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Vertical components in fibre powers of analytic spaces

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Abstract

We study the relationship between degeneracies of the family of fibres of an analytic mapping and the existence of vertical components in fibre powers of the mapping. Our main result is the following criterion for openness of complex analytic maps: Let $f: X \to Y$ be an analytic map of analytic spaces, with X being puredimensional and Y being locally irreducible of dimension n. Then f is open if and only if there are no isolated algebraic vertical components in the nth fibre power $X^{\{n\}}$.

Keywords: Fibre powers; Vertical components; Openness; Flatness

1. Introduction

The purpose of this paper is to discuss the relationship between degeneracies of the family of fibres of an analytic mapping (as expressed by a failure of openness or flatness) and the existence of *vertical* components in fibre powers of the mapping. There are in fact two natural notions of a *vertical* component (*algebraic* and *geometric*) and we are interested also in the relationship between them.

Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. An irreducible (isolated or embedded) component W of X_ξ is called *algebraic vertical* if there exists a nonzero element $a \in \mathcal{O}_{Y,\eta}$ such that (the pullback of) a belongs to the associated prime $\mathfrak{p}$ in $\mathcal{O}_{X,\xi}$ corresponding to W. Equivalently, W is *algebraic vertical* if an arbitrarily small representative of W is mapped into a proper analytic subset of a neighbourhood of η in Y.

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We say that W is *geometric vertical* if an arbitrarily small representative of W is mapped into a nowhere dense subset of a neighbourhood of η in Y, or equivalently, if the hypergerm (in the sense of Galligo and Kwieciński, see [6]) $f(W)$ has empty interior in Y_η with the transcendental topology.

The concept of a *vertical* component comes up naturally as an equivalent of torsion in algebraic geometry: Let $f: X \to Y$ be a polynomial map of algebraic varieties with Y irreducible. Then the coordinate ring of the source, $A(X)$ has nonzero torsion as a module over the coordinate ring $A(Y)$ of the target if and only if there exists a nonzero element $a \in A(Y)$ such that its pullback f^*a is a zerodivisor in $A(X)$. Since the set of zerodivisors equals the union of the associated primes, it follows from "prime avoidance" (see, e.g., [4, Section 3.2]) that $A(X)$ has nonzero torsion over $A(Y)$ if and only if there exists an irreducible (isolated or embedded) component of X whose image under f is contained in a proper algebraic subset of Y (or, equivalently, is nowhere dense in Y). There are two natural ways of generalizing this property of irreducible components to the analytic case. For a morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces (with Y_η irreducible), one can either consider the components of the source that are mapped into nowhere dense subgerms of the target (the *geometric vertical* components), or the components that are mapped into proper analytic subgerms of the target (the *algebraic vertical* components).

The geometric approach has proved to be a very powerful tool in analytic geometry; e.g., in work of Kwieciński [9], Kwieciński and Tworzewski [10], and Galligo and Kwieciński [6]. Note that in principle the existence of the *algebraic vertical* components is a weaker condition than the presence of the *geometric vertical* ones. Indeed, any *algebraic vertical* component (over an irreducible target) is *geometric vertical*, since a proper analytic subset of a locally irreducible analytic set has empty interior. The converse is not true though, as can be seen in the following example of Osgood (cf. [7, Kap. II, §5]):

$$f : \mathbb{C}^2 \ni (x, y) \mapsto \bigl(x, xy, xye^y\bigr) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in $\mathbb{C}^3$, but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target.

Remark 1.1. On the other hand, the algebraic approach has an advantage that all the statements about *algebraic vertical* components (as opposed to *geometric vertical*) can be restated in terms of torsion freeness of the local rings. Namely, $f_\xi : X_\xi \to Y_\eta$ *has no* (*isolated or embedded*) *algebraic vertical components if and only if the local ring* $\mathcal{O}_{X,\xi}$ *is a torsionfree* $\mathcal{O}_{Y,\eta}$*-module*.

