Solutions for Problem Set 10 MATH 4122/9022

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10.1 Since $\{f_n\}$ is Cauchy in measure, there exists a subsequence $\{f_{n_k}\}$ such that

$$\mu(\{x : \left| f_{n_{k+1}} - f_{n_k} \right| > 1/2^k\}) < 1/2^k.$$

(just choose $a = \varepsilon = 1/2^k$ and apply the definition of being Cauchy in measure). So, if we denote $E_k := \{x : \left| f_{n_{k+1}} - f_{n_k} \right| > 1/2^k\}$ we have $\mu(E_k) < 1/2^k$. Since $\cup_{j=k+1}^{\infty} E_j \subset \cup_{j=k}^{\infty} E_j$, it follows that $\mu(\cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j) = \lim_{k\to\infty} \mu(\cup_{j=k}^{\infty} E_j) \leq \lim_{k\to\infty} \sum_{j=k}^{\infty} 1/2^j = \lim_{k\to\infty} 1/2^{k-1} = 0$. Let $E := \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j$. If $x \notin E$ there exists k s.t. $x \notin \cup_{j=k}^{\infty} E_j$, hence for all $j, l \geq k$ $\left| f_{n_j}(x) - f_{n_l}(x) \right| < 1/2^{k-1}$, i.e. $\{f_{n_k}(x)\}$ is a Cauchy sequence. Then, for $x \notin E$, $\lim_{j\to\infty} f_{n_j}(x)$ exists. Define

$$f(x) := \begin{cases} \lim_{j\to\infty} f_{n_j}(x), & \text{if } x \notin E \\ 0, & \text{if } x \in E, \end{cases}$$

which is well defined and measurable (we leave this last statement to be proved by the reader, as an exercise). It follows that $f_{n_j} \to f$ a.e. as $j \to \infty$, since $\mu(E) = 0$. For a fixed j and $x \notin \cup_{l=j}^{\infty} E_l$ we have, as we proved above, $\left| f_{n_j}(x) - f_{n_l}(x) \right| \leq 1/2^{j-1}, \forall l \geq j$, hence $\left| f_{n_j}(x) - f(x) \right| = \left| f_{n_j}(x) - \lim_{l\to\infty} f_{n_l}(x) \right| = \lim_{l\to\infty} \left| f_{n_j}(x) - f_{n_l}(x) \right| \leq 1/2^{j-1}$. It follows that $\{x : \left| f_{n_j}(x) - f(x) \right| \geq 1/2^{j-1}\} \subset \cup_{l=j}^{\infty} E_l$, hence $\mu(\{x : \left| f_{n_j}(x) - f(x) \right| \geq 1/2^{j-1}\}) \leq \mu(\cup_{l=j}^{\infty} E_l) \leq 1/2^{j-1}$ which converges to 0 as $j \to \infty$. This proves that the subsequence $\{f_{n_j}\}$ converges to f in measure. Of course, we have $|f_n(x) - f(x)| \leq \left| f_n(x) - f_{n_j}(x) \right| + \left| f_{n_j}(x) - f(x) \right|$, so if $|f_n(x) - f(x)| > a$ then $\left| f_n(x) - f_{n_j}(x) \right| \geq a/2$ or $\left| f_{n_j}(x) - f(x) \right| \geq a/2$. This implies that $\{x : |f_n(x) - f(x)| > a\} \subset \{x : \left| f_n(x) - f_{n_j}(x) \right| \geq a/2\} \cup \{x : \left| f_{n_j}(x) - f(x) \right| \geq a/2\}$. Applying the facts that $\{f_n\}$ is Cauchy in measure and f_{n_j} converges in measure to f, the sets on the right hand side have arbitrarily small measure (for j, n large enough), which proves the statement.

