PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 11, Pages 3165–3170
S 0002-9939(02)06422-5
Article electronically published on March 25, 2002

NATURAL BOUND IN KWIECIŃSKI'S CRITERION FOR FLATNESS

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Kwieciński has proved a geometric criterion for flatness: A morphism $f : X \to Y$ of germs of analytic spaces is not flat if and only if its i-fold fibre power $f^{\{i\}} : X^{\{i\}} \to Y$ has a vertical component, for *some* i. We show how to bound i using Hironaka's local flattener: If f is not flat, then $f^{\{d\}}$ has a vertical component, where d is the minimal number of generators of the ideal in $\mathcal{O}_Y$ of the flattener of X.

1. Introduction

Let $f : Z \to Y$ be a morphism of germs of analytic spaces, and let W be an irreducible component of Z. Let $\mathcal{P}$ be the associated prime of the zero ideal in the local ring $\mathcal{O}_Z$, corresponding to W. We say that the component W is **vertical** if there exists a nonzero $a \in \mathcal{O}_Y$ with $f^*a \in \mathcal{P}$ (see also the remark following Lemma 3.1).

In [11, Thm. 1.1], Kwieciński shows that assuming Y is irreducible, i.e. the local ring $\mathcal{O}_Y$ is a domain, the following conditions are equivalent:

(i) *$f : X \to Y$ is flat;*
(ii) *for any $i \geq 1$, the canonical map $\underbrace{X \times_Y \ldots \times_Y X}_{i\ times} \to Y$ has no (isolated or embedded) vertical components.*

From now on we will write $X^{\{i\}}$ for the space $X \times_Y \ldots \times_Y X$ (i times) and $f^{\{i\}}$ for the canonical map $X^{\{i\}} \to Y$.

Note that if f is flat, then $f^{\{i\}}$ is flat for all i, since flatness is preserved by any base change ([9, §6, Prop. 8]) and the composition of flat maps is flat. Therefore the implication (i) ⇒ (ii) is an immediate consequence of the definition of flatness in terms of relations (see e.g. [2, Prop. 7.3]). This in fact is the only place where the irreducibility assumption is needed (cf. the example in this section below).

Implication (ii) ⇒ (i) does not require irreducibility of Y, by Kwieciński's Lemma 3.1 below. Thus, for any nonflat map $f : X \to Y$ of germs of analytic spaces there is a positive integer i such that $X^{\{i\}}$ has a *vertical* component. The proof given by Kwieciński is based on Hironaka's criterion for flatness (Thm. 2.2 below, see also [2, Thm. 7.9]). Hironaka uses this criterion to prove the existence of the *local flattener* (see [2, Thm. 7.12]), which we use to give a precise power

Received by the editors March 19, 2001 and, in revised form, June 11, 2001.
2000 *Mathematics Subject Classification.* Primary 14B25, 13C11.
Key words and phrases. Fibre product, vertical component, local flattener.

needed in condition (ii) of Kwieciński's theorem. The *local flattener* for a morphism $f : X \to Y$ of germs of analytic spaces is, by definition, the maximal subgerm P of Y such that $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_P$ is $\mathcal{O}_P$-flat, i.e. $f|f^{-1}(P) : f^{-1}(P) \to P$ is a flat morphism (cf. [2, Thm. 7.12]). Our main result is the following:

Theorem 1.1. *Let $f : X \to Y$ be a nonflat morphism of germs of analytic spaces. Let Q be the ideal in $\mathcal{O}_Y$ of the flattener of X, and let d be the minimal number of generators of Q. Then the canonical map $f^{\{d\}} : X^{\{d\}} \to Y$ has a vertical component.*

Of course, this leaves open the question of calculation. First results have been given by Vasconcelos [15], and Galligo and Kwieciński [6] under certain restrictions on X and Y: Galligo and Kwieciński assert that, assuming X, Y are reduced, X is of pure dimension, and Y is smooth of dimension n, $f : X \to Y$ is flat if and only if the canonical map $f^{\{n\}} : X^{\{n\}} \to Y$ has no *geometric vertical* components ([6, Thm. 6.1]). An analogous result in the algebraic category, but without the pure dimension assumption on X, was obtained by Vasconcelos in the case $\dim Y = 2$ ([15, Prop. 6.1]).