In view of Remark 1.1, an interesting question is under what conditions are the two approaches equivalent. In the next section we show that one can expect a positive answer to this question under some minor constraints on the source space. Our main result, Theorem 2.2, gives a criterion for openness in terms of *vertical* components in fibre powers under the assumption that the source be puredimensional. The result is a consequence of our Proposition 2.1. Though technical, this proposition is interesting in itself and plays an important role in providing a relationship between the isolated irreducible components in the fibre power $X^{\{n\}}_{\xi^{\{n\}}}$ of X_ξ over Y_η, and a filtration of Y by the fibre dimension of f.

As we show in Example 2.5, the geometric and the algebraic approaches are not equivalent in general; that is, there are examples of bad behaviour of analytic mappings that can be detected by means of *geometric vertical* components but not by the *algebraic* ones. The fundamental reasons for this are that, in general, the *algebraic vertical* components do not detect a hidden Gabrielov irregularity, and that a fibre power of a regular map may itself be irregular. These are the main obstructions to extending results on *geometric vertical* components to *algebraic vertical* ones. Section 3 is devoted to clarifying the problem.

In the last section we take on the problem of flatness. We briefly discuss criteria for flatness in terms of *vertical* components of fibre powers and give a generalization of results on flatness by Auslander and by the author (see Theorem 4.3). We also raise an open question of reformulating the Galligo–Kwieciński criterion as a statement about *algebraic vertical* components. As follows from our result on openness, in the context of flatness the most important role is played by the *embedded* components in the fibre powers. Thus, in order to strengthen the Galligo–Kwieciński theorem one needs to understand behaviour of the *embedded* algebraic vertical components and their relationship with the *embedded* geometric vertical components, which is still unclear.

2. Vertical components and openness

We begin with a technical observation carrying the kernel of all the subsequent results. In Sections 2 and 3 we will always keep the following assumptions: Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, let X_ξ be of pure dimension m, and let Y_η be irreducible of dimension n. Let V be a locally irreducible neighbourhood of η in Y, and let U be a neighbourhood of ξ in X such that $f(U) \subset V$ and $U = U_1 \cup \cdots \cup U_s$ consists of finitely many isolated irreducible components of dimension m through ξ that are precisely the representatives in U of the isolated irreducible components of the germ X_ξ. (In Sections 2 and 3 we are not interested in the nilpotent structure of X_ξ. In fact, if one takes into account the embedded components as well, one obtains criteria for flatness instead. See Section 4 for details.)

Let $\operatorname{fbd}_x f = \dim_x f^{-1}(f(x))$ be the fibre dimension of f at a point. Let $l = \min\{\operatorname{fbd}_x f\colon\ x \in U\}$ and let $k = \max\{\operatorname{fbd}_x f\colon\ x \in U\}$. For $l \leqslant j \leqslant k$, define $A_j = \{x \in U\colon\ \operatorname{fbd}_x f \geqslant j\}$. Then, for each j, A_j is analytic in U (by the Cartan–Remmert Theorem, see [12]) and $U = A_l \supset A_{l+1} \supset \cdots \supset A_k$. Define $B_j = f(A_j) = \{y \in V\colon \dim f^{-1}(y) \geqslant j\}$, for $l \leqslant j \leqslant k$. Note that, except for B_k (cf. proof of Proposition 2.1 below), the B_j may not even be semianalytic in general. Nonetheless, there is an interesting connection between the filtration $V \supset B_l \supset B_{l+1} \supset \cdots \supset B_k$ and the isolated irreducible components of the nth fibre power $U^{\{n\}}$ that we describe below. The induced map from $U^{\{n\}}$ to V will be denoted by $f^{\{n\}}$, and $\xi^{\{n\}} = (\xi, \ldots, \xi) \in U^{\{n\}}$.

