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Generic fibre product of one-dimensional manifolds

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Abstract

The generic fibre product $M\times_{\mathbb{R}}N$ of smooth manifolds M and N over $\mathbb{R}$ is itself a smooth manifold. It can therefore be characterized by the number of its connected components. We give such a characterization in the case of compact one-dimensional manifolds in terms of relations among the critical values of maps $f: M \to \mathbb{R}$ and $g: N \mapsto \mathbb{R}$. A simple efficient algorithm is provided.

MSC: 54C30; 57R45

Keywords: Fibre product; One-manifolds; Critical values

1. Introduction and main result

Let $f:X \to Z$, $g:Y \to Z$ be arbitrary maps. By the *fibre product* of X and Y *over* Z we mean the set $X\times_Z Y = \{(x, y) \in X \times Y\colon\ f(x) = g(y)\}$. The *induced map* $f\times_Z g: M\times_Z N \to Z$ is given by $(x, y) \mapsto f(x)$.

The fibre product plays an important role in algebraic and analytic geometry, where it is used for schemes and analytic spaces as well as in the theory of categories. It is not very popular in topology though, mainly due to the fact that the fibre product of smooth manifolds over arbitrary maps need not be smooth itself. Nonetheless, we have an important property that the *generic* fibre product of smooth manifolds remains in the class, as we recall in Section 2. I.e., for a pair $(f, g) \in \mathcal{C}^r(M, \mathbb{R}) \times \mathcal{C}^r(N, \mathbb{R})$ from some open and dense subset of the space, the fibre product $M\times_{\mathbb{R}}N$ of $\mathcal{C}^\infty$ manifolds is a $\mathcal{C}^r$ manifold, $2 \leqslant r \leqslant \infty$. It can therefore be characterized by the number of its connected components.

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The main results of the paper—Theorems 3.5 and 3.6 below—give such a characterization in the case of compact one-dimensional manifolds. (Compactness is a natural condition, since for non-compact manifolds the number of components of their fibre product is usually infinite.) We show that the number of components of $M\times_{\mathbb{R}}N$ depends only on relations among the critical values of maps $f:M\to\mathbb{R}$ and $g:N\to\mathbb{R}$, and that every component is uniquely determined by some collection of the critical points of these maps.

Since both M and N can be identified with S^1, their fibre product is then a finite collection of disjoint circles smoothly embedded into $T^2=S^1\times S^1$. Therefore, in general there are two kinds of components of $M\times_{\mathbb{R}}N$, namely contractible and non-contractible ones, that require distinct characterizations.

First of all, it may happen that values of f are bounded by values of g, i.e., there are critical points $y_b, y_e\in\mathcal{C}_g$ such that $g(y_b)<f(x)<g(y_e)$ for all $x\in M$. Theorem 3.6 asserts that the pairs $(y_b,y_e)\in\mathcal{C}_g\times\mathcal{C}_g$ of this type correspond to the non-contractible components of $M\times_{\mathbb{R}}N$, characterized by the property that the global extrema of the induced map $f\times_{\mathbb{R}}g$ restricted to the component are equal to the global extrema of f on M. It is worth pointing out that all the non-contractible components belong to the same homotopy class (unique for a given pair (f,g)), either the class of the parallel or of the meridian (see Remark 3.7).

The contractible components are generated by the pairs of triples of critical points $((x_b,x_m,x_e),(y_b,y_m,y_e))\in{\mathcal{C}_f}^3\times{\mathcal{C}_g}^3$ such that f has local minima at x_b, x_e and a local maximum at x_m, g has local maxima at y_b, y_e and a local minimum at y_m, and the following inequalities hold

$$f(x_b),\ f(x_e)<g(y_m)<f(x_m)<g(y_b),\ g(y_e).$$

Theorem 3.5 shows the way the contractible components correspond to the pairs $((x_b,x_m,x_e),(y_b,y_m,y_e))$ above. Counting of the components therefore becomes very straightforward, as it simply reduces to finding all the couples of critical points with the required properties satisfied by their critical values.

Throughout the paper we assume strong topologies in the spaces $\mathcal{C}^r(M,\mathbb{R})$ and $\mathcal{C}^r(N,\mathbb{R})$, denoted by $\mathcal{C}^r_S(M,\mathbb{R})$ and $\mathcal{C}^r_S(N,\mathbb{R})$ respectively (see, e.g., [1, Section 2.1]). M and N are always assumed to be $\mathcal{C}^\infty$ manifolds without boundary.

