1)_{j=1}^{\infty}$ by its infinite convergent subsequence does not affect Y_0.) We may further assume that $d_H(Y_j^1 \cap \overline{U}_1, \widetilde{Y}_0^1) \leq 2^{-j}$, as above.

We claim that $\widetilde{Y}_0^1 \subset Y_0^1$. Indeed, there exist points $\{a_1, \dots, a_r\} \subset Y_0$ such that $Y_0^1 = \bigcap_{k=1}^r S_{a_k}$, by compactness of $\overline{U}_2$ and Remark 2.1(i). Therefore, there exist r sequences $(a_k^j)_{j=1}^{\infty}$ such that $a_k^j \in Y_j$ and $\lim_{j\to\infty} a_k^j = a_k$, $k = 1, \dots, r$ (see Remark 3.1).

From the analytic dependence of Segre varieties S_z on the parameter z, we conclude that

$$\lim_H \Big(\bigcap_{k=1}^r S_{a_k^j} \Big) \subset \bigcap_{k=1}^r S_{a_k} = Y_0^1;$$

for if $z \in \lim_H \bigcap_{k=1}^r S_{a_k^j}$, we can find $z^j \in \bigcap_{k=1}^r S_{a_k^j}$ such that $\lim_{j\to\infty} z^j = z$, hence

$$\varrho(z, \bar{a}_k) = \lim_j \varrho(z^j, \bar{a}_k^j) = 0$$

for each $k \in \{1, \cdots, r\}$.

Also, since $a_k^j \in Y_j$, for every fixed j we have $Y_j^1 \subset \bigcap_{k=1}^r S_{a_k^j}$.

From this we conclude that $\lim_H Y_j^1 \subset Y_0^1$, which proves the claim.

We now claim that $Y_0 \cap U_1 \subset Y_0^1 \cap U_1$. Indeed, since the $Y_j \cap U_1$ are irreducible positive-dimensional complex analytic sets in U_1, and subsets of X, we have $Y_j \cap U_1 \subset Y_j^1 \cap U_1$, by Lemma 2.3(ii). Therefore, by Remark 3.1, $\lim_H(Y_j \cap U_1) \subset \lim_H(Y_j^1 \cap U_1) = \widetilde{Y}_0^1$, and hence $Y_0 \cap U_1 = \lim_H(Y_j \cap U_1) \subset Y_0^1 \cap U_1$, by the previous claim. In particular, the set $Y_0^1 \cap U_1$ is not empty. Let

$$Y_0^2 = \bigcap_{z \in Y_0^1 \cap U_1} S_z.$$

Then $Y_0^2 \subset U_2$ is a complex analytic set, such that $Y_0^2 \cap U_1 \subset X$ and $\dim_{p_0} Y_0^2 \geq d$. Indeed, since $Y_0 \cap U_1 \subset Y_0^1 \cap U_1$, Lemma 2.3 implies that $Y_0^2 \cap U_1 \subset X$. Given $z \in Y_0 \cap U_1$, we have $w \in S_z$ for every $w \in Y_0^1$, by the definition of Y_0^1. Hence $z \in S_w$ for every $w \in Y_0^1 \cap U_1$, by (2.1), and so $z \in Y_0^2$. Therefore $Y_0 \cap U_1 \subset Y_0^2$. It thus suffices to show that the Hausdorff dimension of $(Y_0)_{p_0}$ is at least $2d'$. This is a consequence of [15, Thm. 4.2], but one can also argue directly as follows.

Recall that, for every $j \geq 1$, Y_j is an irreducible d'-dimensional complex analytic subset of U_2 (where $d' \geq d$) passing through p_j and such that $Y_j \cap U_1 \subset X$. By (3.1), we have

$$d_H(Y_j \cap \overline{U}_1, Y_{j+k} \cap \overline{U}_1) < 2^{-(j-1)}. \tag{3.2}$$

Since $\lim_{j\to\infty} p_j = p_0$, it follows that, for every $\delta = (\delta_1, \dots, \delta_n)$ with $\delta_l > 0$, all but finitely many Y_j have non-empty intersection with a polydisc

$$P(p_0, \delta) = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_l - p_{0l}| < \delta_l \right\}.$$

For every j, there exist δ and a generic system of coordinates

$$z = (z_1, \dots, z_{d'}, z_{d'+1}, \dots, z_n)$$

at p_0, such that $Y_j \cap P(p_0, \delta)$ has a proper and surjective projection onto the $(z_1, \dots, z_{d'})$-variables (see Remark 2.1(iv)). By (3.2), we may choose a positive δ and a system of coordinates z at p_0 such that all but finitely many of the $Y_j \cap P(p_0, \delta)$ simultaneously have proper and surjective projection onto the $(z_1, \dots, z_{d'})$-variables. Therefore the same must be true for the Hausdorff limit $Y_0 \cap P(p_0, \delta)$, by Remark 3.1. Thus the Hausdorff dimension of $(Y_0)_{p_0}$ is at least $2d' \geq 2d$, and hence $p_0 \in \mathcal{A}^d$, which completes the proof of the proposition. ∎

4 Finiteness and Noetherianity in Analytic Families

In this section we prove two finiteness properties for intersections of elements in a family of analytic sets that will be used in the proof of Theorem 1.4. We begin with some basic facts about semi and subanalytic sets.

Recall that a subset E of a real analytic manifold M is called *semianalytic* if it is locally defined by finitely many real analytic equations and inequalities. More precisely, for each $p \in M$, there is a neighbourhood U of p, and real analytic in U functions f_i, g_{ij}, where $i = 1, \dots, r$, $j = 1, \dots, s$, such that

$$E \cap U = \bigcup_{i=1}^{r} \Big(\bigcap_{j=1}^{s} \{x \in U : g_{ij}(x) > 0 \text{ and } f_i(x) = 0\} \Big).$$

A real analytic set is clearly semianalytic. A *subanalytic* subset E of a real analytic manifold M is one that can be locally represented as the proper projection of a semianalytic set. More precisely, for every $p \in M$, there exist a neighbourhood U of p in M, a real analytic manifold N, and a relatively compact semianalytic set $Z \subset M \times N$

such that $E \cap U = \pi(Z)$, where $\pi\colon M \times N \to M$ is the natural projection. In particular, semianalytic sets are subanalytic. For details on semi and subanalytic sets, we refer the reader to [3].