Proposition 2.1. *Under the above assumptions, let $U^{\{n\}} = \bigcup_{i \in I} W_i$ be the decomposition into finitely many isolated irreducible components through $\xi^{\{n\}}$. Then*

(a) *For each $j = l, \dots, k$, there exist components $W_{i_j,1}, \dots, W_{i_j,p_j}$ of $U^{\{n\}}$ such that*

$$B_j = \bigcup_{q=1}^{p_j} f^{\{n\}}(W_{i_j,q}).$$

(b) *If $y \in B_j$ with $\dim f^{-1}(y) = s$ $(s \geqslant j)$, Z is an irreducible component of the fibre $(f^{\{n\}})^{-1}(y)$ of dimension ns, and W is an irreducible component of $U^{\{n\}}$ containing Z, then $f^{\{n\}}(W) \subset B_j$.*

Proof. Fix $j \geqslant l+1$ (the statement is trivial for $j = l$ as $B_l = f(U)$). Pick any $y \in B_j$. Then $\dim f^{-1}(y) = s$ for some $s \geqslant j$. Let Z be an irreducible component of the fibre $(f^{\{n\}})^{-1}(y)$ of dimension ns, and let W be an irreducible component of $U^{\{n\}}$ containing Z. We will show that $f^{\{n\}}(W) \subset B_j$.

Suppose to the contrary that $W \cap (U^{\{n\}} \setminus (f^{\{n\}})^{-1}(B_j)) \neq \emptyset$, that is, suppose that there exists $z = (x_1, \dots, x_n) \in W$ such that $f(x_i) \in V \setminus B_j$ for $i = 1, \dots, n$. Then $\mathrm{fbd}_{x_i} f \leqslant j-1$, $i = 1, \dots, n$, and hence $\mathrm{fbd}_z f^{\{n\}} \leqslant n(j-1) = nj - n$. In particular, the generic fibre dimension of $f^{\{n\}}|W$ is not greater than $nj - n$. Since $\operatorname{rank}(f^{\{n\}}|W) \leqslant \dim V = n$, then $\dim W \leqslant (nj - n) + n = nj$.

Now we have: $W \supset Z$, $\dim W \leqslant nj$, $\dim Z = ns \geqslant nj$, and both W and Z irreducible. This is only possible when $W = Z$, and hence $f^{\{n\}}(W) = f^{\{n\}}(Z) = \{y\} \subset B_j$, a contradiction. Therefore $f^{\{n\}}(W) \subset B_j$, which completes the proof of part (b) of our proposition.

Part (a) follows immediately, since for any $y \in B_j$ and any irreducible component Z of $(f^{\{n\}})^{-1}(y)$ of the highest dimension, there exists an isolated irreducible component W of $U^{\{n\}}$ that contains Z. □

Now we are ready to establish our criterion for openness:

Theorem 2.2. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be puredimensional and let Y_η be irreducible of dimension n. Then the following conditions are equivalent*:

(i) *f_ξ is open,*
(ii) *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated geometric vertical components,*
(iii) *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated algebraic vertical components.*

Remark 2.3. In light of Remark 1.1, the equivalence (i) $\Leftrightarrow$ (iii) in the above theorem could be restated as follows:

> *$f_\xi : X_\xi \to Y_\eta$ is open if and only if the* reduced *local ring $(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}})_{\mathrm{red}}$ is a torsionfree $\mathcal{O}_{Y,\eta}$-module.*

(Compare with Theorem 4.2 and Remark 4.6 in Section 4.)

Proof of Theorem 2.2. First, note that if $f_\xi : X_\xi \to Y_\eta$ is open, then

$$f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$$

is open for every $i \geqslant 1$, and hence every isolated irreducible component of $X^{\{i\}}_{\xi^{\{i\}}}$ is mapped onto Y_η. This proves (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is trivial, as every *algebraic vertical* component is *geometric vertical* over an irreducible target (see Section 1).

For the proof of (iii) $\Rightarrow$ (i) we assume that $\dim X = m$ and suppose that f_ξ is not open. Let V and $U = U_1 \cup \cdots \cup U_s$ be as before. Consider the set $A = A_k = \{x \in U\colon \operatorname{fbd}_x f = k\}$. Then A is analytic in U, and $\xi \in A$ as the fibre dimension is Zariski upper semi-continuous. Since the fibre dimension of f is constant on A, it follows by Remmert's Rank Theorem (see [12]) that for arbitrarily small neighbourhood $\widetilde{U}$ of ξ in U, $f(\widetilde{U} \cap A)$ is locally analytic in V, of dimension $\dim_\xi A - k$. Without loss of generality can assume that $B_k = f(A)$ is analytic in V (by shrinking V and U if necessary).

