A NOTE ON A DEGREE SUM CONDITION FOR LONG CYCLES IN GRAPHS

JANUSZ ADAMUS

Abstract. We conjecture that a 2-connected graph G of order n, in which $d(x)+d(y) \geq n-k$ for every pair of non-adjacent vertices x and y, contains a cycle of length $n-k$ $(k < n/2)$, unless G is bipartite and $n-k$ is odd. This generalizes to long cycles a well-known degree sum condition for hamiltonicity of Ore. The conjecture is shown to hold for $k=1$.

1. Introduction

The subject of this note is the following conjecture, in which we generalize to long cycles a well-known degree sum condition for hamiltonicity of Ore [4]. All graphs considered are finite, undirected, with no loops or multiple edges.

Conjecture 1.1. *Let G be a 2-connected graph of order $n \geq 3$, $n \neq 5, 7$, and let $k < n/2$ be an integer. If*

$$d(x) + d(y) \geq n - k$$

for every pair of non-adjacent vertices x and y, then G contains a cycle of length $n-k$, unless G is bipartite and $n - k \equiv 1 \pmod 2$.

Remark 1.2. The conjecture is sharp. First of all, a quick look at C_5 and C_7 ensures that the assumption $|G| \neq 5, 7$ is necessary. Secondly, it is easy to see that without the 2-connectedness assumption, there could be no long cycles at all. Consider, for instance, a graph G obtained from disjoint cliques $H_1 = K_{\lfloor n/2 \rfloor}$ and $H_2 = K_{\lceil n/2 \rceil}$ by joining a single vertex x_0 of H_2 with every vertex of H_1. Finally, the bound for the degree sum of non-adjacent vertices is best possible, as shown in the example below.

Example 1.3. Let G be a graph obtained from the complete bipartite graph $K_{(n-k-1)/2,(n+k+1)/2}$ by joining all the vertices in the smaller colour class. Then $d(x) + d(y) \geq n - k - 1$ for every pair of non-adjacent vertices x and y, and G contains no cycle of length greater than $n - k - 1$.

Our main result is the following theorem that implies Conjecture 1.1 for $k = 1$, as shown in Section 2. The proof of Theorem 1.4 is given in the last section.

Theorem 1.4. *Let G be a 2-connected graph of order $n \geq 3$, in which*

$$d(x) + d(y) \geq n - 1$$

for every pair of non-adjacent vertices x and y.

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(i) *If n is even, then G is hamiltonian.*
(ii) *If n is odd, then G contains a cycle of length at least $n-1$.*

Moreover, G is not hamiltonian only if the minimal degree of its n-closure, $Cl_n(G)$, equals $(n-1)/2$. In this case, $Cl_n(G)$ is a maximal non-hamiltonian graph.

Recall that the *n-closure* $Cl_n(G)$ of G is a graph obtained from G by succesively joining all pairs (x, y) of non-adjacent vertices satisfying $d(x) + d(y) \geq n$.

2. Long cycles in graphs

Proposition 2.1. *Conjecture 1.1 holds for $k = 1$.*

For the proof, we will need the following result of [3]:

Theorem 2.2 (Haggkvist-Faudree-Schelp)**.** *Let G be a hamiltonian graph on n vertices. If G contains more than $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ edges, then G is pancyclic or bipartite.*

Proof of Proposition 2.1. By Theorem 1.4, we may assume that G is hamiltonian. Suppose first that G is a 2-connected non-bipartite hamiltonian graph of order n, in which $d(x) + d(y) \geq n-1$ whenever $xy \notin E(G)$.

Consider a vertex x of minimal degree $d(x) = \delta(G)$ in G. Write $\delta = \delta(G)$. Then G has precisely $n-1-\delta$ vertices non-adjacent to x, each of degree at least $n-1-\delta$. The remaining $\delta+1$ vertices are of degree at least δ each, hence

$$\|G\| \geq \frac{1}{2}[(\delta+1)\delta + (n-1-\delta)^2] .$$

As $\delta \geq 2$, one immediately verifies that

$$\frac{1}{2}[(\delta+1)\delta + (n-1-\delta)^2] > \frac{(n-1)^2}{4} + 1 ,$$

whenever $n \neq 5$.

It remains to consider the case of G a bipartite 2-connected hamiltonian graph of order n. But then n must be even, for otherwise G would contain an odd cycle. Thus $n - 1 \equiv 1 \pmod 2$, which completes the proof. □

For convenience, let us finally recall two well-known results, that we shall need in the proof of Theorem 1.4:

Theorem 2.3 (Dirac [2])**.** *Let G be a graph of order $n \geq 3$ and minimal degree $\delta(G) \geq n/2$. Then G is hamiltonian.*

Theorem 2.4 (Bondy-Chvatal [1])**.** *Let G be a graph of order n and suppose that there is a pair of non-adjacent vertices x and y of G such that $d(x) + d(y) \geq n$. Then G is hamiltonian if and only if $G + xy$ is hamiltonian.*

Corollary 2.5. *A graph G is hamiltonian if and only if its n-closure $Cl_n(G)$ is so.*

3. Proof of Theorem 1.4

Proof of part (i). Suppose there exists an even integer $n \geq 4$ for which the assertion of the theorem does not hold. Let G be a maximal non-hamiltonian 2-connected graph of order n, in which $d(x) + d(y) \geq n - 1$ whenever $xy \notin E(G)$.