2. Some facts from differential topology

Let M and N be $\mathcal{C}^\infty$ manifolds of dimensions m and n respectively, and let $f:M\to\mathbb{R}$, $g:N\to\mathbb{R}$ be arbitrary $\mathcal{C}^r$ functions, $2\leqslant r\leqslant\infty$. Denote by $\mathcal{C}_f$ (respectively $\mathcal{C}_g$) the set of critical points of f (respectively g). Let $\Delta=\{(a,a)\colon a\in\mathbb{R}\}$ be the diagonal in $\mathbb{R}^2$.

Observe that the map $f\times g:M\times N\to\mathbb{R}^2$ is transverse to Δ if and only if $f(x)\neq g(y)$, for any pair of critical points $(x,y)\in\mathcal{C}_f\times\mathcal{C}_g$. We recall the following two well known results from differential topology (see, e.g., [1, Theorems 1.3.3 and 3.2.1]):

Proposition 2.1. *Let $f:M\to P$ be a $\mathcal{C}^r$ map, $r\geqslant 1$, and $A\subset P$ a $\mathcal{C}^r$ submanifold. If f is transverse to A, then $f^{-1}(A)$ is a $\mathcal{C}^r$ submanifold of M.* (*The codimension of $f^{-1}(A)$ in M is the same as the codimension of A in P.*)

Proposition 2.2. *Let M, P be $\mathcal{C}^\infty$ manifolds, $A \subset P$ a closed $\mathcal{C}^\infty$ submanifold. Let $1 \leqslant r \leqslant \infty$. Then the set $\pitchfork^r(M, P; A)$ of $\mathcal{C}^r$ mappings $f : M \to P$ which are transverse to A, is open and dense in $\mathcal{C}^r_S(M, P)$.*

From these propositions and the remark above:

Corollary 2.3. *The set of pairs $(f, g) \in \mathcal{C}^r(M, \mathbb{R}) \times \mathcal{C}^r(N, \mathbb{R})$ such that $f(x) \neq g(y)$, for any $(x, y) \in \mathcal{C}_f \times \mathcal{C}_g$, is open and dense in $\mathcal{C}^r_S(M, \mathbb{R}) \times \mathcal{C}^r_S(N, \mathbb{R})$.*

For any pair (f, g) of functions satisfying the above condition, the fibre product $M \times_{\mathbb{R}} N$ is a $\mathcal{C}^r$ submanifold of $M \times N$ (of codimension 1).

For the purpose of application in the next section, we slightly shrink our family of pairs (f, g) of functions in question. Namely, we state the following definition:

Definition 2.4. Let $2 \leqslant r \leqslant \infty$. We say that a pair of Morse functions $(f, g) \in \mathcal{C}^r(M, \mathbb{R}) \times \mathcal{C}^r(N, \mathbb{R})$ is *good*, if it satisfies the following conditions:

(i) $f(x_1) \neq f(x_2)$, for arbitrary distinct $x_1, x_2 \in \mathcal{C}_f$,
(ii) $g(y_1) \neq g(y_2)$, for arbitrary distinct $y_1, y_2 \in \mathcal{C}_g$,
(iii) $f(x) \neq g(y)$, for any $x \in \mathcal{C}_f$, $y \in \mathcal{C}_g$.

We denote the set of such pairs by $\mathcal{G}(M, N)$.

Since the Morse functions on a manifold M form a dense open subset in $\mathcal{C}^r_S(M, \mathbb{R})$, $2 \leqslant r \leqslant \infty$ (see, e.g., [1, Theorem 6.1.2]), we see that the set $\mathcal{G}(M, N)$ is again open and dense in $\mathcal{C}^r_S(M, \mathbb{R}) \times \mathcal{C}^r_S(N, \mathbb{R})$.

3. Characterization

Let M and N be connected, compact, one-dimensional $\mathcal{C}^\infty$ manifolds, and let $f : M \to \mathbb{R}$, $g : N \to \mathbb{R}$ be $\mathcal{C}^r$ mappings such that $(f, g) \in \mathcal{G}(M, N)$. Then by the previous section, their fibre product $M \times_{\mathbb{R}} N$ is a compact one-dimensional $\mathcal{C}^r$ manifold, so it can be characterized by the number of its connected components.