Observe that B_k is a *proper* analytic subset of V: First, notice that k cannot be the generic fibre dimension for all the components $U_1, \ldots, U_s$ simultaneously, since then f would be open. Hence, for some $j \leqslant s$, the generic fibre dimension of $f|U_j$ is at most $k-1$, and therefore $\dim V \geqslant \dim U_j - (k-1) = m - k + 1$.

Let $A = A^1 \cup \cdots \cup A^q$ be the decomposition of the set A into isolated irreducible components. Fix $i \leqslant q$. The set

$$\widetilde{A^i} = \operatorname{Reg}(A^i) \setminus \bigcup_{j \neq i} A^j$$

is open in A. Let $a \in \widetilde{A^i}$ and let Z be an irreducible component of the fibre $f^{-1}(f(a))$ of dimension k containing a. Then $Z \cap \widetilde{A^i}$ is open in Z, and hence of dimension k. Since this is true for an arbitrary point a from an open subset of the component A^i, the generic fibre dimension of $f|A^i$ equals k and $\dim f(A^i) \leqslant m - k$, as $\dim A^i \leqslant \dim U = m$. Thus, $\dim f(A) = \dim \bigcup_{i=1}^q f(A^i) \leqslant m - k$, whereas $\dim V \geqslant m - k + 1$, so $B_k = f(A)$ must be properly contained in V.

Finally, observe that $\eta \in B_k$, as ξ lies in A. Let Z be an irreducible component of the fibre $(f^{\{n\}})^{-1}(\eta)$ of dimension nk, and let W be an irreducible component of $U^{\{n\}}$ containing Z. Then by Proposition 2.1(b), $f^{\{n\}}(W) \subset B_k$, hence W is algebraic vertical. $\square$

Remark 2.4. Note that since openness is a local property, Theorem 2.2 can easily be "globalized" to the case of an analytic map $f : X \to Y$ of analytic spaces, where X is puredimensional and Y is locally irreducible of dimension n. Thus, our result is a stronger version of the theorem by Kwieciński and Tworzewski asserting that openness is equivalent to the lack of *geometric vertical* components in the nth fibre power $X^{\{n\}}$ (see [10, Theorem 3.2]).

However, unlike with *geometric vertical* components, the puredimensionality constraint on X in our theorem is unavoidable. If X is not puredimensional, it may happen that the exceptional fibres of one component are generic for a component of higher dimension, and therefore they do not give rise to an isolated *algebraic vertical* component in any fibre power. This phenomenon is illustrated in the following example, where not only is the morphism f not open, but also it is not even regular in the sense of Gabrielov when restricted to one of the components (cf. Section 3).

Example 2.5. Let $X = X_1 \cup X_2$, where $X_1 = \{(x, y, s, t, z) \in \mathbb{C}^5\colon\ s = t = z = 0\}$ and $X_2 = \{(x, y, s, t, z) \in \mathbb{C}^5\colon\ x = 0\}$. Define $f : X \to Y = \mathbb{C}^3$ as

$$f(x, y, s, t, z) = \bigl(x + s, xy + t, xye^y + z\bigr).$$

Observe that (the germ at the origin of) $f|X_1$ is an Osgood mapping and hence it is not regular in the sense of Gabrielov. Therefore f is not open. But the exceptional fibre $\{x = s = t = z = 0\}$ of $f|X_1$ is in no sense exceptional for $f|X_2$. One can easily verify that in any fibre power of X over Y, any isolated (!) irreducible component is either *purely* geometric vertical (with the image equal to that of $f|X_1$) or maps onto (the germ at the origin of) Y and so is not vertical in neither sense.