By maximality of G, $G + xy$ is hamiltonian for every pair of non-adjacent vertices $x, y \in V(G)$. Hence, by Theorem 2.4, we must have

$$(*) \qquad d(x) + d(y) = n - 1 \quad \text{whenever } xy \notin E(G)\,.$$

The minimal degree $\delta(G)$ of G satisfies inequality $\delta(G) < n/2$, by Theorem 2.3, hence, in particular, $n - 1 - \delta(G) \geq \delta(G) + 1$.

Pick $x \in V(G)$ with $d(x) = \delta(G)$. There are precisely $n - 1 - \delta(G)$ vertices in G non-adjacent to x, each of degree $n - 1 - \delta(G)$, by $(*)$. Put $V = \{v \in V(G) : xv \notin E(G)\}$. Pick $y \in V$. As $d(y) = n - 1 - \delta(G)$, there are precisely $\delta(G)$ vertices in G non-adjacent to y, each of degree $\delta(G)$, by $(*)$ again. Put $U = \{u \in V(G) : uy \notin E(G)\}$. Then $|U| = \delta(G)$, $|V| = n - 1 - \delta(G)$, and $U \cap V = \emptyset$, because vertices in U are of degree $\delta(G)$ and those in V are of degree $n - 1 - \delta(G) > \delta(G)$. It follows that there exists a vertex z in G such that $V(G) = U \cup V \cup \{z\}$ is a partition of the vertex set of G.

We will now show that $d(z) = n - 1$: Observe first that $d(z) > \delta(G)$. Indeed, if $d(z) = \delta(G)$, then by $(*)$, z is adjacent to every vertex in U, as $2\delta(G) < n - 1$. But z is also adjacent to y, as $z \notin U$, hence $d(z) \geq |U| + 1 = \delta(G) + 1$; a contradiction. Consequently, z is adjacent to every vertex in V, by $(*)$ again, as $d(z) + (n - 1 - \delta(G)) > n - 1$. Hence $d(z) \geq |V| = n - 1 - \delta(G)$. On the other hand, z is adjacent to x, as $z \notin V$, which yields $d(z) \geq |V| + 1 = n - \delta(G)$. This last inequality paired with $(*)$ implies that z is adjacent to every other vertex in G, as required.

Next observe that $u_1u_2 \in E(G)$ for every pair of vertices u_1, u_2 in U, as $d(u_1) + d(u_2) = 2\delta(G) < n - 1$. It follows that $N(u) \supset U \cup \{z\} \setminus \{u\}$, and hence, by comparing cardinalities, $N(u) = U \cup \{z\} \setminus \{u\}$ for every $u \in U$.

Similarly, $v_1v_2 \in E(G)$ for every pair v_1, v_2 in V, hence $N(v) = V \cup \{z\} \setminus \{v\}$ for every $v \in V$. Therefore $G = G_1 \cup G_2$, where G_1 is a complete graph of order $\delta(G) + 1$ spanned on the vertices of $U \cup \{z\}$, and G_2 is a complete graph of order $n - \delta(G)$ spanned on $V \cup \{z\}$. Then z is a cutvertex, contradicting the assumption that G be 2-connected.

Proof of part (ii). Suppose there exists a 2-connected graph of odd order $n \geq 3$, in which $d(x) + d(y) \geq n - 1$ for every pair of non-adjacent vertices x and y, that does not contain neither a Hamilton cycle nor a cycle of length $n - 1$. Let G be maximal such a graph of order n. By maximality of G, $G + xy$ contains a cycle of length at least $n - 1$ whenever $xy \notin E(G)$. Hence G contains a path of length at least $n - 2$ between any two of its non-adjacent vertices.

Pick a pair of non-adjacent vertices x and y. By a theorem of Pósa, G contains a Hamilton $x - y$ path P, and hence, by Theorem 2.4, the sum $d(x) + d(y)$ actually equals $n - 1$. Write $P = u_1u_2 \ldots u_n$, where $u_1 = x$ and $u_n = y$.