By a *geometric vertical* component of a morphism $f : Z \to Y$ we mean a component W of Z such that, for arbitrarily small representatives $\widetilde{W}$, $\widetilde{Y}$, $\tilde{f}$ of W, Y, f, respectively, the image $\tilde{f}(\widetilde{W})$ has empty interior in $\widetilde{Y}$ with transcendental topology (cf. [6] and [11]). Note that although the notions of *vertical* and *geometric vertical* coincide in the algebraic case over irreducible germ Y (as the image of an algebraic set under a polynomial morphism is always constructible), they are not the same in the analytic setup.

Clearly, over irreducible Y, every *vertical* component is *geometric vertical*, but the converse is false in general. Consider for instance the Osgood mapping $f : \mathbb{C}^2_0 \to \mathbb{C}^3_0$ defined as $(x, y) \mapsto (x, xy, xye^y)$ (see e.g. [7]). Then for an arbitrary neighbourhood U of the origin in $\mathbb{C}^2$, $f(U)$ has empty interior in $\mathbb{C}^3$, but there is no proper analytic subgerm of $\mathbb{C}^3_0$ containing $(f(U))_0$, and hence f has no *vertical* components in our sense.

Observe that in general, i.e. without the irreducibility assumption on Y, the equivalence from Kwieciński's theorem is no longer valid. Consider for instance the identity mapping on the space $X = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$, which is obviously flat while each of the irreducible components of X is vertical.

2. Diagram of initial exponents and Hironaka's criterion for flatness

We briefly recall here basic facts regarding the diagram of initial exponents. For details we refer to [2].

Let A be a local analytic $\mathbb{C}$-algebra, say $A = \mathbb{C}\{y_1, \dots, y_m\}/J$, with the maximal ideal $\mathfrak{m}$. Let L be a total ordering of monomials in $t = (t_1, \dots, t_n)$ with coefficients in A which is compatible with addition of exponents. We write t^β for $t_1^{\beta_1} \dots t_n^{\beta_n}$, where $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$. Let $A\{t\} = \mathbb{C}\{y, t\}/J \cdot \mathbb{C}\{y, t\}$ be the ring of convergent power series in t with coefficients in A. For a series $F = \sum_{\beta \in \mathbb{N}^n} a_\beta t^\beta \in A\{t\}$ define its *evaluation at* 0 as $F(0) = \sum_{\beta \in \mathbb{N}^n} a_\beta(0) t^\beta \in A/\mathfrak{m}\{t\} = \mathbb{C}\{t\}$, and for an ideal I in $A\{t\}$ define $I(0) = \{F(0) : F \in I\}$, the *evaluated* ideal. The *support* of F is defined as $\operatorname{supp} F = \{\beta \in \mathbb{N}^n : a_\beta \neq 0\}$, and $\nu_L(F) = \min_L\{\beta \in \operatorname{supp} F\}$ denotes the *initial exponent* of F (with respect to L). Similarly, $\operatorname{supp} F(0) = \{\beta \in \mathbb{N}^n : a_\beta(0) \neq 0\}$ and $\nu_L(F(0)) = \min_L\{\beta \in \operatorname{supp} F(0)\}$, for the evaluated series.

Let I be an ideal in $A\{t\}$. The *diagram of initial exponents* of I (with resp. to L) is defined as $N_L(I) = \{\nu_L(F) : F \in I\} \subset \mathbb{N}^n$.

Let $f : X \to Y$ be a morphism of germs of analytic spaces. Without loss of generality can assume that X is a subgerm of $\mathbb{C}^n_0$, for some $n \geq 1$. We can then embed X into $Y \times \mathbb{C}^n_0$ via the graph of f. Therefore the local ring $\mathcal{O}_X$ of X can be thought of as a quotient of the local ring of $Y \times \mathbb{C}^n_0$, i.e. $\mathcal{O}_X = \mathcal{O}_Y\{t\}/I$, for some ideal I in $\mathcal{O}_Y\{t\}$, where $t = (t_1, \dots, t_n)$. Let $\Delta = \mathbb{N}^n \setminus N_L(I(0))$ be the complement of the diagram of initial exponents of the evaluated ideal $I(0)$, and define $\mathcal{O}_Y\{t\}^\Delta = \{F \in \mathcal{O}_Y\{t\} : \operatorname{supp} F \subset \Delta\}$. Now consider the canonical projection $\mathcal{O}_Y\{t\} \to \mathcal{O}_X$ and its restriction to $\mathcal{O}_Y\{t\}^\Delta$, called κ. The two results below, due to Hironaka, are crucial for our considerations.