Remark 2.6. By Theorem 2.2, the existence of isolated *geometric vertical* components in $X^{\{n\}}_{\xi^{\{n\}}}$ is equivalent to the existence of isolated *algebraic vertical* components, provided X_ξ is puredimensional. Nevertheless, it is not true that every isolated *geometric vertical* component in $X^{\{n\}}_{\xi^{\{n\}}}$ is *algebraic vertical*. As we show in Proposition 3.1 and Example 3.3 at the end of the next section, this is not even true in the case of a smooth domain.

3. Fibre powers and Gabrielov regularity

In this section we show that the fibre product doesn't behave well with respect to Gabrielov regularity. Recall that a morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces is called *Gabrielov regular* if $\dim_\eta f(Z) = \dim_\eta \overline{f(Z)}$ for arbitrarily small representative Z of X at ξ, where $\overline{f(Z)}$ denotes the Zariski closure of $f(Z)$ in a representative of Y at η (see, e.g., [13, §1]).

The following result is an immediate corollary of Proposition 2.1. Define $S = \{y \in V\colon\ \dim f^{-1}(y) > l\}$, where as before $l = \min\{\mathrm{fbd}_x f\colon\ x \in U\}$. Then $\dim S < \dim V$, since U is puredimensional.

Proposition 3.1. *Suppose that* $\dim_\eta \overline{S} = n$, *where* $\overline{S}$ *denotes the Zariski closure of* S *in* V. *Then* $X^{\{n\}}_{\xi^{\{n\}}}$ *contains an isolated* purely geometric *vertical component, i.e., a geometric vertical component which is not algebraic vertical.*

As a consequence, there exist Gabrielov regular mappings $f_\xi : X_\xi \to Y_\eta$ such that, for some $i \geqslant 1$, $f^{\{i\}}_{\xi^{\{i\}}}$ is not Gabrielov regular when restricted to one of the isolated irreducible components (see Example 3.3 below).

Proof of Proposition 3.1. By Proposition 2.1, there exist irreducible components $W_1, \ldots, W_p$ of $U^{\{n\}}$ such that

$$S = \bigcup_{i=1}^{p} f^{\{n\}}(W_i).$$

(Recall that $S = B_{l+1}$ according to the notation from Proposition 3.1.) We claim that $\dim_\eta \overline{f^{\{n\}}(W_j)} = n$ for some $j \in \{1, \ldots, p\}$. Indeed, if $\dim_\eta \overline{f^{\{n\}}(W_i)} < n$ for all i, then we would have

$$n = \dim_\eta \overline{S} = \dim_\eta \overline{\bigcup_{i=1}^{p} f^{\{n\}}(W_i)} = \max\bigl\{\dim_\eta \overline{f^{\{n\}}(W_i)}\colon\ i = 1, \ldots, p\bigr\} < n,$$

a contradiction. So obtained W_j is not algebraic vertical, and it is geometric vertical, since $\dim_\eta f^{\{n\}}(W_j) \leqslant \dim_\eta S < n$. □

The proposition yields a necessary and sufficient condition for $X^{\{n\}}_{\xi^{\{n\}}}$ having purely geometric vertical components in the case of dominating mappings:

Corollary 3.2. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces with X_ξ being of pure dimension m and Y_η being irreducible of dimension n, and assume f is dominating, i.e., $m - l = n$. Then $X^{\{n\}}_{\xi^{\{n\}}}$ has an isolated purely geometric vertical component if and only if* (*the germ at η of*) *Y equals* (*the germ at η of*) *the Zariski closure of S.*

Proof. Let V and U be as before. If $\dim_\eta \overline{S} < n$, then there exists a proper locally analytic subset $\widetilde{V} \subset V$ containing S and such that $\dim_\eta \widetilde{V} < n$.

