Problem Set 10
March 28, 2018.

1. Exercises 10.1–10.5.

2. Exercise 15.2.

3. Exercise 15.4.

4. Exercise 15.6.

5. (a) Use Jensen's Inequality to prove that

$$\sqrt[n]{y_1 \cdots y_n} \;\le\; \frac{y_1 + \cdots + y_n}{n}\,,$$

for all $n \in \mathbb{Z}_+$ and $y_1, \ldots, y_n \in (0, \infty)$.
Hint: Apply Jensen's Inequality with the convex function $\varphi(t) = e^t$, $X = \{x_1, \ldots, x_n\}$, and $\mu(x_i) = 1/n$.

(b) Generalize the above argument to prove that

$$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \le \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n\,,$$

for all $n \in \mathbb{Z}_+$, and $y_1, \ldots, y_n, \alpha_1, \ldots, \alpha_n \in (0, \infty)$ with $\sum_{i=1}^n \alpha_i = 1$.

6. Prove that, if $1 \le p \le \infty$ and if $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $L^p(\mu)$ with limit f, then (f_n) has a subsequence which converges pointwise almost everywhere to f.

7. Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$, and $f : X \to [0, \infty)$ is measurable. If

$$A = \int_X f\,d\mu\,,$$

prove that

$$\sqrt{1 + A^2} \;\le\; \int_X \sqrt{1 + f^2}\,d\mu \;\le\; 1 + A\,.$$

8. Let m denote the Lebesgue measure on $\mathbb{R}$.

(a) Find a sequence of Lebesgue integrable functions $f_n : \mathbb{R} \to \mathbb{R}$, such that $\int_{\mathbb{R}} f_n\,dm \longrightarrow \infty$ but $\|f_n\|_\infty \longrightarrow 0$, as $n \to \infty$.

(b) Find a sequence of Lebesgue integrable functions $g_n : \mathbb{R} \to \mathbb{R}$, such that $\int_{\mathbb{R}} g_n\,dm \longrightarrow 0$ but $\|g_n\|_\infty \longrightarrow \infty$, as $n \to \infty$.

(c) Find a sequence of continuous functions $h_n : [0, 1] \to \mathbb{R}$, such that $h_n(x) \longrightarrow 0$ for all $x \in [0, 1]$, $\int_{\mathbb{R}} h_n\,dm \longrightarrow 1$ and $\|h_n\|_\infty \longrightarrow \infty$, as $n \to \infty$.