The class of semianalytic (resp. subanalytic) sets is closed under natural topological operations: locally finite unions and intersections, set-theoretic differences, complements, topological closures, and interiors of semianalytic (resp. subanalytic) sets are semianalytic (resp. subanalytic). Subanalytic sets are furthermore closed under the operation of taking proper projections to linear subspaces.

Remark 4.1 An important property of subanalytic sets is that the number of connected components of fibres of a projection is locally bounded (see, *e.g.*, [3, Thm. 3.14]). If S is a relatively compact subanalytic subset of $\mathbb{R}^m \times \mathbb{R}^n$, and $D \subset \mathbb{R}^m$ is compact, then there is a positive integer k_D such that the number of connected components of the set $\pi^{-1}(x)$ is bounded above by k_D for all $x \in D$, where π is the restriction to S of the canonical projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

Lemma 4.2 *Let S be a subanalytic subset of $\mathbb{C}^m \times \mathbb{C}^n$. Let Ω_1 and Ω_2 be relatively compact open subsets of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, and let $D_1 \subset \mathbb{C}^m$ and $D_2 \subset \mathbb{C}^n$ be open polydiscs such that $\overline{D}_1 \subset \Omega_1$ and $\overline{D}_2 \subset \Omega_2$. Suppose that for every point $a \in D_1$, the set $S_a = \{b \in \Omega_2 : (a, b) \in S\}$ is a complex analytic subset of Ω_2. Then there is a positive integer N such that, for every $a \in D_1$, the analytic set $S_a \cap D_2$ has at most N irreducible components.*

Proof By Remark 2.1(i), it suffices to show that there is a positive integer N such that for every $a \in D_1$, the set $(S_a \cap D_2)^{\text{reg}}$ has at most N connected components. Using Remark 4.1, the latter would be a consequence of the subanalyticity of the set

$$\{(a, b) \in D_1 \times D_2 : b \in (S_a \cap D_2)^{\text{reg}}\}.$$

Remark 2.1(iii) ensures that this set is precisely the set of pairs (a, b) in $D_1 \times D_2$ for which there is a natural number d and a choice of coordinate indices $(j_1, \dots j_d) \in \{1, \dots, n\}^d$ such that there is a number $\varepsilon > 0$ small enough so that for all $(z_{j_1}, \dots, z_{j_d}) \in \mathbb{C}^d$ with $|z_{j_l} - b_{j_l}| < \varepsilon$ $(l = 1, \dots, d)$ there is a unique $b' = (b'_1, \dots, b'_n)$ satisfying $b' \in Y \cap \mathbb{B}(b, \varepsilon)$ and $b'_{j_l} = z_{j_l}$ $(l = 1, \dots, d)$.

The set $\{(a, b) \in D_1 \times D_2 : b \in (S_a \cap D_2)^{\text{reg}}\}$ is thus the proper projection (*there exists*) of the complement of the proper projection (*for all*) of the complement of the proper projection of a semianalytic set, and is therefore subanalytic. ∎

Using this lemma, we can now prove the following proposition.

Proposition 4.3 *Under the notation of the previous lemma, there is a positive integer L such that for any set $A \subset D_1$ there is an L-tuple $(a_1, \dots, a_L) \in A^L$ for which*

$$\Big(\bigcap_{a \in A} S_a\Big) \cap D_2 = S_{a_1} \cap \cdots \cap S_{a_L} \cap D_2.$$

Proof Given $l \geq 1$ and $(a_1, \cdots, a_l) \in (D_1)^l$, let $N(l; a_1, \cdots, a_l)$ denote the $(n+1)$-tuple of natural numbers whose k-th coordinate is the number of irreducible components of dimension $n - k + 1$ of $S_{a_1} \cap \cdots \cap S_{a_l} \cap D_2$.

Applying Lemma 4.2 to the subanalytic set

$$\left\{ (a_1, \dots, a_l, b) \in (\mathbb{C}^m)^l \times \mathbb{C}^n : b \in S_{a_1} \cap \dots \cap S_{a_l} \right\},$$

we conclude that the number of such components of any dimension is bounded above independently of the choice of $(a_1, \dots, a_l)$ (but *a priori* not independently of l). Hence $N(l; a_1, \dots, a_l)$ is well defined for all $(a_1, \dots, a_l) \in (D_1)^l$, and the set

$$\{N(l; a_1, \dots, a_l) : (a_1, \dots, a_l) \in (D_1)^l\}$$

is a finite subset of $\mathbb{N}^{n+1}$.

Let us order $\mathbb{N}^{n+1}$ lexicographically. Observe that

$$N(l; a_1, \dots, a_l) \geq_{\text{lex}} N(l+1; a_1, \dots, a_{l+1})$$

for any $(a_1, \dots, a_{l+1}) \in (D_1)^{l+1}$. Indeed, by intersecting $S_{a_1} \cap \dots \cap S_{a_l}$ with $S_{a_{l+1}}$ we may only decrease lexicographically the number of irreducible components: an irreducible component Z_μ of $S_{a_{l+1}}$ either contains all the irreducible components of $S_{a_1} \cap \dots \cap S_{a_l}$, in which case our $(n+1)$-tuple is not affected, or else there is an irreducible component W_ν of $S_{a_1} \cap \dots \cap S_{a_l}$, of dimension, say, k, such that $Z_\mu \cap W_\nu \subsetneq W_\nu$. In the latter case, by Remark 2.1(ii), the set $Z_\mu \cap W_\nu$ is of dimension strictly smaller than k, and so the number of k-dimensional components in $S_{a_1} \cap \dots \cap S_{a_{l+1}}$ is strictly less than that in $S_{a_1} \cap \dots \cap S_{a_l}$.