We shall show that the number of the components of $M \times_{\mathbb{R}} N$ depends only on relations among the critical values of maps f and g, and to determine this number one does not even need exact knowledge of those values. Recall that for a *good* pair (f, g) we always have $f(x) < g(y)$ or $f(x) > g(y)$, for any $x \in \mathcal{C}_f$ and $y \in \mathcal{C}_g$.

Identifying M and N with S^1, we obtain that $M \times_{\mathbb{R}} N$ is a finite set of disjoint circles smoothly embedded into T^2. The components can then be either contractible or not and it turns out that the two types require distinct characterizations in terms of critical points of the maps f and g.

In Theorems 3.5 and 3.6 we show that every component is uniquely assigned to some collection of critical points of mappings f and g. The rules of constructing these collections, described in Definitions 3.3 and 3.4, provide an easy efficient algorithm for computing the number of the components of $M \times_{\mathbb{R}} N$.

Let $\mathcal{C}_f = \{x_1, \ldots, x_m\}$, $\mathcal{C}_g = \{y_1, \ldots, y_n\}$ be the sets of critical points of the maps f and g, ordered according to some orientations on M and N. Denote by $x^f_{\min}$ (respectively $x^f_{\max}$) the point at which f admits its global minimum (respectively maximum), and by $y^g_{\min}$, $y^g_{\max}$, the analogous points for g. Observe that the induced map $f \times_{\mathbb{R}} g : M \times_{\mathbb{R}} N \to \mathbb{R}$, $(x, y) \mapsto f(x)$, admits a local minimum (respectively maximum) at a point (x_0, y_0) if and only if either f has a local minimum (respectively maximum) at x_0 or g has a local minimum (respectively maximum) at y_0. Moreover, the only critical points of $f \times_{\mathbb{R}} g$ are these at which $f \times_{\mathbb{R}} g$ admits local extrema, since this is the case for f and g. Denote by $\mathcal{C}(f \times_{\mathbb{R}} g)$ the set of critical points of $f \times_{\mathbb{R}} g$.

Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be the family of the connected components of the manifold $M \times_{\mathbb{R}} N$. (Obviously, Λ is a finite set, so in particular the components of $M \times_{\mathbb{R}} N$ are open.) For a component S_λ let $(x^\lambda_{\min,1}, y^\lambda_{\min,1}), \ldots, (x^\lambda_{\min,r}, y^\lambda_{\min,r})$ (respectively $(x^\lambda_{\max,1}, y^\lambda_{\max,1}), \ldots, (x^\lambda_{\max,s}, y^\lambda_{\max,s})$) be the points at which $f \times_{\mathbb{R}} g|S_\lambda$ admits its global minimum (respectively maximum). Notice that if $x^\lambda_{\min,i} \in \mathcal{C}_f$, for some $i \leqslant r$, then the global minimum of $f \times_{\mathbb{R}} g|S_\lambda$ equals $f(x^\lambda_{\min,i})$ and $x^\lambda_{\min,1} = \cdots = x^\lambda_{\min,r} =: x^\lambda_{\min}$, since the critical values of f are pairwise distinct. If $x^\lambda_{\min,i} \notin \mathcal{C}_f$, then $y^\lambda_{\min,i} \in \mathcal{C}_g$, whence the global minimum of $f \times_{\mathbb{R}} g|S_\lambda$ equals $g(y^\lambda_{\min,i})$ and $y^\lambda_{\min,1} = \cdots = y^\lambda_{\min,r} =: y^\lambda_{\min}$ for the same reason. Similarly for the global maximum of $f \times_{\mathbb{R}} g|S_\lambda$.

Given a component S_λ, two cases are possible:

(a) $(x^\lambda_{\min}, x^\lambda_{\max} \in \mathcal{C}_f)$ or $(y^\lambda_{\min}, y^\lambda_{\max} \in \mathcal{C}_g)$ or else
(b) $(x^\lambda_{\min} \in \mathcal{C}_f$ and $y^\lambda_{\max} \in \mathcal{C}_g)$ or $(x^\lambda_{\max} \in \mathcal{C}_f$ and $y^\lambda_{\min} \in \mathcal{C}_g)$.

In the first case we say that the global extrema of $f \times_{\mathbb{R}} g|S_\lambda$ *come from the same map*, and in the second one that the global extrema of $f \times_{\mathbb{R}} g|S_\lambda$ *come from different maps*. Denote by $S(M \times_{\mathbb{R}} N)^n$ the set of the components of the first kind, and by $S(M \times_{\mathbb{R}} N)^c$ the components of the second kind.