Proposition 2.1 ([9, §6, Prop.9])**.** *The natural map $\kappa : \mathcal{O}_Y\{t\}^\Delta \to \mathcal{O}_X = \mathcal{O}_Y\{t\}/I$ is surjective.*

Theorem 2.2 ([9, §6, Prop.10])**.** *With the notations above, the map $f : X \to Y$ is flat if and only if κ is bijective.*

Observe that $\ker\kappa = \{F \in \mathcal{O}_Y\{t\}^\Delta : F \in I\}$, i.e. $\ker\kappa$ consists of these elements of the ideal I whose supports lie entirely in Δ.

Remark 2.3. Let Q be the ideal in $\mathcal{O}_Y$ of the flattener of X. Then by the proof of [2, Thm. 7.12], Q is generated by all the coefficients a_β of all the series $F = \sum_{\beta\in\Delta} a_\beta t^\beta$ from $\ker\kappa$.

3. Kwieciński's lemma

Our proof of Theorem 1.1 is based on the following result due to Kwieciński ([11, Lemma 3.2]).

Lemma 3.1. *Let $f : X \to Y$ be a nonflat morphism of germs of analytic spaces. Then there is a positive integer i such that there exists a nonzero $b \in \mathcal{O}_{X^{\{i\}}}$ and a nonzero $a \in \mathcal{O}_Y$, with $ab = 0$.*

Observe that according to our definition, the condition above is equivalent to $X^{\{i\}}$ having a vertical component. Indeed, for $a \in \mathcal{O}_Y$ is a zerodivisor in $\mathcal{O}_{X^{\{i\}}}$ iff it belongs to some of the associated primes of the zero ideal in the local ring $\mathcal{O}_{X^{\{i\}}}$.

Note also that without any assumptions on Y, flatness implies the following condition: For any $i \geq 1$, if $a \in \mathcal{O}_Y$ is a zerodivisor in $\mathcal{O}_{X^{\{i\}}}$, then it is a zerodivisor in $\mathcal{O}_Y$. (By the definition of flatness in terms of relations.)

We will now sketch the main steps of the proof of Lemma 3.1: Since f is not flat, Theorem 2.2 together with Proposition 2.1 imply that $\ker\kappa \neq \{0\}$. Pick any nontrivial $F = \sum_{\beta\in\Delta} a_\beta t^\beta$ from $\ker\kappa$. Let $a_{\beta_1}, \dots, a_{\beta_i}$ be distinct nonzero coefficients of F which generate the ideal in $\mathcal{O}_Y$ of all the coefficients of the series F. One then shows that $F = a_{\beta_1}F_1 + \dots + a_{\beta_i}F_i$, where $F_j(t) = t^{\beta_j} + \sum_{\beta\in\Delta\setminus\{\beta_1,\dots,\beta_i\}} f_\beta t^\beta \in \mathcal{O}_Y\{t\}$, $j = 1, \dots, i$. Define $a_j = a_{\beta_j}$ and $h_j = \kappa(F_j)$ for $j = 1, \dots, i$. It follows that $a_1h_1 + \dots + a_ih_i = 0$, but $h_1(0), \dots, h_i(0)$ are linearly independent (over $\mathbb{C}$) in $\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X = \mathbb{C}\{t\}/I(0)$, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_Y$.

Next consider the following commutative diagram of canonical maps of $\mathcal{O}_Y$ -modules, where both tensor products are taken i times and we assume that Y is a germ at 0:

$$\begin{array}{ccccc}
\bigwedge_{\mathcal{O}_Y}^{i} \mathcal{O}_X & \xrightarrow{\rho} & \mathcal{O}_X \otimes_{\mathcal{O}_Y} \ldots \otimes_{\mathcal{O}_Y} \mathcal{O}_X & \xrightarrow{\lambda} & \mathcal{O}_{X\{i\}} \\
\downarrow & & \downarrow & & \downarrow \mu \\
\bigwedge_{\mathbb{C}}^{i} \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X & \xrightarrow{\bar{\rho}} & \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X & \xrightarrow{\bar{\lambda}} & \mathcal{O}_{(f^{-1}(0))^i}
\end{array}$$

Put $a = a_1$ and $b = \lambda \circ \rho(h_1 \wedge \cdots \wedge h_i)$. Since $a_1 h_1$ is an $\mathcal{O}_Y$-linear combination of $h_2, \ldots, h_i$, then $ab = \lambda \circ \rho((a_1 h_1) \wedge h_2 \cdots \wedge h_i) = 0$. Finally $b \neq 0$, because $\mu(b) = \bar{\lambda} \circ \bar{\rho}(h_1(0) \wedge \cdots \wedge h_i(0))$ is nonzero, as $h_1(0), \ldots, h_i(0)$ are linearly independent and $\bar{\rho}, \bar{\lambda}$ are injective.