Suppose for a contradiction that the number L from the proposition does not exist. Then for every $l \geq 1$, the set

$$T_l := \left\{ (a_1, \dots, a_l) \in D_1^l : N(1; a_1) >_{\text{lex}} N(2; a_1, a_2) >_{\text{lex}} \dots >_{\text{lex}} N(l; a_1, \dots, a_l) \right\}$$

is nonempty. Assume $N(l)$ is the (lexicographic) maximum among the tuples $N(l; a_1, \dots, a_l)$ as $(a_1, \dots, a_l) \in T_l$, and let $(b_1^l, \dots, b_l^l) \in T_l$ be such that $N(l) = N(l; b_1^l, \dots, b_l^l)$. It follows that

$$N(l) \geq_{\text{lex}} N(l; b_1^{l+1}, \dots, b_l^{l+1}) >_{\text{lex}} N(l+1; b_1^{l+1}, \dots, b_{l+1}^{l+1}) = N(l+1),$$

for all $l \geq 1$. Hence there exists a strictly decreasing infinite sequence of $(n+1)$-tuples

$$N(1) >_{\text{lex}} N(2) >_{\text{lex}} \dots >_{\text{lex}} N(l) >_{\text{lex}} \cdots,$$

which contradicts the fact that $\geq_{\text{lex}}$ is a well-ordering of $\mathbb{N}^{n+1}$. ∎

For $1 \leq d < n$, and $\lambda \in \Lambda(d, n)$, let

$$\pi_\lambda = \pi_{\lambda_1, \dots, \lambda_d} : \mathbb{C}^n \to \mathbb{C}^d \tag{4.1}$$

be the canonical projection from $\mathbb{C}^n$ onto (its linear subspace spanned by) the variables $z_\lambda = (z_{\lambda_1}, \dots, z_{\lambda_d})$. Let $z_\mu = (z_{\mu_1}, \dots, z_{\mu_{n-d}})$ be the $(n-d)$-tuple of the remaining variables (that is, $\{1, \dots, n\} = \{\lambda_1, \dots, \lambda_d\} \cup \{\mu_1, \dots, \mu_{n-d}\}$, with $1 \leq \mu_1 < \dots < \mu_{n-d} \leq n$).

Corollary 4.4 *Under the notation of Lemma 4.2, there exists a positive integer κ such that for every non-empty $A \subset D_1$ and any λ, the number of irreducible components of a fibre of $\pi_\lambda|_{(\bigcap_{a\in A} S_a \cap D_2)}$ is bounded above by κ.*

Proof Use Proposition 4.3 to replace $\bigcap_{a\in A} S_a \cap D_2$ by some $S_{a_1} \cap \cdots \cap S_{a_L}$ and then apply Lemma 4.2 to the sets

$$\{(a_1, \dots, a_L, z_\lambda, z_\mu) \in (D_1^L \times \mathbb{C}^d) \times \mathbb{C}^{n-d} : z \in S_{a_1} \cap \cdots \cap S_{a_L} \cap D_2\}. \qquad \blacksquare$$

5 Proofs of the Main Theorems

We first prove Theorem 1.4, from which the semianaliticity in Theorem 1.1 will follow.

5.1 Proof of Theorem 1.4

Fix $d \geq 1$. We give the proof of Theorem 1.4 for this given dimension.

(i) $\Rightarrow$ (ii). Let $p \in \mathcal{A}^d$ be an arbitrary point, and let U_1 and U_2 be neighbourhoods of p as defined in Section 2. Then there exists a complex analytic set Y in a neighbourhood of p, of dimension d, which is contained in X and passes through p. We may assume that Y is irreducible, and hence, by Lemma 2.3(ii), $\bigcap_{z\in Y\cap U_1} S_z$ contains $Y \cap U_1$.

Let $(z_1, \dots, z_n)$ be the coordinates in $\mathbb{C}^n$. We will show that, for every $\varepsilon > 0$ and $\kappa > 0$, there exists $\lambda \in \Lambda(d, n)$ for which there is a κ-grid $\mathcal{G}_\lambda^\kappa$ with d-dimensional base z_λ such that $\mathcal{G}_\lambda^\kappa \subset \mathbb{B}(p, \varepsilon)$.

Fix $\varepsilon > 0$. By Remark 2.1(iv) there are a small polydisc $D \subseteq \mathbb{B}(p, \varepsilon) \cap U_1$ such that $Y \cap D$ is a complex manifold and $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda(d, n)$ such that $Y \cap D$ is the graph of a holomorphic mapping in variables z_λ. In particular, any set

$$\{z_\nu \in \pi_\lambda(D), \nu = (\nu_1, \dots, \nu_d) \in \{1, \dots, \kappa+1\}^d : \\ \text{for all } \nu, \nu', j,\ \nu_j = \nu'_j \Leftrightarrow \pi_{\lambda_j}(z_\nu) = \pi_{\lambda_j}(z_{\nu'})\}$$

is pulled back by $\pi_\lambda|_{Y\cap D}$ to a set

$$\mathcal{G}_\lambda^\kappa = \{p_\nu : \nu = (\nu_1, \dots, \nu_d) \in \{1, \dots, \kappa+1\}^d\}$$

satisfying Definition 1.3(ii) (π_λ and π_{λ_j} are as in (4.1)). But as noted earlier,

$$\bigcap_{z\in Y\cap U_1} S_z \supset Y \cap U_1,$$

which shows that $\mathcal{G}_\lambda^\kappa$ also satisfies (i) of Definition 1.3.

(ii) $\Rightarrow$ (i). Let $q \in X \cap V$ be arbitrary, and let U_1 and U_2 be neighbourhoods of q as defined in Section 2. Let $\kappa \geq 1$ be an upper bound for the number of irreducible components of any fibre of $\bigcap_{z\in Z} S_z \cap U_2$ for any projection π_λ, $\lambda \in \Lambda(d,n)$, as Z ranges over the subsets of U_1. Corollary 4.4 applied to the set $\{(a,\overline{b}) \in \mathbb{C}^n \times \mathbb{C}^n : \varrho(a,\overline{b}) = 0\}$ ensures that this upper bound is finite.

Let $p \in U_1 \cap X$. Suppose that for any $\varepsilon > 0$, there exists a κ-grid with d-dimensional base z_λ for some $\lambda = (\lambda_1, \dots, \lambda_d)$,

$$\mathcal{G}_\lambda^\kappa = \left\{ p_\nu : \nu = (\nu_1, \dots, \nu_d) \in \{1, \dots, \kappa+1\}^d \right\}$$

contained in $\mathbb{B}(p,\varepsilon)$. Without loss of generality we may assume that the open ε-ball $\mathbb{B}(p,\varepsilon)$ is contained in U_1.

