ON THE CYCLE STRUCTURE OF HAMILTONIAN k-REGULAR BIPARTITE GRAPHS OF ORDER $4k$

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Abstract. It is shown that a hamiltonian $n/2$-regular bipartite graph G of order $2n > 8$ contains a cycle of length $2n-2$. Moreover, if such a cycle can be chosen to omit a pair of adjacent vertices, then G is bipancyclic.

In [2], Entringer and Schmeichel gave a sufficient condition for a hamiltonian bipartite graph to be bipancyclic.

Theorem 1. *A hamiltonian bipartite graph G of order $2n$ and size $\|G\| > n^2/2$ is bipancyclic (that is, contains cycles of all even lengths up to $2n$).*

Interestingly enough, a non-hamiltonian graph with this same bound on the size may contain no long cycles whatsoever. Consider for instance, for n even, a graph obtained from the disjoint union of $H_1 = K_{n/2,n/2}$ and $H_2 = K_{n/2,n/2}$ by joining a single vertex of H_1 with a vertex of H_2.

In the present note, we are interested in the cycle structure of a hamiltonian bipartite graph of order $2n$, whose every vertex is of degree $n/2$. One immediately verifies that the size of such a graph is precisely $n^2/2$, so the above theorem does not apply. Instead, we prove the following result.

Theorem 2. *If G is a hamiltonian $n/2$-regular bipartite graph of order $2n > 8$, then G contains a cycle C of length $2n-2$. Moreover, if C can be chosen to omit a pair of adjacent vertices, then G is bipancyclic.*

Our motivation for presenting Theorem 2 is that, although concerning a narrow class of graphs, it plays an important role in the general study of long cycles in balanced bipartite graphs [1]. We find it also quite amusing that the proof below relies entirely on the combinatorics of the adjacency matrix.

It should be noted that Tian and Zang [3] proved that a hamiltonian bipartite graph of order $2n \geq 120$ and minimal degree greater than $\frac{2n}{5}+2$ is necessarily bipancyclic. This result leaves open the case of $|G| = n < 60$, in which the above theorem seems to be best to date.

Proof. Suppose to the contrary that there is a hamiltonian $n/2$-regular bipartite graph G on $2n$ vertices ($n \geq 5$), without a cycle of length $2n-2$. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ be the colour classes of G, and let H be a Hamilton cycle in G; say, $H = x_1y_1x_2y_2 \dots x_ny_nx_1$. Let $E = E(G)$ be the edge set of G. The requirement that G contain no C_{2n-2} implies that, for every $i = 1, \dots, n$,

$$x_iy_{i-2} \notin E\,, \quad x_iy_{i+1} \notin E\,, \quad \text{and} \tag{1}$$

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$$\text{if } x_iy_j \in E \text{ for some } j \in \{i+2,\dots,n+i-3\}\,, \quad \text{then } x_{i+1}y_{j+1} \notin E\,. \tag{2}$$

(All indices are understood modulo n.)

Consider the $n\times n$ adjacency matrix $A_G = [a^i_j]_{1\le i,j\le n}$, where $a^i_j = 1$ if $x_iy_j \in E$, and $a^i_j = -1$ otherwise. Notice that, from adjacency on H and by (1),

$$a^i_{i-1} = a^i_i = 1 \quad \text{and} \quad a^i_{i-2} = a^i_{i+1} = -1 \quad \text{for all } i, \tag{3}$$

and by (2),

$$a^i_j = 1 \;\Rightarrow\; a^{i+1}_{j+1} = -1 \quad \text{for } i=1,\dots,n,\; j=i+2,\dots,i-3. \tag{4}$$

As every x_i has precisely $n/2$ neighbours, the entries of each row of A_G sum up to 0; i.e., $\sum_{j=1}^n a^i_j = 0$. Therefore, by (4), we also have

$$a^i_j = -1 \;\Rightarrow\; a^{i+1}_{j+1} = 1 \quad \text{for } i=1,\dots,n,\; j=i+2,\dots,i-3. \tag{5}$$

The properties (3), (4) and (5) imply that A_G (and hence G itself) is uniquely determined by the entries $a^1_3,\dots,a^1_{n-2}$, and more importantly, that the sum of entries of the first column of A_G equals

$$\begin{aligned} a^1_1 + a^1_n + a^1_{n-1} - a^1_{n-2} + a^1_{n-3} - a^1_{n-4} + \dots - a^1_4 + a^1_3 + a^1_2 \\ = a^1_3 - a^1_4 + \dots + a^1_{n-3} - a^1_{n-2}, \end{aligned}$$

given that $a^1_1 + a^1_2 + a^1_{n-1} + a^1_n = 0$.

