Solutions for Problem Set 2 MATH 4122/9022

Octavian Mitrea

February 11, 2018

3.8 By the definition of σ-algebra, $\mathcal{B}$ contains all sets of the form $A \cup N$, where $A \in \mathcal{A}$ and $N \in \mathcal{N}$. Therefore, it suffices to show that the family $\mathcal{F} = \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$ is itself a σ-algebra. Clearly, $\varnothing \in \mathcal{F}$. Given $A \cup N \in \mathcal{F}$, let $C \in \mathcal{A}$ be such that $N \subset C$ and $\mu(C) = 0$. Then, $C \setminus N$ (and all its subsets) are in $\mathcal{N}$, and $N^c = C^c \cup (C \setminus N)$, hence $(A \cup N)^c = A^c \cap N^c = (A^c \cap C^c) \cup (A^c \cap (C \setminus N)) \in \mathcal{F}$. Given $\{A_i \cup N_i\}_{i=1}^{\infty} \subset \mathcal{F}$, let $C_i \in \mathcal{A}$ be such that $N_i \subset C_i$ and $\mu(C_i) = 0$, for all i. Then, $\bigcup_i (A_i \cup N_i) = \bigcup_i A_i \cup \bigcup_i N_i \in \mathcal{F}$, since $\bigcup_i N_i \subset \bigcup_i C_i$ and $\mu(\bigcup_i C_i) \leq \sum_i \mu(C_i) = 0$.

The rest of the statements are straightforward, provided $\bar{\mu}$ is well defined. Suppose then that $B = A_1 \cup N_1 = A_2 \cup N_2$, where $A_1, A_2 \in \mathcal{A}$, $N_1 \subset C_1, N_2 \subset C_2$, $C_1, C_2 \in \mathcal{A}$, $\mu(C_1) = \mu(C_2) = 0$. Then, $A_1 \cup C_1 \cup C_2 = A_2 \cup C_1 \cup C_2$, so $\mu(A_1) \leq \mu(A_1 \cup C_1 \cup C_2) = \mu(A_2 \cup C_1 \cup C_2) \leq \mu(A_2) + \mu(C_1) + \mu(C_2) = \mu(A_2)$. Similarly, $\mu(A_2) \leq \mu(A_1)$, so $\mu(A_1) = \mu(A_2)$, as required.

3.9 For every $k \in \mathbb{Z}_+$, let $X_k := (-k, k)$, $\mathcal{C}_k := \{A \in \mathcal{B} | A \subset X_k,\ m(A) = n(A)\}$ and let $\mathcal{A}_k$ be the algebra generated by the open intervals in X_k. Then, $\mathcal{A}$ is the collection of all possible iterations of finite unions and intersections of open intervals and their complements in X_k. Since $m((-k,k)) = n((-k,k)) < \infty$, then, for any $-k \leq a < b \leq k$, $m(X_k \setminus (a,b)) = m((-k,k)) - m((a,b)) = n((-k,k)) - n((a,b)) = n(X_k \setminus (a,b))$. It follows that $\mathcal{A}_k \subset \mathcal{C}_k$. It is straightforward to show that $\mathcal{C}_k$ is a monotone class, and hence $\mathcal{M}(\mathcal{A}_k) \subset \mathcal{C}_k$ (where $\mathcal{M}(C)$ denotes the monotone class generated by C). Then, by the Monotone Class Theorem, $\sigma(\mathcal{A}_k) = \mathcal{M}(\mathcal{A}_k) \subset \mathcal{C}_k$. But $\sigma(\mathcal{A}_k) = \mathcal{B} \cap \mathcal{P}(X_k)$, so $\mathcal{C}_k = \mathcal{B} \cap \mathcal{P}(X_k)$. Let now $A \in \mathcal{B}$ be arbitrary. Then, for every $k \in \mathbb{Z}_+$, $A \cap X_k \in \mathcal{C}_k$ and hence

$$m(A) = \lim_{k\to\infty} m(A \cap X_k) = \lim_{k\to\infty} n(A \cap X_k) = n(A)\,.$$

3.10 For the σ-finite case, let $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ be our measurable space and consider m to be the counting measure and $n := 2m$. Both m and n are σ-finite measures, $m = n$ on $\mathcal{C} := \{A \subset \mathbb{N} \mid A \text{ is infinite}\}$, $\sigma(\mathcal{C}) = \mathcal{P}(\mathbb{N})$, but m and n do not agree on finite subsets of $\mathbb{N}$, so the answer is NO. For the finite case, let $X := \{1,2,3\}, \mathcal{A} := \{\varnothing, \{1\}, \{2,3\}, X\}$. The pair $(X, \mathcal{A})$ is clearly a σ-algebra. Let m be the counting measure, and define

$$n(A) = \begin{cases} 0, & A = \varnothing, \\ 1, & A = \{1\}, \\ 3, & A = \{2,3\}, \\ 4, & A = X. \end{cases}$$

Both m and n are finite measures, $m(\{1\}) = n(\{1\})$, $\sigma(\{1\}) = \mathcal{A}$, but $m(\{2,3\}) \neq n(\{2,3\})$, so the answer is again NO.