For a component S_λ, let $M_\lambda = p_1(S_\lambda)$ and $N_\lambda = p_2(S_\lambda)$, where $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are the canonical projections. Then M_λ (respectively N_λ) is a connected, compact subset of M (respectively N).

For $x', x'' \in M$, $x' \neq x''$, let $\langle x', x''\rangle_+$ be an arc connecting x' and x'' according to the orientation, and $\langle x', x''\rangle_-$ an arc joining x' with x'' opposite to the orientation on M. Similarly we define $\langle y', y''\rangle_+$ and $\langle y', y''\rangle_-$ for $y', y'' \in N$, $y' \neq y''$.

Remark 3.1. The elements of $S(M \times_{\mathbb{R}} N)^c$ are precisely the contractible components of $M \times_{\mathbb{R}} N$, as each of them is contained in some rectangle $\langle x_b, x_e\rangle_+ \times \langle y_b, y_e\rangle_+$ (cf. the proof of Theorem 3.5 below). $S(M \times_{\mathbb{R}} N)^n$ consists of the homotopically nontrivial components of $M \times_{\mathbb{R}} N$. Indeed, for any $S_\lambda \in S(M \times_{\mathbb{R}} N)^n$, either $p_1(S_\lambda) = M$ or $p_2(S_\lambda) = N$ (see the proof of Theorem 3.6).

Lemma 3.2. *Let the points $x_b, x_e \in M$, $y_b, y_e \in N$ be such that $f(x_b) = g(y_b)$, $f(x_e) = g(y_e)$, $f(x_b) < f(x) < f(x_e)$ for $x \in \langle x_b, x_e\rangle_+$, and $g(y_b) < g(y) < g(y_e)$ for $y \in$*

$\langle y_b, y_e\rangle_+$. Then there exists a path $\varphi:[0,1]\to M\times_{\mathbb{R}}N$ connecting (x_b, y_b) and (x_e, y_e) such that $p_1(\varphi([0,1])) = \langle x_b, x_e\rangle_+$ and $p_2(\varphi([0,1])) = \langle y_b, y_e\rangle_+$.

Proof. Consider the rectangle $P = \langle x_b, x_e\rangle_+ \times \langle y_b, y_e\rangle_+$. Let S_λ be the component of $M\times_{\mathbb{R}}N$ passing through (x_b, y_b). By assumption, $f(x) > f(x_b)$ for $x \in \operatorname{int}\langle x_b, x_e\rangle_+$, and $g(y) > g(y_b)$ for $y \in \operatorname{int}\langle y_b, y_e\rangle_+$, which implies that S_λ enters inside of P at the point (x_b, y_b). Being a simple closed curve, S_λ has to leave P at another point $(x_0, y_0) \in \partial P$. By definition of the fibre product, $f(x_0) = g(y_0)$. By our assumptions on f and g, there are only two points on ∂P satisfying the last condition, namely (x_b, y_b) and (x_e, y_e). Therefore $(x_0, y_0) = (x_e, y_e)$ and φ is just a parametrization of the part of S_λ lying between (x_b, y_b) and (x_e, y_e).

We now define some concepts necessary for further considerations. Fix arbitrary $x_b \neq x_m \neq x_e$ elements of $\mathcal{C}_f$, and $y_b \neq y_m \neq y_e$ elements of $\mathcal{C}_g$ such that f has a local maximum at x_m and local minima at x_b, x_e, and g has a local minimum at y_m and local maxima at y_b, y_e, or to the contrary: f has a local minimum at x_m and local maxima at x_b, x_e, and g has a local maximum at y_m and local minima at y_b, y_e. Moreover assume that $x_m \in \langle x_b, x_e\rangle_+$ and $y_m \in \langle y_b, y_e\rangle_+$.

Definition 3.3. The pair of triples of points $((x_b, x_m, x_e), (y_b, y_m, y_e))$ satisfying the above conditions is called *reduced*, if:

(1) $f(x_b), f(x_e) < g(y_m) < f(x_m) < g(y_b), g(y_e)$,
$g(y_m) < f(x) < f(x_m)$, for $x \in \mathcal{C}_f \cap (\operatorname{int}\langle x_b, x_e\rangle_+ \setminus \{x_m\})$
$g(y_m) < g(y) < f(x_m)$, for $y \in \mathcal{C}_g \cap (\operatorname{int}\langle y_b, y_e\rangle_+ \setminus \{y_m\})$
in the case when f has a local maximum at x_m, or else

(2) $f(x_b), f(x_e) > g(y_m) > f(x_m) > g(y_b), g(y_e)$,
$g(y_m) > f(x) > f(x_m)$, for $x \in \mathcal{C}_f \cap (\operatorname{int}\langle x_b, x_e\rangle_+ \setminus \{x_m\})$
$g(y_m) > g(y) > f(x_m)$, for $y \in \mathcal{C}_g \cap (\operatorname{int}\langle y_b, y_e\rangle_+ \setminus \{y_m\})$
in the opposite case.