Let $Y^1 = \bigcap_{z\in\mathcal{G}_\lambda^\kappa} S_z$ and $Y^2 = \bigcap_{z\in Y^1\cap U_1} S_z$. By Lemma 2.3(i), $\mathcal{G}_\lambda^\kappa \subset Y^2$; moreover, $Y^2 \subset X$ by Definition 1.3(i) and Lemma 2.3(iii).

For λ as above, we denote by $\lambda^{(\delta)}$ the δ-tuple $(\lambda_1, \dots, \lambda_\delta) \in \Lambda(\delta,n)$ of the first δ components of λ, $\delta \in \{1, \dots, d\}$. We will consider the fibres $\pi_{\lambda^{(\delta)}}^{-1}(\pi_{\lambda^{(\delta)}}(p_\nu))$ at points $p_\nu \in \mathcal{G}_\lambda^\kappa$, with the convention that $\pi_{\lambda^{(0)}}^{-1}(\pi_{\lambda^{(0)}}(p_\nu)) = V$.

Let us prove by descending induction on $\delta \in \{0, \dots, d\}$ that for each $p_\nu \in \mathcal{G}_\lambda^\kappa$ the fibre

$$\pi_{\lambda^{(\delta)}}^{-1}\left(\pi_{\lambda^{(\delta)}}(p_\nu)\right) \cap Y^2$$

contains an irreducible component of dimension $\geq d-\delta$ that passes through a $p_{\nu'} \in \mathcal{G}_\lambda^\kappa$ with $\pi_{\lambda^{(\delta)}}(p_\nu) = \pi_{\lambda^{(\delta)}}(p_{\nu'})$ (the latter equality being vacuously true if $\delta = 0$).

- For $\delta = d$, it suffices to take any irreducible component of $\pi_\lambda^{-1}(\pi_\lambda(p_\nu)) \cap Y^2$ passing through p_ν (which exists, since $p_\nu \in Y^2$).
- Suppose the result holds for $\delta + 1$. Then the collection of subsets of V

$$\left\{ \pi_{\lambda^{(\delta+1)}}^{-1}(\pi_{\lambda^{(\delta+1)}}(p_\mu)) \cap Y^2 : p_\mu \in \mathcal{G}_\lambda^\kappa,\ \pi_{\lambda^{(\delta)}}(p_\mu) = \pi_{\lambda^{(\delta)}}(p_\nu) \right\}$$

has $\kappa + 1$ pairwise disjoints elements (one for each $\pi_{\lambda^{(\delta+1)}}(p_\mu)$), each containing an irreducible component of dimension $\geq d - (\delta+1)$ and each contained in $\pi_{\lambda^{(\delta)}}^{-1}(\pi_{\lambda^{(\delta)}}(p_\nu)) \cap Y^2$. By the definition of κ and the pigeonhole principle, there is an irreducible component X_ν of $\pi_{\lambda^{(\delta)}}^{-1}(\pi_{\lambda^{(\delta)}}(p_\nu)) \cap Y^2$ and two indices μ and μ' such that $\pi_{\lambda^{(\delta+1)}}(p_\mu) \neq \pi_{\lambda^{(\delta+1)}}(p_{\mu'})$, and there is an irreducible component X_μ (resp. $X_{\mu'}$) of $\pi_{\lambda^{(\delta+1)}}^{-1}(\pi_{\lambda^{(\delta+1)}}(p_\mu))$ (resp. of $\pi_{\lambda^{(\delta+1)}}^{-1}(\pi_{\lambda^{(\delta+1)}}(p_{\mu'}))$) of dimension $\geq d - (\delta+1)$ with

$$X_\mu \subset X_\nu \text{ and } X_{\mu'} \subset X_\nu.$$

Since $X_\mu \cap X_{\mu'} = \varnothing$, we get $\dim X_\nu \geq d - \delta$, for else X_ν would be the union of proper analytic subsets X_μ, $X_{\mu'}$ and $\overline{X_\nu \setminus (X_\mu \cup X_{\mu'})}$, with $\dim X_\mu = \dim X_{\mu'} = \dim X_\nu$, contradicting the irreducibility of X_ν (Remark 2.1(ii)).

The case $\delta = 0$ of the induction provides a point $p_{\nu'} \in \mathbb{B}(p,\varepsilon) \cap \mathcal{A}^d$. Therefore, p is an accumulation point of $\mathcal{A}^d$, and hence $p \in \mathcal{A}^d$ by Proposition 3.2.

Finally, for any point $q \in V$, there is a pair of neighbourhoods $U_1^q \Subset U_2^q \Subset V$ such that for every $w \in U_1^q$, S_w is a complex analytic subset of U_2^q of dimension at least $n-1$ (*cf.* Section 2). Since V is relatively compact in the domain of convergence of ϱ, the set $X \cap V$ can be covered by a finite collection of open sets $U_1^{q_\alpha}$, $\alpha = 1, \dots, N$. Taking the maximum value among the κ associated with each $U_2^{q_\alpha}$ will give the uniform κ, as claimed in Theorem 1.4. ∎

5.2 Proof of Theorem 1.1

Theorem 1.4 gives us a description of $\mathcal{A}^d$, $d \geq 1$, as a subanalytic set. This description will be shown to actually define a semianalytic set, which will prove Theorem 1.1.

Let $p \in X$ be arbitrary. Let $\varrho(z, \bar{z})$ be any defining function of X given by a convergent power series in a polydisc neighbourhood V of p. Let κ be as in Theorem 1.4. Define

$$\Sigma_1 = \left\{ (z_1, \dots, z_{\kappa+1}) \in V^{\kappa+1} : \varrho(z_\mu, \bar{z}_\nu) = 0, 1 \leq \mu, \nu \leq \kappa + 1 \right\}.$$

Then Σ_1 is a real analytic subset of $V^{\kappa+1}$. Let

$$\Delta_1 = \left\{ (z_1, \dots, z_{\kappa+1}) \in (\mathbb{C}^n)^{\kappa+1} : z_1 = \cdots = z_{\kappa+1} \right\},$$

and consider the set

$$S_1 = \overline{\Sigma_1 \setminus \{(z_1, \dots, z_{\kappa+1}) \in V^{\kappa+1} :\ z_\nu = z_{\nu'} \ \text{ for some } \nu \neq \nu'\}} \ \cap\ \Delta_1.$$

The closure of a semianalytic set being semianalytic, S_1 is a semianalytic subset of the diagonal Δ_1. One easily checks that the projection to the first coordinate of a semianalytic subset of the diagonal is itself semianalytic. But $\mathcal{A}^1 \cap V$ is precisely the projection of S_1 to the first coordinate by Theorem 1.4.