10.2 The symmetry and nonnegativity of d are obvious. Also, $d(f, g) = 0$ implies $\int |f - g| / (1 + |f - g|) d\mu = 0$, hence $|f - g| = 0$ a.e. Lastly, for the triangle inequality evaluate the expression $D := d(f, h) + d(h, g) - d(f, g)$. After elementary calculations, we find that D has positive denominator and its numerator is given by $|f-h|+|h-g|-|f-g|+|f-h||h-g|+|h-g||f-h|$ which is nonnegative (apply the triangle inequality to the first three terms). Suppose now that $f_n \to f$ in measure. The function $t \mapsto t/(1+t)$ is increasing so, for all $\varepsilon > 0$, $d(f_n, f) = \int_X \frac{|f_n - f|}{1+|f_n - f|} d\mu = \int_{|f_n - f|<\varepsilon} \frac{|f_n - f|}{1+|f_n - f|} d\mu + \int_{|f_n - f|\geq\varepsilon} \frac{|f_n - f|}{1+|f_n - f|} d\mu \leq \varepsilon\mu(X) + \mu(|f_n - f| \geq \varepsilon)$, where the last term converges to 0 as $n \to \infty$, because of the convergence in measure of f_n. It follows that $\lim_{n\to\infty} d(f_n, f) \leq \varepsilon\mu(X)$ for all $\varepsilon > 0$, which implies $\lim_{n\to\infty} d(f_n, f) = 0$ since $\mu(X) < \infty$. For the converse, fix $\varepsilon > 0$. Then $\frac{\varepsilon}{1+\varepsilon}\mu(\{|f_n - f| \geq \varepsilon\}) = \frac{\varepsilon}{1+\varepsilon}\int_{\{|f_n - f|\geq\varepsilon\}} d\mu \leq \int_{\{|f_n - f|\geq\varepsilon\}} \frac{|f_n - f|}{1+|f_n - f|} d\mu \leq \int_X \frac{|f_n - f|}{1+|f_n - f|} d\mu \to 0$, where we used that on $\{|f_n - f| \geq \varepsilon\}$ we have $\frac{|f_n - f|}{1+|f_n - f|} \geq \frac{\varepsilon}{1+\varepsilon}$.

10.3 Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ such that $\int f_{n_k} \to \liminf_{n\to\infty} \int f_n$ (such subsequence always exists by the definition of lim inf of a sequence of numbers). As a subsequence of a sequence converging in measure to f, $\{f_{n_k}\}$ converges in measure to f. Hence, there exists a subsequence $\{g_n\}$ of $\{f_{n_k}\}$ (which we denoted as g_n to avoid the proliferation of indices) that converges to f a.e. Also, $\int g_n \to \liminf_{n\to\infty} \int f_n$ (as a subsequence of $\{f_{n_k}\}$). So, $\int f = \int \lim_{n\to\infty} g_n = \int \liminf_{n\to\infty} g_n \leq \liminf_{n\to\infty} \int g_n = \lim_{n\to\infty} \int g_n = \liminf_{n\to\infty} \int f_n$ which proves the statement.

10.4 Denote by $f_n := \chi_{A_n}$ for all n and let $\{g_n\}$ be a subsequence of $\{f_n\}$ such that $g_n \to f$ a.e.. Every function g_n satisfies $g_n(X) \subset \{0, 1\}$, where X is the underlining set of the measure space. Then, $f(X) \subset \{0, 1\}$ a.e. since $g_n \to f$ a.e.. Let $E := \{x \in X : \lim_{n\to\infty} g_n(x) = f(x)\}$. Then, $A := E \cap \{f = 1\} = \cup_n \cap_{m\geq n} \{g_m = 1\}$ is measurable and $f = \chi_A$ a.e..

10.5 For every $\varepsilon > 0$ let F_ε be the measurable set on which f_n converges to f uniformly and $\mu(F_\varepsilon^c) < \varepsilon$. Define $A := \cap_m F_{1/m}^c$, for $m \in \mathbb{Z}_+$, which is clearly measurable. Then $\mu(A) \leq \mu(F_{1/m}^c) \leq 1/m$ for all m, hence $\mu(A) = 0$. If $x \notin A$ then there exists m such that $x \in F_{1/m}$, hence $f_n(x) \to f(x)$. Since $\mu(A) = 0$ this means that $f_n \to f$ a.e..