Now suppose that values of the map f are bounded by values of g, i.e., that there exist $y', y'' \in N$ such that for any $x \in M$ we have $g(y') < f(x) < g(y'')$.

Definition 3.4. We will say that the pair $(y_b, y_e) \in \mathcal{C}_g \times \mathcal{C}_g$ *covers M*, if:

(a) $(g(y_b) < f(x) < g(y_e)$, for any $x \in M)$ or $(g(y_e) < f(x) < g(y_b)$, for any $x \in M)$,
(b) there exist $x', x'' \in M$ such that $f(x') < g(y) < f(x'')$ for any $y \in \mathcal{C}_g \cap \operatorname{int}\langle y_b, y_e\rangle_+$.

Similarly we define a pair (x_b, x_e) which *covers N*, in the case when values of g are bounded by values of f. Of course these two cases exclude each other.

In the example presented on Fig. 1 there are two pairs *covering N*, namely (x_5, x_2) and (x_2, x_5), and one *reduced* pair: $((x_2, x_3, x_4), (y_3, y_4, y_1))$.

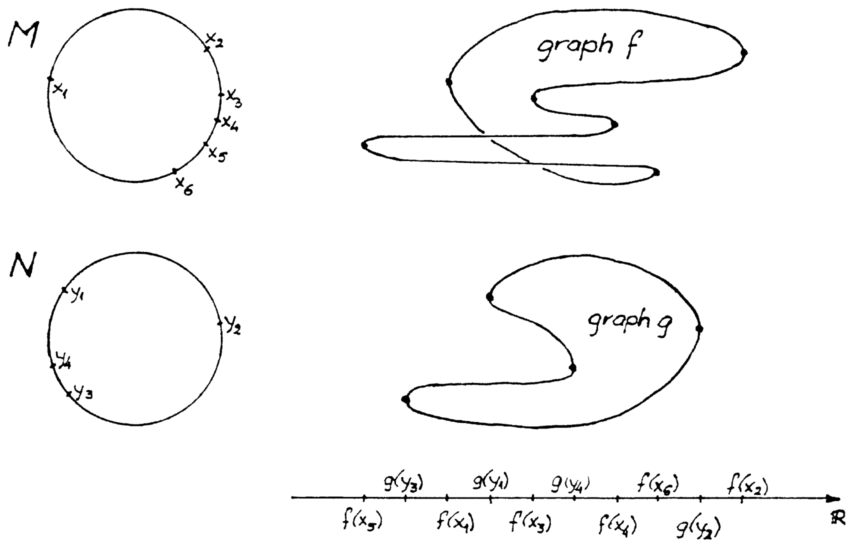

Fig. 1.

Theorem 3.5. *There is a bijection between the set of reduced pairs* $((x_b, x_m, x_e), (y_b, y_m, y_e))$ *and the set* $S(M \times_{\mathbb{R}} N)^c$ *of contractible components of the fibre product* $M \times_{\mathbb{R}} N$.

Proof. Let S_λ be a contractible component of $M \times_{\mathbb{R}} N$. Without loss of generality can assume that $x^\lambda_{\max} \in \mathcal{C}_f$ and $y^\lambda_{\min} \in \mathcal{C}_g$, i.e., that the global minimum of $f \times_{\mathbb{R}} g|S_\lambda$ *comes from* g and its global maximum *comes from* f. Then the map $f \times_{\mathbb{R}} g|S_\lambda$ admits its minimal value at points $(x^\lambda_{\min,1}, y^\lambda_{\min}), \ldots, (x^\lambda_{\min,r}, y^\lambda_{\min})$, and the maximal value at $(x^\lambda_{\max}, y^\lambda_{\max,1}), \ldots, (x^\lambda_{\max}, y^\lambda_{\max,s})$.