Similarly, for $d \geq 2$, define

$$\Sigma_d = \Big\{ (z_{1,\dots,1}, \dots, z_{\kappa+1,\dots,\kappa+1}) \in V^{(\kappa+1)^d} : \\ \varrho(z_\nu, \bar{z}_{\nu'}) = 0,\ \ \nu, \nu' \in \{1, \dots, \kappa+1\}^d \Big\},$$

and for every $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda(d, n)$, put

$$\Theta^d_\lambda = \Big\{ (z_{1,\dots,1}, \dots, z_{\kappa+1,\dots,\kappa+1}) \in V^{(\kappa+1)^d} :\ \text{for all } j \in \{1, \dots, d\} \\ \text{and } (\nu, \nu') \in (\{1, \dots \kappa+1\}^d)^2\,, \pi_{\lambda_j}(z_\nu) = \pi_{\lambda_j}(z_{\nu'}) \Leftrightarrow \nu_j = \nu'_j \Big\}.$$

Then $\Sigma_d \cap \bigcup_{\lambda \in \Lambda(d,n)} \Theta^d_\lambda$ is a semianalytic subset of $V^{(\kappa+1)^d}$. Let

$$\Delta_d = \{(z_{1,\dots,1}, \dots, z_{\kappa+1,\dots,\kappa+1}) \in V^{(\kappa+1)^d} : z_{1,\dots,1} = \cdots = z_{\kappa+1,\dots,\kappa+1}\}$$

and consider the set

$$S_d = \overline{\Big(\Sigma_d \cap \bigcup_{\lambda \in \Lambda(d,n)} \Theta^d_\lambda\Big) \setminus T} \cap \Delta_d,$$

where

$$T = \{(z_{1,\dots,1}, \dots, z_{\kappa+1,\dots,\kappa+1}) \in V^{(\kappa+1)^d} :\ z_\nu = z_{\nu'} \ \text{ for some } \nu \neq \nu'\}.$$

As above, S_d is a semianalytic subset of the diagonal Δ_d, and hence its projection to the first coordinate, which is precisely $\mathcal{A}^d \cap V$ (by Theorem 1.4), is itself semianalytic.

Finally, suppose that X is real algebraic. Then ϱ is a polynomial, and hence the sets Σ_d above are all semialgebraic. It follows that the $\mathcal{A}^d$ are semialgebraic, for all $d \in \mathbb{N}$, which completes the proof of Theorem 1.1. ∎

A Appendix: Points of Infinite Type

In this section we review the basics of D'Angelo's theory of points of finite type. As before, let X denote a closed real analytic subset of an open set in $\mathbb{C}^n$. Our goal is to clarify the definition of type in the case that X is not a smooth hypersurface, and to give a condensed but self-contained proof of the fact that the subset of X of points of infinite type coincides with $\mathcal{A}^1$ (*cf.* [6, §3.3.3, Thm. 4]). We were motivated, in part, by the claims of incompleteness of the D'Angelo argument (see [8]). All the proofs presented in this section (modulo minor technical modifications) originate in D'Angelo [5, 6].

A.1 Order of Contact of a Holomorphic Ideal

Let $\mathcal{O}_p = {}_n\mathcal{O}_p$ denote the ring of germs of holomorphic functions at a point $p = (p_1, \dots, p_n) \in \mathbb{C}^n$. By the Taylor expansion isomorphism, we may identify ${}_n\mathcal{O}_p$ with the ring $\mathbb{C}\{z - p\}$ of convergent power series in $z - p$, where $z = (z_1, \dots, z_n)$ is a system of n complex variables. Let $\mathfrak{m}_p$ denote the maximal ideal of the local ring ${}_n\mathcal{O}_p$. Let Hol_p denote the set of germs of (non-constant) holomorphic mappings from a neighbourhood of 0 in $\mathbb{C}$ to a neighbourhood of p in $\mathbb{C}^n$ (sending 0 to p). Given $f = (f_1, \dots, f_n) \in \mathbb{C}\{\zeta\}^n$, we denote by $\nu(f)$ the order of vanishing of f at 0; *i.e.*, $\nu(f) := \max\{k \in \mathbb{N} : f_j \in \mathfrak{m}^k, j = 1, \dots, n\}$ if $f \neq 0$ in $\mathbb{C}\{\zeta\}^n$, and $\nu(0) := \infty$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{\zeta\}$.

Definition A.1 ([5, Def. 2.6]) Given a proper ideal I in $\mathcal{O}_p$, define

$$\tau^*(I) = \sup_{\gamma \in \mathrm{Hol}_p} \inf_{g \in I} \frac{\nu(g \circ \gamma)}{\nu(\gamma)};$$

$$K(I) = \inf\{k \in \mathbb{N} : \mathfrak{m}_p^k \subset I\};$$

$$D(I) = \dim_{\mathbb{C}}(\mathcal{O}_p / I) \qquad \text{(as a complex vector space)}.$$

The following is a simplified variant of [5, Thm. 2.7].

Lemma A.2 *Suppose that I is a proper ideal in $\mathcal{O}_p$. Then*

$$\tau^*(I) \leq K(I) \leq D(I).$$

Moreover, each of the above constants is finite if and only if the zero-set germ of I is the singleton $\{p\}$.

Proof Let $\mathcal{V}(I)$ denote the zero-set germ of I. By the complex analytic Nullstellensatz (see, *e.g.*, [11, Ch. 3, §4.1]), $\mathcal{V}(I) = \{p\}$ if and only if $\sqrt{I} = \mathfrak{m}_p$, or equivalently (by Noetherianity of $\mathcal{O}_p$), I contains a power of the maximal ideal $\mathfrak{m}_p$. Hence $\mathcal{V}(I)$ equals $\{p\}$ precisely when both $K(I)$ and $D(I)$ are finite. On the other hand, $\mathcal{V}(I) \supsetneq \{p\}$ if and only if there exists a 1-dimensional irreducible complex-analytic germ Y_p at p such that every $g \in I$ vanishes on Y_p. Choosing $\gamma \in \mathrm{Hol}_p$ the Puiseux parametrization of Y_p (see [11, Ch. II, §6.2]), we see that the latter is equivalent to $g \circ \gamma = 0$ for every $g \in I$, that is, $\tau^*(I) = \infty$.

