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Journal of Pure and Applied Algebra 193 (2004) 1–9

JOURNAL OF PURE AND APPLIED ALGEBRA

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Vertical components and flatness of Nash mappings

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Received 29 January 2004; received in revised form 13 February 2004
Communicated by A.V. Geramita

Abstract

We prove the following criterion for flatness of Nash morphisms: Let $f_\xi : X_\xi \to Y_\eta$ be a Nash morphism of reduced Nash germs with X_ξ of pure dimension and Y_η smooth of dimension n. Then f_ξ is flat if and only if the nth analytic tensor power $\underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{n \text{ times}}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.

MSC: 32C07; 13C11

1. Main result

Let Ω be an open set in $\mathbb{C}^m$. An analytic function $f \in \mathcal{O}(\Omega)$ is called a *Nash function* if it is algebraic over the ring of regular functions on Ω. An analytic set X is a *Nash set* if it can locally be defined by Nash functions, and *Nash mappings* are analytic mappings whose all components are Nash functions (see Section 3 for details). Given an analytic mapping $f : X \to Y$ of analytic spaces, with $f(\xi) = \eta$, let $f_\xi : X_\xi \to Y_\eta$ denote the germ of f at ξ, and let $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ be the germ at $\xi^{\{i\}} = (\xi, \dots, \xi)$ of

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[1] Research was supported by NSERC Postdoctoral Fellowship PDF-267954-2003.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.02.009

the induced canonical map from the i-fold fibre power of X over Y. The main result of this paper is the following criterion for flatness of Nash morphisms:

Theorem 1.1. *Let $f_\xi : X_\xi \to Y_\eta$ be a Nash morphism of Nash germs, where X_ξ is reduced of pure dimension and Y_η is smooth of dimension n. Then the following conditions are equivalent*:

(i) *f_ξ is flat*;
(ii) *the canonical map $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ has no algebraic vertical components.*

Remark 1.2. The equivalence can alternatively be formulated as follows:

$$f_\xi \text{ is flat } \Leftrightarrow \underbrace{\mathcal{O}_{X,\xi} \mathop{\hat{\otimes}}_{\mathcal{O}_{Y,\eta}} \cdots \mathop{\hat{\otimes}}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{n \text{ times}} \text{ is a torsion-free } \mathcal{O}_{Y,\eta}\text{-module.}$$

(Compare with Auslander's Theorem 2.2 below.)

The Nash category fits between the algebraic and analytic categories in a way that allows use of transcendental methods to obtain strong algebraic results. Geometrically, Nash sets are built, locally, from analytic branches of algebraic sets. For example, consider a Nash subset of the unit ball in $\mathbb{C}^2$ defined by the equation $y(1+\sqrt{1+x})-1=0$, where $\sqrt{\cdot}$ is a branch of the square root satisfying $\sqrt{1}=1$. Then X is an analytic branch at the origin of an algebraic curve $xy^2+2y+1=0$.

The proof of our result proceeds in three main steps: Firstly, we show that (isolated or embedded) irreducible components of Nash sets are Nash themselves (Lemma 4.1 and Proposition 5.1). Secondly, observe that every Nash mapping of Nash sets is regular in the sense of Gabrielov (Proposition 5.4). Finally, the result follows from the Galligo–Kwieciński criterion for flatness (see Section 2).

2. Motivation

This paper is concerned with the study of the relationship between degeneracies of the family of fibres of an analytic mapping (as expressed by a failure of flatness) and the existence of **vertical** components in fibre powers of the mapping. There are in fact two natural notions of a *vertical* component:

Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. An irreducible (isolated or embedded) component W of X_ξ is called **algebraic vertical** if there exists a nonzero element $a \in \mathcal{O}_{Y,\eta}$ such that (the pullback of) a belongs to the associated prime $\mathfrak{p}$ in $\mathcal{O}_{X,\xi}$ corresponding to W. Equivalently, W is *algebraic vertical* if an arbitrarily small representative of W is mapped into a proper analytic subset of a neighbourhood of η in Y. We say that W is **geometric vertical** if an arbitrarily small representative of W is mapped into a nowhere dense subset of a neighbourhood of η in Y, or equivalently, if the hypergerm (in the sense of Galligo and Kwieciński, see [7]) $f(W)$ has empty interior in Y_η with the transcendental topology.