Consider the projection $M_\lambda = p_1(S_\lambda)$. Observe that $M_\lambda = \langle x^\lambda_1, x^\lambda_2\rangle_+$, for some $x^\lambda_1, x^\lambda_2 \in M$, because otherwise $M_\lambda = M$, hence $x^f_{\min}, x^f_{\max} \in M_\lambda$, which implies that both global extrema of $f \times_{\mathbb{R}} g|S_\lambda$ *come from* f, contrary to our assumption. For the same reason $N_\lambda = \langle y^\lambda_1, y^\lambda_2\rangle_+$ is a proper subset of N.

Notice next that if $(x_0, y_0) \in \mathcal{C}(f \times_{\mathbb{R}} g) \cap S_\lambda$ and $x_0 \in \mathcal{C}_f$, then $y_0 \notin \mathcal{C}_g$, i.e., g is a homeomorphism near y_0. It follows that near x_0, the mapping $x \mapsto (x, g^{-1}(f(x)))$ is also a homeomorphism, hence in particular $x_0 \in \operatorname{int} M_\lambda$. Similarly, if $(x_0, y_0) \in \mathcal{C}(f \times_{\mathbb{R}} g) \cap S_\lambda$ and $y_0 \in \mathcal{C}_g$, then $y_0 \in \operatorname{int} N_\lambda$.

Consider a point $(x^\lambda_{\min,i}, y^\lambda_{\min})$, $1 \leqslant i \leqslant r$. By the above observation, $y^\lambda_{\min} \in \operatorname{int} N_\lambda$, and $g(y) > g(y^\lambda_{\min})$ near $y^\lambda_{\min}$. Hence $(f \times_{\mathbb{R}} g)(x, y) > (f \times_{\mathbb{R}} g)(x^\lambda_{\min,i}, y^\lambda_{\min})$ in a small neighbourhood U of $(x^\lambda_{\min,i}, y^\lambda_{\min})$ in S_λ, and therefore $f(x) > f(x^\lambda_{\min,i})$ for $x \in p_1(U)$. But f is a homeomorphism near $x^\lambda_{\min,i}$, as $x^\lambda_{\min,i} \notin \mathcal{C}_f$, so $x^\lambda_{\min,i} \in \partial M_\lambda$. Therefore, $r = 2$ and $x^\lambda_{\min,1} = x^\lambda_1$, $x^\lambda_{\min,2} = x^\lambda_2$. Similarly, $s = 2$ and $y^\lambda_{\max,1} = y^\lambda_1$, $y^\lambda_{\max,2} = y^\lambda_2$.

Let now $x^\lambda_b, x^\lambda_e \in \mathcal{C}_f \setminus M_\lambda$ be the critical points closest to x^λ_1 and x^λ_2 respectively, and let $y^\lambda_b, y^\lambda_e \in \mathcal{C}_g \setminus N_\lambda$ be closest to y^λ_1 and y^λ_2 respectively. As follows immediately from the construction above, f admits local minima at x^λ_b, x^λ_e, g admits local maxima at y^λ_b, y^λ_e, and

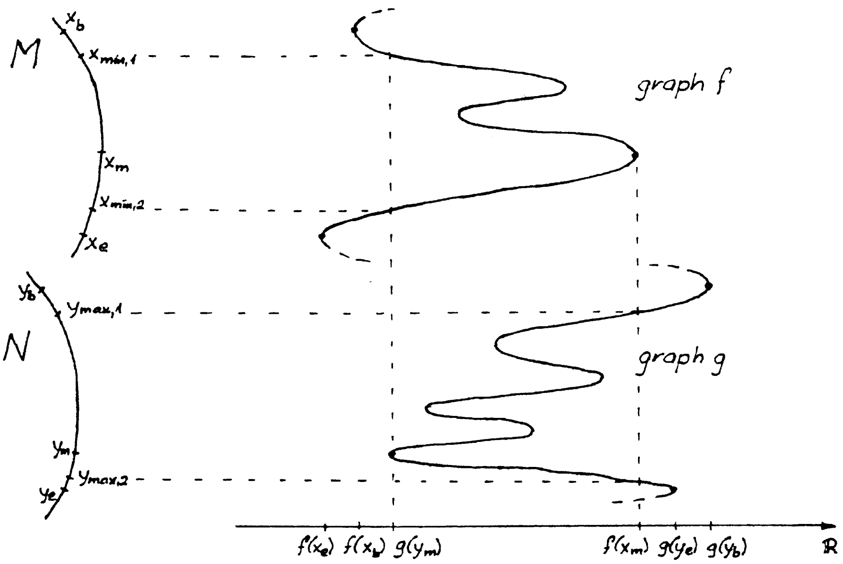

Fig. 2.

the pair $((x_b^\lambda, x_{\max}^\lambda, x_e^\lambda), (y_b^\lambda, y_{\min}^\lambda, y_e^\lambda))$ is *reduced*. Clearly, the pair is uniquely determined by S_λ.