Assume then that $\mathcal{V}(I) = \{p\}$, or equivalently, that I contains a power of the maximal ideal $\mathfrak{m}_p$. Observe that $I \subset J$ implies $\tau^*(I) \geq \tau^*(J)$. Hence, if $I \supset \mathfrak{m}_p^k$, then $\tau^*(I) \leq \tau^*(\mathfrak{m}_p^k)$. The inequality $\tau^*(I) \leq K(I)$ thus follows from the fact that $\tau^*(\mathfrak{m}_p^K) = K$ (as $\mathfrak{m}_p^K$ can be generated by monomials, all of degree K).

Suppose now that $\mathfrak{m}_p^k \not\subset I$. Then there is a multi-index $\beta \in \mathbb{N}^n$ of length $|\beta| = k$, such that $(z-p)^\beta \notin I$. It follows that $(z-p)^\alpha \notin I$ for every $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ satisfying $\alpha_j \leq \beta_j$, $j = 1, \dots, n$. Since there are at least $|\beta| + 1 = k + 1$ of such α's, then $\mathcal{O}_p/I$ contains at least $k+1$ elements linearly independent over $\mathbb{C}$. This proves the inequality $K(I) \leq D(I)$. ■

A.2 The Type of a Real Analytic Principal Ideal

Let $\mathcal{O}_p^{\mathbb{R}} = {}_n\mathcal{O}_p^{\mathbb{R}}$ denote the ring of real-valued real analytic germs at a point $p = (p_1, \dots, p_n) \in \mathbb{C}^n$. Let $\varrho(z, \bar{z}) = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha\beta}(z-p)^\alpha(\overline{z-p})^\beta$ be a power series representation of $\varrho(z, \bar{z}) \in \mathcal{O}_p^{\mathbb{R}}$, convergent in an open neighbourhood of p in $\mathbb{C}^n$. We define the *type of ϱ at p* as

$$\Delta(\varrho, p) = \sup_{\gamma \in \mathrm{Hol}_p} \frac{\nu(\varrho \circ \gamma)}{\nu(\gamma)},$$

where the order of vanishing is taken with respect to the maximal ideal $(\mathrm{Re}(\zeta), \mathrm{Im}(\zeta))$ of the ring $\mathbb{R}\{\mathrm{Re}(\zeta), \mathrm{Im}(\zeta)\}$ of real analytic germs at 0 in $\mathbb{C} \cong \mathbb{R}^2$. It is readily seen that $\Delta(u \cdot \varrho, p) = \Delta(\varrho, p)$ for any invertible $u \in \mathcal{O}_p^{\mathbb{R}}$. Hence, since $\mathcal{O}_p^{\mathbb{R}}$ is a UFD, we may speak of the *type $\Delta(I, p)$ of a principal ideal $I = (\varrho)$* in $\mathcal{O}_p^{\mathbb{R}}$.

Let X be a smooth real analytic hypersurface in an open neighbourhood U of a point p in $\mathbb{C}^n$. Then, after shrinking U if necessary, there is a unique (up to multiplication by an invertible $u \in \mathcal{O}_p^{\mathbb{R}}$) real analytic $\varrho \in \mathcal{O}_p^{\mathbb{R}}$ with $d\varrho(p) \neq 0$ and $X = \{z \in U : \varrho(z, \bar{z}) = 0\}$. One defines (see [5, Def. 2.16], [6, §3.3.3]) the *type of X at p* as $\Delta(X, p) := \Delta(\varrho, p)$. However, the type of a real analytic set X is not well defined if X is not a hypersurface. Indeed, if the real codimension of X at p is greater than 1, there is no canonical choice of a single defining function, and given two distinct defining functions ϱ_1, ϱ_2 for X in a neighbourhood of p there need not exist an invertible u with $\varrho_2 = u \cdot \varrho_1$. Consequently, the family of ideals $I(\varrho, U, p)$ associated with X_p (see below) is not an invariant of X_p, but only of the principal ideal $(\varrho) \cdot \mathcal{O}_p^{\mathbb{R}}$. (Thus D'Angelo's [6, §3.3.2, Prop. 5] only applies to smooth real hypersurfaces.) Nonetheless, we can state the following definition.

Definition A.3 Let X be a closed real analytic subset of an open set in $\mathbb{C}^n$, and let $\varrho(z, \bar{z})$ be any real analytic function in a neighbourhood U of a point $p \in X$ satisfying $X \cap U = \{z \in U : \varrho(z, \bar{z}) = 0\}$. We say that p is a *point of finite type of X*, when $\Delta(\varrho, p) < \infty$. Otherwise, p is called a *point of infinite type of X*.

Remark A.4 By Proposition A.8, the notion of a point of finite type is well defined, *i.e.*, independent of the choice of a defining function. Indeed, if ϱ_1 and ϱ_2 are two real analytic functions defining X in a neighbourhood of a point $p \in X$, then $\Delta(\varrho_1, p) = \infty$ if and only if $\Delta(\varrho_2, p) = \infty$, because both equalities are equivalent to

X_p containing a positive-dimensional complex analytic germ.

