Assignment 4

due: November 20

1. Find the following limits (using l'Hospital's Rules). Show your work.

(a) $\lim_{x\to\infty} \frac{e^x}{x^{\ln x}}$

(b) $\lim_{x\to 0^+} x^x$

(c) $\lim_{x\to 1^-} (1-x)\tan(\frac{\pi x}{2})$

(d) $\lim_{x\to 1} \left(\frac{\sin x}{\sin 1}\right)^{\frac{1}{x-1}}$

(e) $\lim_{x\to\infty} \left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$

(f) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(g) $\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$.

2. Find derivatives of the following functions. Show your work.

(a) $y = x^{f(x)}$, where $f(x) = e^x$

(b) $y = \left(\frac{\sin(mx)}{\sin(nx)}\right)^{g(x)}$, where $g(x) = (mx)^{nx}$.

3. Prove the Mean Value Theorem as a corollary to Rolle's Theorem.

[Hint: Given a function $f(x)$ continuous on the interval $[a, b]$ and differentiable on (a, b), consider the function $h(x)$ defined as

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)\,.$$

Verify that $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and satisfies $h(a) = h(b)$. Thus, $h(x)$ satisfies the assumptions of Rolle's Theorem. Use the conclusion of Rolle's Theorem for $h(x)$ to complete the proof.]

4. A function f is called *weakly increasing* when $x_1 < x_2$ implies $f(x_1) \le f(x_2)$, for any x_1, x_2 in the domain of f. Similarly, f is *weakly decreasing* when $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$, for any x_1, x_2 in the domain of f.

Let f be a function differentiable on an open interval I. Use the Mean Value Theorem to prove the following statements.

(a) If $f'(x) \ge 0$ for all $x \in I$, then f is weakly increasing on I.

(b) If $f'(x) \le 0$ for all $x \in I$, then f is weakly decreasing on I.