Next we show that every *reduced* pair $((x_b, x_m, x_e), (y_b, y_m, y_e))$ generates some contractible component S_λ. Fix such a pair and assume without loss of generality that f has a local maximum at x_m, g has a local minimum at y_m, $x_m \in \langle x_b, x_e\rangle_+$, and $y_m \in \langle y_b, y_e\rangle_+$. Let $x_{\min,1} \in \langle x_b, x_m\rangle_+$ and $x_{\min,2} \in \langle x_m, x_e\rangle_+$ be the unique points satisfying $f(x_{\min,1}) = f(x_{\min,2}) = g(y_m)$. Similarly, let $y_{\max,1} \in \langle y_b, y_m\rangle_+$ and $y_{\max,2} \in \langle y_m, y_e\rangle_+$ be such that $f(x_m) = g(y_{\max,1}) = g(y_{\max,2})$ (see Fig. 2).

Now Lemma 3.2 implies that there exist paths in $M \times_{\mathbb{R}} N$ connecting $(x_{\min,1}, y_m)$ with $(x_m, y_{\max,1})$, $(x_m, y_{\max,1})$ with $(x_{\min,2}, y_m)$, $(x_{\min,2}, y_m)$ with $(x_m, y_{\max,2})$, and $(x_m, y_{\max,2})$ with $(x_{\min,1}, y_m)$. In other words, all the four points lie on the same component S_λ. As $S_\lambda \subset \langle x_{\min,1}, x_{\min,2}\rangle_+ \times \langle y_{\max,1}, y_{\max,2}\rangle_+$, the minimal and maximal values of $f \times_{\mathbb{R}} g|S_\lambda$ come from y_m and x_m respectively, and hence S_λ is a contractible component generated by our pair. $\square$

Suppose now that values of f are bounded by values of g, i.e., that there exist $y', y'' \in N$ such that $g(y') < f(x) < g(y'')$ for all $x \in M$, or to the contrary: values of f bound values of g. Then there exist non-contractible components of the fibre product $M \times_{\mathbb{R}} N$ and we have the following

Theorem 3.6. *Every non-contractible component S_λ, for which global extrema of $f \times_{\mathbb{R}} g|S_\lambda$ come from f (respectively g) is generated by the unique pair $(y_b^\lambda, y_e^\lambda)$ which covers M (respectively pair $(x_b^\lambda, x_e^\lambda)$ covering N).*

Proof. Let $S_\lambda \in S(M\times_{\mathbb{R}}N)^n$ be a non-contractible component and assume that the global extrema of $f\times_{\mathbb{R}} g|S_\lambda$ *come from* f. We shall show that $M_\lambda = p_1(S_\lambda) = M$. Let $x^\lambda_{\min}$ be any point at which $f|M_\lambda$ admits its global minimum, $x^\lambda_{\max}$ any point at which $f|M_\lambda$ admits its global maximum. Note that N_λ is a proper subset of N (as it does not contain the points $y^g_{\min}, y^g_{\max}$), and hence $\partial N_\lambda = \{y^\lambda_1, y^\lambda_2\}$, for some $y^\lambda_1, y^\lambda_2 \in N$. Let $y^\lambda_{\min} \in N_\lambda$ be any point such that $f\times_{\mathbb{R}} g|S_\lambda$ has global minimum at $(x^\lambda_{\min}, y^\lambda_{\min})$. Since $x^\lambda_{\min} \in \mathcal{C}_f$, then, as in the proof of Theorem 3.5, we have $y^\lambda_{\min} \in \partial N_\lambda$. Similarly, if $y^\lambda_{\max} \in N_\lambda$ is such that $f\times_{\mathbb{R}} g|S_\lambda$ has global maximum at $(x^\lambda_{\max}, y^\lambda_{\max})$, then $y^\lambda_{\max} \in \partial N_\lambda$. Therefore (up to the order) $y^\lambda_{\min} = y^\lambda_1$ and $y^\lambda_{\max} = y^\lambda_2$ are unique.