15.2 It suffices to show the result for nonnegative functions in L^p (for an arbitrary function in L^p apply the result to its positive and negative parts). Let $f \geq 0$ be such function and let us first suppose that $1 \leq p < \infty$. By Proposition 5.14 in the textbook there is an increasing sequence of simple functions $\{s_n\}$ such that $\lim_{n\to\infty} s_n = f$. Clearly, the functions s_n are in L^p for all n (note that in general a simple function is not necessarily in L^p, for $1 \leq p < \infty$). Since

$s_n \le f$ and $f \ge 0$ we have $|f - s_n|^n \le |f|^p$ which is integrable since $f \in L^p$. By the dominated convergence theorem we have $\lim_{n\to\infty} \int |f - s_n|^p \, d\mu = 0$ which proves the statement for $1 \le p < \infty$. For $p = \infty$, note that each element of L^∞ has a bounded representative so we can assume f to be bounded. By reviewing the proof of Proposition 5.14, it is easy to see that, if f is bounded, the convergence $s_n \to f$ is uniform (where $\{s_n\}$ are the simple function given by Proposition 5.14), that is $\forall M \ge 0, \exists n_0$ s.t. $\forall n > n_0 \, |s_n - f| < M$. But this easily implies that $\| s_n - f \|_\infty \to 0$.

15.4 For $0 < \varepsilon < \| f \|_\infty$ define $A_\varepsilon := \{x \in [0,1] : |f(x)| \ge \| f \|_\infty - \varepsilon\}$. Then $\| f \|_p = (\int_{[0,1]} |f|^p \, dm)^{1/p} \ge (\int_{A_\varepsilon} |f|^p \, dm)^{1/p} \ge (\int_{A_\varepsilon} (\| f \|_\infty - \varepsilon)^p dm)^{1/p} = (\| f \|_\infty - \varepsilon) m(A_\varepsilon)^{1/p}$ (of course, $m(A_\varepsilon)$ is finite). This implies

$$\liminf_{p\to\infty} \| f \|_p \ge \| f \|_\infty. \tag{0.1}$$

Let $p > q$ and note that $|f| \le \| f \|_\infty$ a.e.. Then, $\| f \|_p = (\int_{[0,1]} |f|^{p-q} |f|^q \, dm)^{1/p} \le \| f \|_\infty^{\frac{p-q}{p}} (\int_{[0,1]} |f|^q \, dm)^{1/p} = \| f \|_\infty^{\frac{p-q}{p}} \| f \|_q^{q/p}$, which implies

$$\limsup_{p\to\infty} \| f \|_p \le \| f \|_\infty. \tag{0.2}$$

From (0.1) and (0.2) we get the statement.

15.6 We work in the real Lebesgue measure space. For the first case, let $f(x) := x^{-1/q}\chi_{(0,1)}$. Then $\| f \|_q = (\int \left| x^{-1/q}\chi_{(0,1)} \right|^q \, dm)^{1/q} = (\int_{(0,1)} (1/x) \, dm)^{1/q} = \infty$. But $\| f \|_p = (\int \left| x^{-1/q}\chi_{(0,1)} \right|^p \, dm)^{1/p} = (\int_{(0,1)} (1/x)^{p/q} \, dm)^{1/p} < \infty$, so $f \in L^p$ but $f \notin L^q$. For the second case, it is easy to verify (similarly) that$f(x) := x^{-1/p}\chi_{(1,\infty)}$ is an element of L^q but not one of L^p.

5. We prove directly (b) since (a) follows from (b) by setting $\alpha_1 = \cdots = \alpha_n = 1/n$. Let $X := \{x_1, \ldots, x_n\}$ endowed with the σ-algebra of all subsets of X and the finite measure given by $\mu(x_i) := \alpha_i$, $i = 1, \ldots, n$. Define $\varphi(t) := e^t$ and $f : X \to \mathbb{R}$ as $f(x_i) := \log(y_i)$, $i = 1, \ldots, n$ (note that, by hypothesis, we have $y_i > 0$ for all i). By Jensen's inequality applied to the convex function φ and to f, we get

$$\exp\left(\sum_{i=1}^{n} \alpha_i \log(y_i)\right) \le \sum_{i=1}^{n} \alpha_i \exp(\log(y_i))$$

which leads to the desired inequality.