The concept of a vertical component comes up naturally as an equivalent of torsion in algebraic geometry and the two notions of a vertical component coincide in the algebraic case (over an irreducible target). However, it is no longer so in the analytic category. In principle, the existence of the *algebraic vertical* components is a weaker condition than the presence of the *geometric vertical* ones. Indeed, any *algebraic vertical* component (over an irreducible target) is *geometric vertical*, since a proper analytic subset of a locally irreducible analytic set has empty interior. The converse is not true though, as can be seen in the following example of Osgood (cf. [8, Kap.II, Section 5]):

$$f:\mathbb{C}^2 \ni (x,y) \mapsto (x, xy, xye^y) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in $\mathbb{C}^3$, but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target.

The geometric vertical components have proved to be a powerful tool in analytic geometry (see [7,11,12]). On the other hand, the algebraic approach, introduced in [1,2], has an advantage that all the statements about *algebraic vertical* components (as opposed to *geometric vertical*) can be restated in terms of torsion freeness of the local rings:

Remark 2.1. $f_\xi : X_\xi \to Y_\eta$ has no (isolated or embedded) algebraic vertical components if and only if the local ring $\mathcal{O}_{X,\xi}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.

(This follows from "prime avoidance", see e.g. [6, Section 3.2].)

Also, it seems plausible that algebraic properties of analytic morphisms, like flatness, could be controlled by means of algebraic vertical components rather than the geometric vertical ones. Some results in this direction were obtained in [2]. Here we give the affirmative answer for Nash mappings.

The study of vertical components originates in the following fundamental result of Auslander.

Theorem 2.2 (Auslander [4, Theorem 3.2]). *Let R be an unramified regular local ring of dimension $n>0$ and let M be a finite R-module. Then M is R-free if and only if the nth tensor power $M^{\otimes n}$ is a torsionfree R-module.*

(Auslander's result was later extended by Lichtenbaum [13] to arbitrary regular local rings.)

Recall that in the case of finite modules, freeness is equivalent to flatness. Also, for finite modules M and N over a local analytic algebra R, their analytic tensor product, denoted by $M \hat{\otimes}_R N$, equals the ordinary one, $M \otimes_R N$. In particular, if $f_\xi : X_\xi \to Y_\eta$ is a finite morphism of germs of analytic spaces, with Y_η smooth of dimension n, then Theorem 2.2 asserts that f_ξ is flat if and only if the local ring $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}} = \underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{n \text{ times}}$ of the nth fibre power of X_ξ over Y_η is a torsion-free $\mathcal{O}_{Y,\eta}$-module (or, equivalently, $X^{\{n\}}_{\xi^{\{n\}}}$ has no algebraic vertical components).

Recently, a remarkable generalization of Theorem 2.2 to the case of nonfinite mappings was obtained by Galligo and Kwieciński.

Theorem 2.3 (Galligo and Kwieciński [7, Theorem 6.1]). *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be reduced of pure dimension and let Y_η be smooth of dimension n. Then the following conditions are equivalent*:

(i) *f_ξ is flat*;
(ii) *the canonical map $f^{\{n\}}_{\xi\{n\}} : X^{\{n\}}_{\xi\{n\}} \to Y_\eta$ has no* (*isolated or embedded*) *geometric vertical components.*

(The result was first proved in the algebraic case, for $n = 2$ and for arbitrary X, by Vasconcelos, see [19, Proposition 6.1].)

Note that Theorem 2.3 cannot be rephrased in terms of torsion freeness of analytic tensor powers of $\mathcal{O}_{X,\xi}$, as the vertical components in question are *geometric* and not *algebraic*. Also, we do not know a proof of the algebraic version of Theorem 2.3 (for $n > 2$) that does not involve transcendental methods.