A.3 Holomorphic Decomposition

Consider $\varrho(z,\bar z) = \sum_{|\alpha|+|\beta|\geq 1} c_{\alpha\beta}(z-p)^\alpha(\overline{z-p})^\beta$ a real analytic function vanishing at p, with the power series convergent in the polydisc $\{z : |z_j - p_j| < \delta_j\}$. Let $\delta = (\delta_1,\dots,\delta_n)$, and let $0 < t < 1$. One can associate with ϱ functions

$$\begin{aligned} h(z) &= 4\sum_{|\alpha|\geq 1} c_{\alpha 0}(z-p)^\alpha,\\ f^\beta(z) &= \sum_{|\alpha|\geq 1} c_{\alpha\beta}(t\delta)^\beta(z-p)^\alpha + (z-p)^\beta(t\delta)^{-\beta},\\ g^\beta(z) &= \sum_{|\alpha|\geq 1} c_{\alpha\beta}(t\delta)^\beta(z-p)^\alpha - (z-p)^\beta(t\delta)^{-\beta}, \end{aligned}$$

for all $\beta \in \mathbb{N}^n$, $|\beta| \geq 1$. It is easy to see that $h(z)$ and all the $f^\beta(z)$, $g^\beta(z)$ are holomorphic in the polydisc $\{z : |z_j - p_j| < t\delta_j\}$, and that $\|f(z)\|^2 = \sum_{|\beta|\geq 1} |f^\beta(z)|^2$, $\|g(z)\|^2 = \sum_{|\beta|\geq 1} |g^\beta(z)|^2$ are real analytic in the same polydisc. One may thus consider $f = (f^\beta)_{|\beta|\geq 1}$ and $g = (g^\beta)_{|\beta|\geq 1}$ as holomorphic functions with values in the Hilbert space l^2. Moreover, ϱ admits a *holomorphic decomposition* of the form

$$4\varrho(z,\bar z) = 2\mathrm{Re}(h(z)) + \|f(z)\|^2 - \|g(z)\|^2. \tag{A.1}$$

For a unitary transformation $U\colon l^2 \to l^2$, consider an ideal $I(\varrho, U, p)$ in $\mathcal{O}_p$ generated by $h(z)$ and by the components $f^\beta(z) - \sum_{\sigma\in\mathbb{N}^n} u_{\beta\sigma} g^\sigma(z)$ of $f - U(g)$, where $u_{\beta\sigma}$ are the entries of the (matrix of) U.

Lemma A.5 (*cf.* [5, Thm. 3.5]) *The following inequality holds:*

$$\Delta(\varrho,p) \;\leq\; 2\sup_U \tau^*\big(I(\varrho,U,p)\big),$$

where the supremum is taken over all unitary transformations $U\colon l^2 \to l^2$.

Proof Suppose that $\gamma \in \mathrm{Hol}_p$ is such that $\nu(\varrho\circ\gamma) > 2k$ for some integer $k \geq 1$. It suffices to find a unitary $U\colon l^2 \to l^2$ for which $\tau^*(I(\varrho,U,p)) > k/\nu(\gamma)$. We have $J^{2k}(\varrho\circ\gamma) = 0$, where, for a germ $f \in \mathbb{R}\{x,y\}$, $J^s(f)$ denotes the *s-jet* of f, that is, the image of f under the homomorphism $J^s\colon \mathbb{R}\{x,y\} \to \mathbb{R}\{x,y\}/(x,y)^{s+1}$ of $\mathbb{R}\{x,y\}$-modules. For simplicity of notation assume that $p = 0$. Then

$$\varrho(\gamma(\zeta),\overline{\gamma(\zeta)}) = \Big(\sum_{|\alpha|\geq 1} c_{\alpha 0}\gamma(\zeta)^\alpha + \sum_{|\beta|\geq 1} c_{0\beta}\overline{\gamma(\zeta)}^\beta\Big) + \sum_{|\alpha|,|\beta|\geq 1} c_{\alpha\beta}\gamma(\zeta)^\alpha\overline{\gamma(\zeta)}^\beta.$$

Since the bracket on the right-hand side of this equation contains only pure terms and all the other (non-zero) terms contain positive powers of both ζ and $\bar\zeta$, it follows from $J^{2k}(\varrho \circ \gamma) = 0$ that the $2k$-th jet of the bracket is zero. The content of the bracket is precisely $2\mathrm{Re}(h \circ \gamma)$, hence $J^{2k}(h \circ \gamma) = 0$, and consequently $J^{2k}(\|f \circ \gamma\|^2 - \|g \circ \gamma\|^2) = 0$ by (A.1). One checks by direct computation that the latter implies $\|J^k(f \circ \gamma)\|^2 = \|J^k(g \circ \gamma)\|^2$. Then, by Lemma A.6, there is a unitary $U : l^2 \to l^2$ such that $J^k(f \circ \gamma) - U(J^k(g \circ \gamma)) = 0$. Since $J^k(f \circ \gamma) - U(J^k(g \circ \gamma)) = J^k[(f - U(g)) \circ \gamma]$, it follows that $\nu((f^\beta - \sum_{\sigma \in \mathbb{N}^n} u_{\beta\sigma} g^\sigma) \circ \gamma) > k$ for all $|\beta| \geq 1$. Therefore $\nu(F \circ \gamma) > k$ for every generator F of $I(\varrho, U, p)$, which proves $\tau^*(I(\varrho, U, p)) > k/\nu(\gamma)$. ■

Lemma A.6 (*cf.* [6, §3.3.1, Prop. 4]) *Let $F, G \colon B \to l^2$ be holomorphic mappings on an open ball in $\mathbb{C}^q$, with $\|F\|^2 = \|G\|^2$. Suppose there exists $k \in \mathbb{N}$ such that all the components of F and G are polynomials of degree at most k. Then there is a unitary operator $U : l^2 \to l^2$ satisfying $F = U(G)$.*

Proof Write $F = \sum F_\alpha z^\alpha$, $G = \sum G_\alpha z^\alpha$. By expanding and equating the norms squared, one obtains relations

$$(F_\alpha, F_\beta) = (G_\alpha, G_\beta) \tag{A.2}$$

for all multi-indices α, β, where $(\cdot\,,\,\cdot)$ denotes the inner product in l^2. Since all the components of F and G are polynomials of degree at most k, it follows that $\mathrm{span}(F_\alpha)$ and $\mathrm{span}(G_\alpha)$ are finite-dimensional vector spaces. Moreover, by (A.2), they are of the same dimension. Hence one can define $U \colon \mathrm{span}(G_\alpha) \to \mathrm{span}(F_\alpha)$ by setting $U(G_\alpha) = F_\alpha$ on a maximal linearly independent set. Then U is a well-defined linear transformation and an isometry from $\mathrm{span}(G_\alpha)$ to $\mathrm{span}(F_\alpha)$. By defining U to be an isometry from the orthogonal complement of $\mathrm{span}(G_\alpha)$ to the orthogonal complement of $\mathrm{span}(F_\alpha)$, one obtains an operator with the required properties. ■