Now suppose that $M_\lambda \neq M$, i.e., $M_\lambda = \langle x^\lambda_1, x^\lambda_2\rangle_+$, for some $x^\lambda_1, x^\lambda_2 \in M$, and choose $y', y'' \in N$ so that $(x^\lambda_1, y'), (x^\lambda_2, y'') \in S_\lambda$. Then S_λ is contained in the rectangle $P = \langle x^\lambda_1, x^\lambda_2\rangle_+ \times \langle y^\lambda_{\min}, y^\lambda_{\max}\rangle_+$ and passes through the (pairwise distinct) points (x^λ_1, y'), $(x^\lambda_{\min}, y^\lambda_{\min})$, (x^λ_2, y''), $(x^\lambda_{\max}, y^\lambda_{\max})$ lying on the four edges of P. As S_λ is a simple closed curve, there exists a point $y \in \langle y^\lambda_{\min}, y^\lambda_{\max}\rangle_+$, distinct from $y^\lambda_{\min}$ and such that $(x^\lambda_{\min}, y) \in S_\lambda$, which contradicts the uniqueness of $y^\lambda_{\min}$. Thus $M_\lambda = M$, hence in particular $x^\lambda_{\min} = x^f_{\min}$ and $x^\lambda_{\max} = x^f_{\max}$.

Finally, let $y^\lambda_b \in \mathcal{C}_g \setminus N_\lambda$ be the critical point next to $y^\lambda_{\min}$ and let $y^\lambda_e \in \mathcal{C}_g \setminus N_\lambda$ be next to $y^\lambda_{\max}$. The pair $(y^\lambda_b, y^\lambda_e)$ is uniquely determined by S_λ and satisfies the conditions of Definition 3.4.

Consider now any pair (y_b, y_e) *covering* M. By Definition 3.4 we can assume that g has a local minimum at y_b and a local maximum at y_e, and that there exists exactly one point $y' \in \langle y_b, y_e\rangle_+$ for which $g(y') = f(x^f_{\min})$ and exactly one point $y'' \in \langle y_b, y_e\rangle_+$ for which $g(y'') = f(x^f_{\max})$. By a similar argument as in the second part of the proof of Theorem 3.5, one obtains a loop in $M\times_{\mathbb{R}}N$ containing $(x^f_{\min}, y')$ and $(x^f_{\max}, y'')$, being in fact some component $S_\lambda \in S(M\times_{\mathbb{R}}N)^n$ generated by the pair (y_b, y_e). □

Remark 3.7. All the non-contractible components of $M\times_{\mathbb{R}}N \subset T^2$ belong to the same homotopy class (unique for a given pair of maps (f, g)). It is either the class of the parallel or of the meridian.

Indeed, if S_λ is a component such that global extrema of $f\times_{\mathbb{R}} g|S_\lambda$ come from f, then $M_\lambda = M$ and $N_\lambda = \langle y^\lambda_1, y^\lambda_2\rangle_+$, for some (distinct) $y^\lambda_1, y^\lambda_2 \in N$ (see the proof of Theorem 3.6). Hence the homotopy class of S_λ is the same as that of $S^1 \times \{1\} \subset T^2$. Similarly, if global extrema of $f\times_{\mathbb{R}} g|S_\lambda$ come from g, $M_\lambda = \langle x^\lambda_1, x^\lambda_2\rangle_+$ (with $x^\lambda_1 \neq x^\lambda_2$) and $N_\lambda = N$, i.e., S_λ is homotopy equivalent to $\{1\} \times S^1 \subset T^2$.

Remark 3.8. In order to completely classify the fibre products $S^1\times_{\mathbb{R}}S^1$ as submanifolds of T^2, one also needs to know whether some of the contractible components lie inside the others. These inclusions can be trivially checked given all the *reduced* pairs of triples of critical points. For, if a component S_λ is generated by $((x^\lambda_b, x^\lambda_m, x^\lambda_e), (y^\lambda_b, y^\lambda_m, y^\lambda_e))$ and S_τ is generated by $((x^\tau_b, x^\tau_m, x^\tau_e), (y^\tau_b, y^\tau_m, y^\tau_e))$, then S_λ lies inside S_τ if and only if $x^\tau_b < x^\lambda_b$, $x^\lambda_e < x^\tau_e$, $y^\tau_b < y^\lambda_b$, and $y^\lambda_e < y^\tau_e$ with respect to the cyclic orderings of critical points on M and N (or equivalently, $M_\lambda \subset M_\tau$ and $N_\lambda \subset N_\tau$).

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References

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