In [2] we conjectured that the above theorem can be restated in terms of the *algebraic* vertical components, which should eventually lead to such a proof. That is, we suggest the following:

Conjecture 2.4. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be reduced of pure dimension and let Y_η be smooth of dimension n. Then the following conditions are equivalent*:

(i) *f_ξ is flat*;
(ii) *the canonical map $f^{\{n\}}_{\xi\{n\}} : X^{\{n\}}_{\xi\{n\}} \to Y_\eta$ has no algebraic vertical components.*

We believe there are good reasons to expect the conjecture be true. First, as we showed in [2, Theorem 2.2], the presence of the *isolated* geometric vertical components in $X^{\{n\}}_{\xi\{n\}}$ is equivalent to the existence of *isolated* algebraic vertical components in the same fibre power. Secondly, the existence of an (isolated or embedded) geometric vertical component in the nth fibre power $X^{\{n\}}_{\xi\{n\}}$ implies that there is an (isolated or embedded) algebraic vertical component in *some* fibre power $X^{\{i\}}_{\xi\{i\}}$.

The main result of this paper is a step towards a proof of the general Conjecture 2.4.

3. Toolbox

For reader's convenience, most of the definitions and tools used later on are gathered in this section. We start with recalling the Nash category terminology (see [18] for details):

Let Ω be an open subset of $\mathbb{C}^m$, and let $x = (x_1, \ldots, x_m)$ be a system of m complex variables. A function f analytic on Ω is called a *Nash function* at $x_0 \in \Omega$ if there exist an open neighbourhood U of x_0 in Ω and a polynomial $P(x, y) \in \mathbb{C}[x, y]$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. An analytic function is a Nash function on Ω if it is a Nash function at every point of Ω. An analytic mapping $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{C}^n$ is a *Nash mapping* if each of its components is a Nash function on Ω.

A subset X of Ω is called a *Nash subset* of Ω if for every $x_0 \in \Omega$ there exist an open neighbourhood U of x_0 in Ω and Nash functions $f_1, \ldots, f_s$ on U, such that $X \cap U = \{x \in U : f_1(x) = \cdots = f_s(x) = 0\}$. A germ X_ξ at $\xi \in \mathbb{C}^m$ is a *Nash germ* if there exists an open neighbourhood U of ξ in $\mathbb{C}^m$ such that $X \cap U$ is a Nash subset of U. Equivalently, X_ξ is a Nash germ if its defining ideal can be generated by power series algebraic over the polynomial ring $\mathbb{C}[x]$; that is, $\mathcal{O}_{X,\xi} \cong \mathbb{C}\{x\}/(f_1, \ldots, f_s)\mathbb{C}\{x\}$ with $f_j \in \mathbb{C}\langle x \rangle$, $j = 1, \ldots, s$, where $\mathbb{C}\langle x \rangle$ denotes the algebraic closure of $\mathbb{C}[x]$ in $\mathbb{C}[[x]]$. An analytic mapping $f : \Omega \to \mathbb{C}^n$ (resp. germ f_ξ of f at $\xi \in \Omega$) is a Nash mapping (resp. Nash morphism) if and only if its graph is a Nash subset of $\Omega \times \mathbb{C}^n$ (resp. a Nash germ at $(\xi, f(\xi)) \in \Omega \times \mathbb{C}^n$).

Next, we sketch some consequences of Hironaka's division algorithm (for a thorough treatment we refer to [5]). We use the following notation: If $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, then x^β denotes the monomial $x_1^{\beta_1} \cdots x_m^{\beta_m}$. For $f \in \mathbb{C}\{x\}$, write $f(x) = \sum_{\beta \in \mathbb{N}^m} f_\beta x^\beta$ with $f_\beta \in \mathbb{C}$. Define the *support* of $f \in \mathbb{C}\{x\}$ by $\operatorname{supp} f = \{\beta \in \mathbb{N}^m : f_\beta \neq 0\}$, and the *initial exponent* of f by $\exp(f) = \min\{\beta : \beta \in \operatorname{supp} f\}$, where the minimum is taken with respect to the total ordering of $\mathbb{N}^m$ given by the lexicographic ordering of the $(m+1)$-tuples $(\beta_1 + \cdots + \beta_m, \beta_1, \ldots, \beta_m)$.