4. Proof of Theorem 1.1

Let d be the minimal number of generators of the flattener ideal Q, i.e. $d = \dim_{\mathbb{C}} Q/\mathfrak{m}Q$, where $\mathfrak{m}$ is the maximal ideal in the local ring $\mathcal{O}_Y$. We begin with the following lemma.

Lemma 4.1. *Assume $N_L(I(0)) = \mathbb{N} \times D$ for some $D \subset \mathbb{N}^{n-1}$. Then there is a series $F \in \ker \kappa$ such that the coefficients of F generate the ideal Q.*

Remark 4.2. The condition $N_L(I(0)) = \mathbb{N} \times D$ means that the diagram $N_L(I(0))$ is *trivial in the β_1 direction*. As can readily be seen from the proof below, we do not really need that much but only triviality in *some* of the $\beta_1, \ldots, \beta_n$ directions. The point is that triviality in the β_j direction implies that for a series $F \in \mathcal{O}_Y\{t\}$ with $\operatorname{supp} F \subset \Delta$ and for any power k, the series $t_j^k F$ has support contained in Δ again, since $\operatorname{supp}(t_j^k F) = \operatorname{supp} F + (0, \ldots, k, \ldots, 0)$, with k in the j'th place.

Proof of Lemma 4.1. Suppose to the contrary that for any series $F = \sum a_\beta t^\beta$ from $\ker \kappa$, the coefficients a_β of F do not generate Q. Note that by Remark 2.3, all the coefficients of F belong to Q.

For a series $F \in \ker \kappa$ define $d(F)$ as the maximal number of its coefficients linearly independent (over $\mathbb{C} = \mathcal{O}_Y/\mathfrak{m}\,\mathcal{O}_Y$) modulo $\mathfrak{m}Q$. It follows that for any F, $d(F) < d$, because otherwise the coefficients of some series would generate Q, by Nakayama's Lemma. Let $s = \max\{d(F) \mid F \in \ker \kappa\}$. Pick any $F_1 = \sum_{\beta \in \Delta} a_\beta t^\beta$ from $\ker \kappa$ with $d(F_1) = s$, and let $a_{\beta_1}, \ldots, a_{\beta_s}$ be its coefficients linearly independent modulo $\mathfrak{m}Q$. Then by the definition of $d(F_1)$, for any $\beta \in \operatorname{supp} F_1 \setminus \{\beta_1, \ldots, \beta_s\}$ there exist $r_\beta^1, \ldots, r_\beta^s \in \mathbb{C}$ and $q_\beta \in \mathfrak{m}Q$ such that $a_\beta = r_\beta^1 a_{\beta_1} + \cdots + r_\beta^s a_{\beta_s} + q_\beta$, i.e. all the other coefficients of F are $\mathbb{C}$-linear combinations of $a_{\beta_1}, \ldots, a_{\beta_s}$ modulo $\mathfrak{m}Q$.

Since $s < d$, Remark 2.3, together with Nakayama's Lemma, implies that there exists a series $F_2 = \sum_{\gamma \in \Delta} b_\gamma t^\gamma \in \ker \kappa$ such that for some $\gamma_0 \in \operatorname{supp} F_2$, the coefficient b_{γ_0} of F_2 is linearly independent from $a_{\beta_1}, \ldots, a_{\beta_s}$ modulo $\mathfrak{m}Q$. Fix such a series F_2 and take a positive integer k satisfying inequality

$$(*) \qquad (k, 0, \ldots, 0) > \max\{\beta_1, \ldots, \beta_s\}$$

with respect to the total ordering (induced by) L in $\mathbb{N}^n$.

Define a new series $F_0 = F_1 + t_1^k F_2$ and observe that $F_0 \in \ker\kappa$. Indeed, $F_2 \in \ker\kappa$ if and only if $F_2 \in I$ and $\operatorname{supp} F_2 \subset \Delta$. Therefore Remark 4.2 yields $\operatorname{supp}(t_1^k F_2) \subset \Delta$ (and obviously $t_1^k F_2 \in I$), whence $t_1^k F_2 \in \ker\kappa$ and $F_0 \in \ker\kappa$.

