Assignment 5

due: December 7

1. (a) Prove that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

(b) Use the formulas for $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ to find $\sum_{k=1}^{n} k^4$. [Hint: Consider the sum $\sum_{k=1}^{n}[(1+k)^5 - k^5]$.

2. Evaluate the following sums. Show your work.

(a) $\sum_{k=1}^{m} \left(\sum_{l=1}^{n} \frac{k}{l(l+1)} \right)$

(b) $\sum_{k=1}^{n} \left(\frac{2018}{\sum_{l=1}^{k} l} \right)$.

3. Find the following limit. Show your work.

$$\lim_{n\to\infty} \left(\sqrt{\left(\sum_{i=1}^{n} i \right)} - \frac{n\sqrt{2}}{2} \right).$$

4. Find the limit $\lim_{n\to\infty} S_n$, where

$$S_n = \sum_{k=1}^{n-1} \frac{k^4}{n^5}.$$

[Hint: Consider the function $f(x) = x^4$. Interpret S_n as the sum of areas of certain rectangles with base of length $1/n$ each, and one of the vertices lying on the graph of f. Use it to show that, for every natural $n \geq 2$, $S_n \leq \int_0^1 f(x)\,\mathrm{d}x$ and $S_n \geq \int_0^{1-1/n} f(x)\,\mathrm{d}x$. Finally, apply the Squeeze Theorem.]

5. Let f be *any* linear function on a closed interval $[a, b]$ (i.e., a function of the form $f(x) = Ax + B$ for some $A, B \in \mathbb{R}$). Show that

$$\int_a^b f(x)\,\mathrm{d}x = (b-a) \cdot f(\frac{a+b}{2}).$$

6. Let f be a function defined (piecewise) on the interval $[1, 2018]$ by the formula

$$f(1) = \frac{2}{3} \qquad \text{and} \qquad f(x) = \frac{2}{k(k+2)}, \quad \text{for } x \in (k, k+1], \text{ where } k = 1, \ldots, 2017.$$

Find the area of the region bounded by the graph of f, the x-axis and the lines $x = 1$ and $x = 2018$.