Given an ideal I in $\mathbb{C}\{x\}$, the *diagram of initial exponents* of I is defined as

$$\mathfrak{N}(I) = \{\exp(f) : f \in I \setminus \{0\}\} \subset \mathbb{N}^m.$$

Let $\mathbb{C}\{x\}^{\mathfrak{N}(I)} = \{f \in \mathbb{C}\{x\} : \operatorname{supp} f \cap \mathfrak{N}(I) = \emptyset\}$ denote the set of series supported outside the diagram $\mathfrak{N}(I)$. Then the natural mapping $\kappa : \mathbb{C}\{x\}^{\mathfrak{N}(I)} \to \mathbb{C}\{x\}/I$ is surjective (see [10, Section 6, Proposition 9]).

Remark 3.1. The above result implies in particular that if $f \notin I$ then $f \equiv \hat{f}$ modulo I for some $\hat{f}$ with $\exp(\hat{f}) \notin \mathfrak{N}(I)$.

Let $\mathbb{K}$ be an algebraically closed field. We will use Artin's approximation theorem in the following form:

Theorem 3.2 (Artin [3, Theorem 1.7]). *Let $f(x, y) = 0$ be a system of polynomial equations in $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_N)$ with coefficients in $\mathbb{K}$. Let c be an integer. Given a formal power series solution $\bar{y}(x) \in \mathbb{K}[[x]]$, there is an algebraic solution $y(x) \in \mathbb{K}\langle x \rangle$ such that $\bar{y}(x) \equiv y(x)$ modulo $(x)^c$.*

Finally, recall the notion of regularity in the sense of Gabrielov: A morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces is called *Gabrielov regular* if, for every irreducible component W of X_ξ, $\dim_\eta f(Z) = \dim_\eta \overline{f(Z)}$ for an arbitrarily small representative Z of W at ξ, where $\overline{f(Z)}$ denotes the Zariski closure of $f(Z)$ in a representative of Y at η (see, e.g., [17, Section 1]).

4. Algebraic lemma

Let $x=(x_1,\ldots,x_m)$ be a system of m complex variables, and let R be either the ring of convergent power series $\mathbb{C}\{x\}$ or the ring of formal power series $\mathbb{C}[[x]]$ in variables x. Let, as before, $\mathbb{C}\langle x\rangle$ denote the algebraic closure of $\mathbb{C}[x]$ (i.e., the Henselization of $\mathbb{C}[x]_{(x)}$) in $\mathbb{C}[[x]]$. The following result can be derived from [16, Section 45, Exercise 4] but we give a simple alternative proof.

Lemma 4.1. *Let I be an ideal in $\mathbb{C}\langle x\rangle$ and let $I=Q_1\cap\cdots\cap Q_t$ be its primary decomposition in $\mathbb{C}\langle x\rangle$. Then the extended ideals Q_jR $(j=1,\ldots,t)$ are primary, and hence $IR=(Q_1R)\cap\cdots\cap(Q_tR)$ is a primary decomposition of IR in R.*

Proof. First, we shall show that prime ideals in $\mathbb{C}\langle x\rangle$ extend to prime ideals in R. Suppose $P=(f_1,\ldots,f_s)$ is an ideal in $\mathbb{C}\langle x\rangle$ such that PR is not prime. Then there exist two series $p(x)$ and $q(x)$ in R such that $pq\in PR$, $p\notin PR$, and $q\notin PR$.

As $pq\in PR$, we have $pq=f_1h_1+\cdots+f_sh_s$ for some $h_1,\ldots,h_s\in R$. Without loss of generality can assume that $\exp(p)\notin\mathfrak{N}(P)$ and $\exp(q)\notin\mathfrak{N}(P)$. Indeed, since $p,q\notin PR$, then by Remark 3.1, $p\equiv\hat{p}$ and $q\equiv\hat{q}$ modulo PR for some $\hat{p}$, $\hat{q}$ whose initial exponents lie outside the diagram $\mathfrak{N}(P)$.

Consider the following system of polynomial equations

$$(\dagger)\qquad \begin{cases} P_1(x,y_1)=0\\ \ldots\\ P_s(x,y_s)=0\\ w_1w_2=y_1z_1+\cdots+y_sz_s, \end{cases}$$

where the $P_j(x,y_j)\in\mathbb{C}[x,y_j]$ are chosen so that $P_j(x,f_j(x))=0$, $j=1,\ldots,s$ in a neighbourhood of the origin in $\mathbb{C}^m$.