Remark A.7 We are indebted to the anonymous referee for pointing out a mistake in an earlier version of the above lemma. In fact, the mistake can be traced back to [6, §3.3.1, Prop. 4], where it is not assumed that F and G are polynomial. As it turns out, without this assumption one cannot guarantee that the dimensions of $\mathrm{span}(F_\alpha)$ and $\mathrm{span}(G_\alpha)$ are the same and so are the dimensions of their orthogonal complements. This can be seen readily if one sets, for example, $F = (z, z^2, z^3, \dots)$ and $G = (0, z, z^2, z^3, \dots)$. In the general case, one can still prove that there exists an isometry U (but not necessarily unitary) such that $F \oplus 0 = U(G)$ or $G \oplus 0 = U(F)$, where 0 is a certain (possibly infinite) vector of zeros.

A.4 The Equivalence

Proposition A.8 (*cf.* [6, §3.3.3, Thm. 4]) *Let X be a closed real analytic subset of an open set in $\mathbb{C}^n$, defined in a neighbourhood of a point $p \in X$ by the vanishing of a real analytic function $\varrho(z, \bar{z}) = \sum_{\alpha,\beta} c_{\alpha\beta}(z-p)^\alpha \overline{(z-p)}^\beta$. Then $\Delta(\varrho, p) < \infty$ if and only if the germ X_p contains no positive-dimensional complex analytic germ.*

Proof We follow the argument of Lempert [10]. Suppose X_p contains a 1-dimensional complex-analytic germ Y_p. Choosing $\gamma \in \mathrm{Hol}_p$ the Puiseux parametrization of (an irreducible component of) Y at p, we get $\varrho \circ \gamma = 0$, hence $\Delta(\varrho, p) = \infty$.

Conversely, assume that X_p contains no positive-dimensional complex germs and, for a proof by contradiction, suppose that $\Delta(\varrho, p) = \infty$. Then, by Lemma A.5, there exists a sequence $(U^j)_{j\geq 1}$ of unitary matrices for which

$$\lim_{j\to\infty} \tau^*\big(I(\varrho, U^j, p)\big) = \infty. \tag{A.3}$$

Denoting by $(U^j)^*$ the adjoint of U^j, we have, for every j,

$$I(\varrho, U^j, p) = \big(h, f - U^j(g)\big)\cdot \mathcal{O}_p = \big(h, f - U^j(g), (U^j)^*(f) - g\big)\cdot \mathcal{O}_p, \tag{A.4}$$

since $(U^j)^* = (U^j)^{-1}$ and ideals in $\mathcal{O}_p$ are closed in the topology of coefficient-wise convergence (see, *e.g.*, [9, Thm. 6.3.5]). By (A.3) and Lemma A.2, it follows that

$$\lim_{j\to\infty} D\big(I(\varrho, U^j, p)\big) = \infty. \tag{A.5}$$

The entries $u^j_{\beta\sigma}$ of every U^j with respect to any complete orthonormal set are bounded in absolute value by 1. Hence, it can be assumed that, for all $\beta, \sigma \in \mathbb{N}^n$, the sequence $(u^j_{\beta\sigma})_{j\geq 1}$ has a limit, say, $u^\infty_{\beta\sigma}$. Denote by U^∞ the limit operator $(u^\infty_{\beta\sigma})$, and let $(U^\infty)^* = (\widetilde{u}^\infty_{\beta\sigma})$ denote its adjoint.

Let Y_p be the zero-set germ of the ideal $J = (h, f - U^\infty(g), (U^\infty)^*(f) - g)\cdot \mathcal{O}_p$. The operator norms of U^∞ and of $(U^\infty)^*$ are less than or equal to 1 (however, U^∞ need not be unitary). Therefore, for every z in a (sufficiently small) representative of Y_p, we have

$$\|f(z)\| = \|U^\infty(g(z))\| \leq \|g(z)\| = \|(U^\infty)^*(f(z))\| \leq \|f(z)\|.$$

Thus $Y_p \subset X_p$, by (A.1), and hence Y_p is the germ of the singleton $\{p\}$, by assumption. Consequently $D(J) < \infty$, by Lemma A.2; say, $D(J) = d$.

Now, by noetherianity of $\mathcal{O}_p$, there exists $N \in \mathbb{N}$ such that

$$J = \Big(h, f^\beta - \sum_\sigma u_{\beta\sigma} g^\sigma, g^\beta - \sum_\sigma \widetilde{u}_{\beta\sigma} f^\sigma : |\beta| \leq N\Big).$$

Set

$$I_j = \Big(h, f^\beta - \sum_\sigma u^j_{\beta\sigma} g^\sigma, g^\beta - \sum_\sigma \widetilde{u}^j_{\beta\sigma} f^\sigma : |\beta| \leq N\Big),$$

where $\widetilde{u}^j_{\beta\sigma}$ are the entries of $(U^j)^*$. By the Banach–Steinhaus Theorem, in a sufficiently small neighbourhood of p, all $f^\beta - \sum_\sigma u^j_{\beta\sigma} g^\sigma$ (resp. $g^\beta - \sum_\sigma \widetilde{u}^j_{\beta\sigma} f^\sigma$) converge uniformly to $f^\beta - \sum_\sigma u_{\beta\sigma} g^\sigma$ (resp. to $g^\beta - \sum_\sigma \widetilde{u}_{\beta\sigma} f^\sigma$) as $j \to \infty$. Hence, by the upper semi-continuity of $D(I) = \dim_{\mathbb{C}} \mathcal{O}_p/I$ as a function of I ([14, Ch. II, Prop. 5.3]), we have $D(I_j) \leq d$ for j large enough. On the other hand,

$$I_j \subset \Big(h, f^\beta - \sum_\sigma u^j_{\beta\sigma} g^\sigma, g^\beta - \sum_\sigma \widetilde{u}^j_{\beta\sigma} f^\sigma : \beta \in \mathbb{N}^n\Big) = I(\varrho, U^j, p),$$

where the equality follows from (A.4). Therefore, $D(I(\varrho, U^j, p)) \leq D(I_j) \leq d$, which contradicts (A.5). ∎

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