On the other hand, every column sums up to 0, as each y_j has precisely $n/2$ neighbours. Hence $\sum_{j=3}^{n-2} a^1_j = 0$ and $\sum_{j=3}^{n-2}(-1)^{j+1}a^1_j = 0$, and thus $n-4 = 4l$ for some $l \ge 1$, and $\sum_{k=1}^{2l} a^1_{2k+1} = \sum_{k=1}^{2l} a^1_{2k+2} = 0$. In general, for any $1 \le i_0 \le n$,

$$a^{i_0}_{i_0+2} + a^{i_0}_{i_0+4} + \dots + a^{i_0}_{i_0+n-4} = a^{i_0}_{i_0+3} + a^{i_0}_{i_0+5} + \dots + a^{i_0}_{i_0+n-3} = 0\,. \tag{6}$$

Let now $1 \le i_0 \le n$ be such that $a^{i_0}_{i_0+2} = -1$. In fact, we can choose $i_0 = 1$ or $i_0 = 2$, for if $a^1_3 = 1$, then $a^2_4 = -1$, by (4). We will show that there exists a $k \in \{3,\dots,n-3\}$ such that

$$a^{i_0}_{i_0+k} = a^{i_0+k}_{i_0} = 1\,.$$

Suppose otherwise; i.e., suppose that, for all $3 \le k \le n-3$, $a^{i_0}_{i_0+k} + a^{i_0+k}_{i_0} \in \{0,-2\}$. Notice that, by (4) and (5), $a^{i_0+k}_{i_0} = (-1)^k a^{i_0}_{i_0-k}$ for $k = 3,\dots,n-3$. Hence, in particular, $a^{i_0}_{i_0+4} + a^{i_0}_{i_0+n-4}$, $a^{i_0}_{i_0+6} + a^{i_0}_{i_0+n-6}, \dots, a^{i_0}_{i_0+2l+2} + a^{i_0}_{i_0+2l+2}$ are all non-positive. In light of (6), this is only possible when $a^{i_0}_{i_0+2} = 1$, which contradicts our choice of i_0.

To sum up, we have found i_0 and $k \in \{3,\dots,n-3\}$ with the property that $a^{i_0-1}_{i_0+1} = a^{i_0}_{i_0+k} = a^{i_0+k}_{i_0} = 1$, which is to say that

$$x_{i_0-1}y_{i_0+1} \in E\,,\;\; x_{i_0}y_{i_0+k} \in E\,,\;\text{ and } x_{i_0+k}y_{i_0} \in E\,.$$

Hence a cycle

$$C \;=\; x_{i_0-1}\,y_{i_0+1}\,x_{i_0+2}\dots y_{i_0+k-1}\,x_{i_0+k}\,y_{i_0}\,x_{i_0}\,y_{i_0+k}\,x_{i_0+k+1}\dots y_{i_0-2}\,x_{i_0-1}$$

of length $2n-2$ in G; a contradiction.

For the proof of the second assertion of the theorem, suppose that C can be chosen so that the omitted vertices x' and y' are adjacent in G. Let $G' = G - \{x',y'\}$

be the induced subgraph of G spanned by the vertices of C. Then G' is hamiltonian of order $2(n-1)$ and size

$$\|G'\| = \|G\| - (d_G(x') + d_G(y') - 1) = n^2/2 - n + 1\,,$$

which is greater than $(n-1)^2/2$. Thus G', and hence G itself, is bipancyclic, by Theorem 1.

$\square$

References

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