The system has a solution in R given by

$$y_j=f_j(x),\ z_j=h_j(x),\ w_1=p(x),\ w_2=q(x).$$

Thus, by Artin's Theorem 3.2, for any positive integer c, there exists an algebraic solution $(\tilde{f}_1,\ldots,\tilde{f}_s,\tilde{h}_1,\ldots,\tilde{h}_s,\tilde{p},\tilde{q})$ of $(\dagger)$ such that

$$\tilde{f}_j(x)\equiv f_j(x),\ \tilde{h}_j(x)\equiv h_j(x),\ \tilde{p}(x)\equiv p(x),\quad\text{and}\quad \tilde{q}(x)\equiv q(x)\ \operatorname{mod}(x)^c.$$

Note that each of the polynomial equations $P_j(x,y_j)=0$ admits only finitely many solutions in $y_j(x)\in R$. Therefore, by choosing c sufficiently large, one can assume that $\tilde{f}_j(x)=f_j(x)$ for $j=1,\ldots,s$, whence

$$\tilde{p}\tilde{q}=f_1\tilde{h}_1+\cdots+f_s\tilde{h}_s\in P.$$

Now, we can choose c so that $\exp(p)<x_1^{c_1}\ldots x_m^{c_m}$ and $\exp(q)<x_1^{c_1}\ldots x_m^{c_m}$ for any $(c_1,\ldots,c_m)\in\mathbb{N}^m$ with $\sum_{i=1}^m c_i=c$. Then $\exp(\tilde{p})=\exp(p)$ and $\exp(\tilde{q})=\exp(q)$ lie outside the diagram $\mathfrak{N}(P)$, and hence $\tilde{p},\tilde{q}\notin P$. Consequently, P is not prime itself.

Finally, let Q be a primary ideal in $\mathbb{C}\langle x\rangle$, with $P=\sqrt{Q}\in\operatorname{Spec}\mathbb{C}\langle x\rangle$. Then by above, $\sqrt{QR}=PR$ is prime in R, and thus QR is PR-primary, by [15, 9.C Theorem 13]. This completes the proof of the lemma. □

Remark 4.2. Note that the above result is *not* equivalent to saying that primary factors in any primary decomposition of the extended ideal IR can be generated by algebraic functions. In fact, the latter is not true (even under the non-redundancy assumptions), as was pointed out to the author by Mark Spivakovsky:

Consider, for instance, $I=(x^2,xy)$ in $\mathbb{C}\langle x,y,z\rangle$, and let $f(z)$ be a non-algebraic convergent power series in z. Then

$$I\cdot\mathbb{C}\{x,y,z\}=(x)\cap(y-f(z)x,x^2)$$

and the second factor cannot be generated by elements of $\mathbb{C}\langle x,y,z\rangle$.

5. Proof of the main result

Our Lemma 4.1 implies immediately the following:

Proposition 5.1. *If W is an (isolated or embedded) irreducible component of a Nash germ (resp. set), then W is a Nash germ (resp. set) itself.*

Remark 5.2. The above fact is well-known in case of the isolated components (see [18, Section 2.B] for a geometric argument). However, to our best knowledge, the embedded components have not been accounted for so far.

The idea of the proof of Theorem 1.1 is to show that, like in the algebraic case, in any fibre power of a Nash morphism all the geometric vertical components are algebraic vertical. The difficulty lies in the fact that, despite our assumption that both X_ξ and Y_η be reduced, the fibre powers $X^{\{i\}}_{\xi\{i\}}$ of X_ξ over Y_η may contain embedded components (see example below) and one needs to control the behaviour of the induced map $f^{\{i\}}_{\xi\{i\}}$ along these components as well. This is achieved by combining Propositions 5.1 and 5.4.