Then, for any isolated component W of $U^{\{n\}}$, either $f^{\{n\}}(W) \subset S$ or $W \cap (f^{\{n\}})^{-1}(V \setminus S) \neq \emptyset$. In the first case $f^{\{n\}}(W) \subset \widetilde{V}$, so W is algebraic vertical. In the second case there exists a point $z = (x_1, \ldots, x_n) \in W$ such that $\mathrm{fbd}_{x_i} f = l$, $i = 1, \ldots, n$, hence the image under f of an arbitrarily small neighbourhood of x_i has nonempty interior in V (by Remmert's Rank Theorem and since $m - l = n$). Therefore, the image under $f^{\{n\}}$ of an arbitrarily small neighbourhood of z has nonempty interior in V, i.e., W is not a vertical component in neither sense. □

Finally we show an example of a Gabrielov regular mapping $f : X \to Y$, with smooth X and Y, and such that the Zariski closure of S equals Y. By Proposition 3.1 above, the top fibre power of X over Y has an isolated irreducible *purely geometric vertical* component. This shows that a fibre product of a regular map may itself be nonregular when restricted to an irreducible component.

Example 3.3. Let $X = Y = \mathbb{C}^4$ and define f as follows

$$(x, y, s, t) \mapsto \bigl(x, (x+s)y, x^2y^2e^y, x^2y^2e^{y(1+xe^y)} + st\bigr).$$

It is easy to check that the generic fibre dimension of f equals 0, hence the generic rank of f is 4, and so f is Gabrielov regular.

On the other hand, S contains the image of the set $\mathbb{C}^2 \times \{0\} \times \mathbb{C}$, consisting of points (z_1, z_2, z_3, z_4) of the form $(x,\ xy,\ x^2y^2e^y,\ x^2y^2e^{y(1+xe^y)})$, whose Zariski closure equals $\mathbb{C}^4$ (cf. [7, Kap. II, §5]). Hence also $\overline{S} = \mathbb{C}^4$.

4. Vertical components and flatness

In this section we discuss briefly the results relating flatness of a morphism of germs of analytic spaces and vertical components in fibre products. Our point of departure is the fundamental result of Auslander:

Theorem 4.1 [2, Theorem 3.2]. *Let R be an unramified regular local ring of dimension $n > 0$ and let M be a finite R-module. Then M is R-free if and only if the nth tensor power $M^{\otimes n}$ is a torsionfree R-module.*

(Auslander's result was later extended by Lichtenbaum [11] to arbitrary regular local rings.)

Recall that in the case of finite modules, freeness is equivalent to flatness. Also, for finite modules M and N over a local analytic algebra R, their analytic tensor product, denoted by $M \hat{\otimes}_R N$, equals the ordinary one, $M \otimes_R N$. Thus, the following result obtained by Kwieciński can be viewed as a generalization of Theorem 4.1 to arbitrary (nonfinite) analytic mappings.

Theorem 4.2 [9, Theorem 1.1]. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces and assume that Y_η is irreducible. Then the following conditions are equivalent*:

(i) *f_ξ is flat,*

(ii) *for any $i \geqslant 1$, the canonical map $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ has no* (*isolated or embedded*) *algebraic vertical components.*

Note that by our Remark 1.1, the latter condition is equivalent to saying that

for any $i \geqslant 1$, the ith analytic tensor power of $\mathcal{O}_{X,\xi}$ over $\mathcal{O}_{Y,\eta}$, $\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}$ (*i times*) *is a torsionfree $\mathcal{O}_{Y,\eta}$-module.*

Observe that the implication (i) $\Rightarrow$ (ii) in Theorem 4.2 is immediate, since flatness of f_ξ implies that $f^{\{i\}}_{\xi^{\{i\}}}$ is flat for all $i \geqslant 1$, as flatness is preserved by any base change (see [8, §6, Proposition 8]) and composition of flat maps is flat. Therefore, the interesting part of the theorem is (ii) $\Rightarrow$ (i). It turns out that this implication can be significantly strengthened and one can give an explicit bound in condition (ii) expressed in terms of Hironaka's local flattener.

Recall that for a morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces and a finite $\mathcal{O}_{X,\xi}$-module M, there exists a unique maximal germ of an analytic subspace P of Y_η

such that $\mathcal{O}_P \hat{\otimes}_{\mathcal{O}_{Y,\eta}} M$ is $\mathcal{O}_P$-flat. We call the germ P the *flattener* of M (see, e.g., [3, Theorem 7.12]).