Put $F_0 = \sum_{\beta\in\Delta} c_\beta t^\beta$. By the inequality (*), $\operatorname{supp}(t_1^k F_2) > \max\{\beta_1, \dots, \beta_s\}$, so in particular $c_{\beta_1} = a_{\beta_1}, \dots, c_{\beta_s} = a_{\beta_s}$. Moreover, if $\beta_0 = \gamma_0 + (k, 0, \dots, 0)$, then $\beta_0 \neq \beta_j$, $j = 1, \dots, s$, and

$$c_{\beta_0} = r^1_{\beta_0} a_{\beta_1} + \cdots + r^s_{\beta_0} a_{\beta_s} + q_{\beta_0} + b_{\gamma_0},$$

where $r^j_{\beta_0} \in \mathbb{C}$, $j = 1, \dots, s$, $q_{\beta_0} \in \mathfrak{m}Q$. But b_{γ_0} is linearly independent from $a_{\beta_1}, \dots, a_{\beta_s}$, which implies that c_{β_0} is linearly independent from $c_{\beta_1}, \dots, c_{\beta_s}$. Thus $c_{\beta_0}, c_{\beta_1}, \dots, c_{\beta_s}$ are $s+1$ coefficients of F_0 linearly independent modulo $\mathfrak{m}Q$, whence $d(F_0) \geq s+1$, a contradiction. □

Proof of Theorem 1.1. Suppose first that $N_L(I(0)) = \mathbb{N} \times D$, as in Lemma 4.1. Then we can find $F \in \ker\kappa$, $F = \sum_{\beta\in\Delta} a_\beta t^\beta$ such that among its coefficients are $a_{\beta_1}, \dots, a_{\beta_d}$ linearly independent modulo $\mathfrak{m}Q$. By Nakayama's Lemma, $a_{\beta_1}, \dots, a_{\beta_d}$ generate Q (recall that $d = \dim_{\mathbb{C}} Q/\mathfrak{m}Q$), so in particular they generate all the coefficients of F. Therefore by applying Kwieciński's Lemma 3.1 to this F one obtains a nonzero $a \in \mathcal{O}_Y$ and a nonzero $b \in \mathcal{O}_{X^{\{d\}}}$ with $ab = 0$, i.e. a vertical component in the d'th fibre power $X^{\{d\}}$ (cf. the remark following Lemma 3.1).

Next suppose the diagram $N_L(I(0))$ is not trivial in any direction. Define $\widetilde{X} = \mathbb{C}_0 \times X$, and $\tilde{f} : \widetilde{X} \to Y$ as $\tilde{f} = f \circ \pi$, where $\pi : \widetilde{X} \to X$ is a canonical projection. Let $\tilde{I} = I \cdot \mathcal{O}_Y\{t_0, t_1, \dots, t_n\}$ be the extended ideal, so that $\mathcal{O}_{\widetilde{X}} = \mathcal{O}_Y\{t_0, t_1, \dots, t_n\}/\tilde{I}$. Since every element of $\tilde{I}$ can be expressed in the form $\sum_{i\in\mathbb{N}} F_i(t_1, \dots, t_n) \cdot t_0^i$, with $F_i \in I$, it follows that $N_L(\tilde{I}(0)) = \mathbb{N} \times N_L(I(0))$. Moreover, the flattener ideal for $\tilde{f} : \widetilde{X} \to Y$ is generated by the same elements as the flattener ideal Q, so in particular its minimal sets of generators consist of d elements.

Now Lemma 4.1 applies to the diagram $N_L(\tilde{I}(0))$, as it is trivial in the β_0 direction, and following the first part of this proof we obtain that $\widetilde{X}^{\{d\}}$ has a vertical component. But clearly every irreducible component $\widetilde{Z}$ of $\widetilde{X}^{\{d\}}$ is of the form $Z \times (\mathbb{C}^d)_0$ for some irreducible component Z of $X^{\{d\}}$, which implies that $X^{\{d\}}$ itself has a vertical component. This completes the proof of our theorem. □

Acknowledgements

I thank my Ph.D. supervisor, Professor Edward Bierstone, for endless discussions and many valuable comments that led me to the solution of the problem.

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