Example 5.3 (Hartshorne [9, Chapter III, Exercise 9.3(b)]). For reduced spaces X_1, X_2, and Y, the fibre product $X_1\underset{Y}{\times}X_2$ need not be reduced itself: Let $X\subset\mathbb{C}^4$ be a union of two copies of $\mathbb{C}^2$ that intersect precisely at the origin, say

$$X_1=\{(y_1,y_2,t_1,t_2)\in\mathbb{C}^4: t_1=t_2=0\}$$

and

$$X_2=\{(y_1,y_2,t_1,t_2)\in\mathbb{C}^4: t_1-y_1=t_2-y_2=0\}$$

and let $f:X \to \mathbb{C}^2$ be the projection onto the y variables. Then the fibre power $X^{\{2\}} = X \underset{\mathbb{C}^2}{\times} X$ has an embedded component, namely the origin in $\mathbb{C}^4$.

Proposition 5.4. *Every Nash mapping $f:X \to Y$ of Nash sets is Gabrielov regular.*

Proof. The problem being local, it suffices to consider the case when $f_\xi : X_\xi \to Y_\eta$ is a Nash morphism of germs of Nash sets. Furthermore, by passing to the graph of f, can assume that $\xi = (0,\eta)$, $X_\xi \subset (\mathbb{C}^m \times Y)_\xi$, and f_ξ is a germ at ξ of the canonical projection $\pi : \mathbb{C}^m \times Y \to Y$. This makes f_ξ a germ of a polynomial mapping. Next, observe that X_ξ being Nash, there exists a germ of an algebraic set Z_ξ in $(C^m \times Y)_\xi$ such that $X_\xi \subset Z_\xi$ and $\dim Z_\xi = \dim X_\xi$ (cf. [18, Theorem 2.10]). By Chevalley's Theorem [14, Chapter 7, Section 8.3], the image $f(\widetilde{Z})$ of an arbitrarily small representative $\widetilde{Z}$ of Z at ξ is algebraic constructible, and hence

$$\dim_\eta \overline{f(\widetilde{X})} \leqslant \dim_\eta \overline{f(\widetilde{Z})} = \dim_\eta f(\widetilde{Z}) = \dim_\eta f(\widetilde{X}),$$

which completes the proof. □

Proof of Theorem 1.1. Our theorem now follows immediately from Theorem 2.3 and Propositions 5.1 and 5.4. Indeed, if f_ξ is not flat then there exists a geometric vertical component W in $X^{\{n\}}_{\xi\{n\}}$. But W is a Nash germ, by Proposition 5.1, and hence $f^{\{n\}}_{\xi\{n\}}|W$ is Gabrielov regular, by Proposition 5.4. Thus W is algebraic vertical.

Conversely, if f_ξ is flat then so are all its fibre powers $f^{\{i\}}_{\xi\{i\}}$, as flatness is preserved by any base change (see [10, Section 6, Proposition 8]) and composition of flat maps is flat. Hence, $\mathscr{O}_{X^{\{n\}},\xi^{\{n\}}}$ is a torsion-free $\mathscr{O}_{Y,\eta}$-module, and thus by Remark 2.1, $X^{\{n\}}_{\xi\{n\}}$ has no algebraic vertical components. □

Remark 5.5. Note that the above argument cannot be extended beyond the Nash category. In general, a fibre power of a Gabrielov regular morphism of germs of analytic spaces need not be regular itself: Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces with X_ξ of pure dimension and Y_η irreducible of dimension n. Let V be a locally irreducible representative of Y_η and let U be a pure-dimensional representative of X_ξ such that $f(U) \subset V$. Define $S = \{y \in V : \dim f^{-1}(y) > l\}$, where l is the minimal fibre dimension of f on U, and suppose that $\dim_\eta \bar{S} = n$, where $\bar{S}$ denotes the Zariski closure of S in V. Then the top fibre power $X^{\{n\}}_{\xi\{n\}}$ contains an isolated geometric vertical component W which is not algebraic vertical. In particular, $f^{\{n\}}_{\xi\{n\}}$ is not Gabrielov regular (see [2, Proposition 3.1 and Example 3.3]).

Acknowledgements

The author thanks Professors Edward Bierstone, Ragnar-Olaf Buchweitz, and Mark Spivakovsky for valuable discussions and comments regarding problems treated in this paper.

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