Put $I_x = \{i : xu_{i+1} \in E(G), 1 \leq i \leq n - 1\}$ and $I_y = \{i : u_iy \in E(G), 1 \leq i \leq n - 1\}$. If $I_x \cap I_y \neq \emptyset$, say $i_0 \in I_x \cap I_y$, then G contains a Hamilton cycle

$$u_1u_{i_0+1}u_{i_0+2} \ldots u_nu_{i_0}u_{i_0-1} \ldots u_2u_1\,.$$

We may thus assume that $I_x \cap I_y = \emptyset$. Then, for every $1 \le i \le n-1$, either u_i is adjacent to y or else u_{i+1} is adjacent to x, because $|I_x| + |I_y| = d(x) + d(y) = n-1$. Let $d = d(y)$ and let $v_1, \dots, v_d = y$ be the vertices that lie on P next to the (respective) neighbours of y.

If there exists $j < d$ such that $v_j \notin N(y)$, then $v_j = u_{i_0}$ for some $i_0 \in I_x$. It follows that u_{i_0+1} is adjacent to x, and G contains a cycle of length $n-1$ of the form

$$u_1 u_{i_0+1} u_{i_0+2} \dots u_n u_{i_0-1} u_{i_0-2} \dots u_2 u_1 \,.$$

Therefore we can assume that

$$(\dagger) \qquad v_1, \dots, v_{d-1} \text{ are all adjacent to } y \,.$$

Let z denote the furthermost neighbour of y on P. It follows from (†) that all the vertices between z and y on P are adjacent to y, and hence $z = u_{n-d}$.

Suppose $N(v_j) \subset \{z, v_1, \dots, v_d\}$ for $j \le d$. Then $N(u_i) \subset \{u_1, \dots, u_{n-d-1}, z\}$ for $i \le n-d-1$. Consequently, $d(u_i) \le n-d-1$, $d(v_j) \le d$, and $u_i v_j \notin E(G)$ for $i \le n-d-1$ and $j \le d$. But then $d(u_i) + d(v_j) \ge n-1$ yields

$$d(u_i) = n-d-1 \quad \text{and} \quad d(v_j) = d \quad \text{for} \quad i = 1, \dots, n-d-1, \; j = 1, \dots, d \,.$$

Therefore, as in the proof of part *(i)*, we get that $G = G_1 \cup G_2$, where G_1 is a complete graph of order $n-d$ spanned on the vertices $\{u_1, \dots, u_{n-d-1}, z\}$ and G_2 is a complete graph of order $d+1$ on $\{z, v_1, \dots, v_d\}$. Then z is a cutvertex contradicting our assumptions on G.

It remains to consider the case of some v_{j_0} being adjacent to u_{i_0}, where $i_0 \le n-d-1$. But then again G contains a Hamilton cycle

$$u_1 \dots u_{i_0} v_{j_0} \dots v_d v_{j_0-1} \dots u_{i_0+1} u_1 \,.$$

For the proof of the last assertion of Theorem 1.4, suppose that $n = 2k+1$ is odd and G is a non-hamiltonian 2-connected graph on n vertices, satisfying $d(x) + d(y) \ge n-1$ for every pair of non-adjacent x and y. Then the n-closure of G, $G^* = Cl_n(G)$ is not hamiltonian either, by Theorem 2.5, and we have equality

$$d_{G^*}(x) + d_{G^*}(y) = n-1 \quad \text{whenever} \quad xy \notin E(G^*).$$

Now, if $\delta(G^*) < k = \frac{n-1}{2}$, then $n-1-\delta(G^*) > \delta(G^*)$ and one can repeat the proof of part *(i)* to show that G^* contains a Hamilton cycle, which contradicts the assumptions on G.

Thus $\delta(G^*) = \frac{n-1}{2}$. Moreover, $d_{G^*}(x) + d_{G^*}(y) = n-1 = 2k$ for $xy \notin E(G^*)$ implies that $d_{G^*}(x) = k$ or $d_{G^*}(x) = n-1$ for every vertex x.

Suppose G^* is not maximal among the non-hamiltonian 2-connected graphs on n vertices. Then G^* has a pair of non-adjacent vertices x and y such that $G^* + xy$ is contained in a maximal non-hamiltonian graph H. By maximality of H, $H + uv$ contains a Hamilton cycle for every $uv \notin E(H)$, so Theorem 2.4 implies that $d_H(u) + d_H(v) = n-1$ for every $uv \notin E(H)$.

Notice that $d_{G^*}(x) = k$, as $d_{G^*}(x) < n-1$. Then $d_H(x) \ge k+1$ and hence, for every v non-adjacent to x in G^*, $d_H(x) + d_H(v) \ge d_{G^*}(x) + 1 + d_{G^*}(v) > n-1$, implying $xv \in E(H)$. Therefore H is obtained from G by increasing degrees of at least x and all its non-neighbours in G^*, that is, at least $1 + (n-1-k) = k+1$ vertices. But then H contains at least $k+1$ vertices of degree $n-1$, which means that $\delta(H) \geq k+1 = \frac{n+1}{2}$, and hence H is hamiltonian by Theorem 2.3; a contradiction. $\square$

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Department of Mathematics, Middlesex College, University of Western Ontario, London, Ontario N6A 5B7 Canada

E-mail address: jadamus@uwo.ca