Here we generalize the result of [1] to finite modules over $\mathcal{O}_{X,\xi}$, establishing a clearer connection with Theorem 4.1. In particular, if $X_\xi = Y_\eta$, then we obtain a generalization of Auslander's original result to the case of nonregular rings.

Theorem 4.3 (cf. [1, Theorem 1.1]). *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, let Y_η be irreducible, and let M be a finite $\mathcal{O}_{X,\xi}$-module that is not $\mathcal{O}_{Y,\eta}$-flat. Let Q be the ideal in $\mathcal{O}_{Y,\eta}$ of the flattener of M and let d be the minimal number of generators of Q. Then the dth analytic tensor power of M, $M \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} M$ (d times) has nonzero torsion over $\mathcal{O}_{Y,\eta}$.*

The proof is a straightforward generalization of that of [1, Theorem 1.1].

Another remarkable generalization of Theorem 4.1 to the case of nonfinite mappings was obtained by Galligo and Kwieciński under the original regularity assumption of Auslander.

Theorem 4.4 [6, Theorem 6.1]. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be puredimensional and let Y_η be smooth of dimension n. Then the following conditions are equivalent*:

(i) *f_ξ is flat,*
(ii) *the canonical map $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ has no* (*isolated or embedded*) *geometric vertical components.*

(The result was first proved in the algebraic case, for $n = 2$ and for arbitrary X, by Vasconcelos, see [14, Proposition 6.1].)

Recall that in the algebraic context the lack of vertical components is equivalent to torsionfreeness. That is, the algebraic version of the Galligo–Kwieciński theorem reads as follows:

> *Let A be a regular local ring of dimension n and let B be a puredimensional noetherian A-algebra. Then B is A-flat if and only if the nth tensor power of B over A, $B \otimes_A \cdots \otimes_A B$* ($n$ *times*) *is a torsionfree A-module.*

Note that, although Theorem 4.4 combined with primary decomposition algorithms provides a useful computer algebra tool for testing flatness, it cannot be rephrased in terms of torsionfreeness of analytic tensor powers of $\mathcal{O}_{X,\xi}$, as the vertical components in question are *geometric* and not *algebraic*. Also, we do not know a proof of the algebraic version of Theorem 4.4 (for $n > 2$) that does not involve transcendental methods. Therefore, a natural question arises: Can one replace *geometric vertical* by *algebraic vertical* (possibly in a higher fibre power of X_ξ) in Theorem 4.4? Our results from previous sections allow to put some restrictions on this problem:

Remark 4.5. It is enough to restrict our attention to the case when $f_\xi : X_\xi \to Y_\eta$ is open. For, if f_ξ is not open, then there exists an *isolated* algebraic vertical component in $X^{\{n\}}_{\xi^{\{n\}}}$, by Theorem 2.2. Moreover, if a morphism $f_\xi : X_\xi \to Y_\eta$ is open and nonflat, then a vertical component occurring in $X^{\{n\}}_{\xi^{\{n\}}}$ must be embedded. Indeed, if f_ξ is open, then $f^{\{i\}}_{\xi^{\{i\}}}$ is open for $i \geqslant 1$, and hence all the isolated components are mapped onto Y_η.

It seems plausible that one could generalize the Galligo–Kwieciński theorem exploiting techniques similar to those from Section 2. By the above remark though, one would need to understand the relationship between existence of the *embedded* geometric vertical and the *embedded* algebraic vertical components in fibre powers. This seems to be much more difficult than describing the connection between the isolated components.

Finally, observe that in the (very special) case when $\mathcal{O}_{X,\xi}$ is Cohen–Macaulay, our Theorem 2.2 yields the following nice equivalence:

Remark 4.6. Assume that $\mathcal{O}_{X,\xi}$ is Cohen–Macaulay and Y_η is smooth of dimension n. Then, by [5, Proposition 3.20], $f_\xi : X_\xi \to Y_\eta$ is flat if and only if it is open, that is: *$f_\xi : X_\xi \to Y_\eta$ is flat if and only if $(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}})_{\mathrm{red}}$ is a torsionfree $\mathcal{O}_{Y,\eta}$-module.*

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