6. This is mostly the proof of Theorem 3.11 in Rudin's *Real and Complex Analysis*. First assume $1 \le p < \infty$ and let $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$. There is a subsequence $\{f_{n_i}\}$ such that

$$\| f_{n_{i+1}} - f_{n_i} \| < 1/2^i. \tag{0.3}$$

Define $g_k := \sum_{i=1}^{k} \left| f_{n_{i+1}} - f_{n_i} \right|$ and $g := \sum_{i=1}^{\infty} \left| f_{n_{i+1}} - f_{n_i} \right|$. By (0.3) and Minkowski's inequality we get that $\| g_k \|_p < 1$. By applying Fatou's lemma to $\{g_k^p\}$ we also get that $\| g \|_p \leq 1$. This means that $g < \infty$ a.e. and implies that the series

$$f := f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i})$$

converges absolutely a.e.. Extend f by making it equal to 0 for all x for which the above series does not converge and notice that $f_{n_k} = f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i})$, hence $f_{n_k} \to f$ a.e. as $k \to \infty$. For $p = \infty$, let $A_k := \{| f_k | > \| f_k \|_\infty\}$, $B_{m,n} := \{| f_m - f_n | > \| f_m - f_n \|_\infty\}$ and $E := A_k \cup B_{m,n}$ for all $k, m, n \in \mathbb{Z}_+$. Then $\mu(E) = 0$ and the sequence $\{f_n\}$ converges uniformly to a bounded function f on E^c. Then let $f = 0$ on E.

7. The second inequality follows easily from the fact that $\sqrt{1 + f^2} \leq 1 + f$ (recall that $f \geq 0$) and that $\mu(X) = 1$. For the first inequality notice that the function $x \mapsto \sqrt{1 + x^2}$ is convex. By Jensen's inequality

$$\sqrt{1 + \left(\int_X f \, d\mu \right)^2} \leq \int_X \sqrt{1 + f^2} \, d\mu$$

which proves the statement.

8. Let m denote the Lebesgue measure on $\mathbb{R}$.

(a) For all $n \in \mathbb{Z}_+$ define

$$f_n(x) := \begin{cases} 1/n, \text{ if } -n^2 \leq x \leq n^2, \\ 0, \text{ otherwise.} \end{cases}$$

Then, the functions f_n are simple and satisfy $\int f_n dm = (1/n)\, m([-n^2, n^2]) = 2n^2/n = 2n \to \infty$ as $n \to \infty$. On the other hand, $\| f \|_\infty = 1/n$ and hence $\| f \|_\infty \to 0$ as $n \to \infty$.

(b) The sequence of simple functions defined as

$$g_n(x) := \begin{cases} n, \text{ if } -1/n^2 \leq x \leq 1/n^2, \\ 0, \text{ otherwise,} \end{cases}$$

for all $n \in \mathbb{Z}_+$, satisfies the requirements.

(c) Define the following sequence of continuous functions on $[0, 1]$:

$$h_n(x) := \begin{cases} n^2 x, \text{ if } 0 \leq x \leq 1/n, \\ 2n - n^2 x, \text{ if } 1/n < x \leq 2/n, \\ 0, \text{ otherwise.} \end{cases}$$

Notice that, for all n, the graph of h_n consists in the two equal sides of an isosceles triangle of height n and with base $[0, 2/n]$, together with the segment $[2/n, 1]$ (where $h_n = 0$). Then, $\int h_n dm = \int_0^{1/n} n^2 x dx + \int_{1/n}^{2/n} (2n - n^2 x) dx = 1/2 + 2 - 2 + 1/2 = 1$. Also, $\| h_n \|_\infty = \max h_n = h_n(1/n) = n$, so $\lim_{n\to\infty} \| h_n \|_\infty = \infty$. Lastly, for every $x \in [0, 1]$ there exists $n_x \in \mathbb{Z}_+$ such that $h_n(x) = 0$ for all $n > n_x$, which implies that $h_n \to 0$ pointwise, as $n \to \infty$.