4.3 Clearly, $\mu^*(\varnothing) = 0$ and μ^* is monotone. To show the subadditivity, let $\{A_i\}_{i=1}^\infty \subset \mathcal{P}(X)$ be arbitrary. Without loss of generality, we may assume that $\mu^*(\cup_{i=1}^\infty A_i) < \infty$. Let then $\varepsilon > 0$ be arbitrary, and let $C_i \in \mathcal{A}$ be such that $\mu(C_i) \leq \mu^*(A_i) + \frac{\varepsilon}{2^i}$, for $i \geq 1$. Then,

$$\mu^*(\cup_{i=1}^\infty A_i) = \inf\{\mu(B) | (\cup_{i=1}^\infty A_i) \subset B, B \in \mathcal{A}\} \leq \mu(\cup_{i=1}^\infty C_i) \\ \leq \sum_{i=1}^\infty \mu(C_i) \leq \sum_{i=1}^\infty \mu^*(A_i) + \varepsilon\,.$$

Since ε was arbitrary, we get that $\mu^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu^*(A_i)$, which proves that μ^* is an outer measure.

Next, let $A \in \mathcal{A}$ and let $Z \in \mathcal{P}(X)$ be such that $\mu^*(Z) < \infty$. Note that, for arbitrary $\varepsilon > 0$, there exists $B \in \mathcal{A}$ such that $Z \subset B$ and $\mu^*(Z) + \varepsilon \geq \mu^*(B) = \mu(B) = \mu((B \cap A) \cup (B \cap A^c)) = \mu(B \cap A) + \mu(B \cap A^c) \geq \mu(Z \cap A) + \mu(Z \cap A^c)$, where the last inequality follows from the fact that $Z \subset B$. Since this is true for all $\varepsilon > 0$, it follows that $\mu^*(Z) \geq \mu(Z \cap A) + \mu(Z \cap A^c)$. Lastly, it is immediate that $\mu^*(A) = \mu(A)$ if $A \in \mathcal{A}$.

4.15 ($\Rightarrow$) Since A is μ^*-measurable, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, for all $E \subset X$. In particular, for $E = X$ we get the desired formula, since $\mu^*(X) = l(X)$.

($\Leftarrow$) Let $\mathcal{M}$ denote the σ-algebra of μ^*-measurable sets. Suppose that $A \subset X$ satisfies $\mu^*(A) + \mu^*(A^c) = l(X)$. By regularity of μ^*, we can choose $B, C \in \mathcal{M}$ such that $A \subset B$, $A^c \subset C$, $\mu^*(A) = \mu^*(B)$, and $\mu^*(A^c) = \mu^*(C)$. Then, $\mu^*(B) + \mu^*(C) = \mu^*(A) + \mu^*(A^c) = l(X)$. Moreover, since $B \cup C = X$ and $\mu^*|_{\mathcal{M}}$ is a measure, we have $\mu^*(B \setminus C) + \mu^*(C) = \mu^*(B \cup C) = l(X) = \mu^*(B) + \mu^*(C)$, hence $\mu^*(B \setminus C) = \mu^*(B)$. Similarly, $\mu^*(C \setminus B) = \mu^*(C)$, and consequently, $\mu^*(B \cap C) = 0$.

Let now $E \subset X$ and $\varepsilon > 0$ be arbitrary. Let $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ be any family such that $E \subset \bigcup_i A_i$ and $\sum_{i=1}^{\infty} l(A_i) \leq \mu^*(E) + \varepsilon$. Then,

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E \cap B) + \mu^*(E \cap C) \leq \sum_{i=1}^{\infty} \mu^*(A_i \cap B) + \sum_{i=1}^{\infty} \mu^*(A_i \cap C)$$

$$= \sum_{i=1}^{\infty} [\mu^*(A_i \cap (B \setminus C)) + \mu^*(A_i \cap (B \cap C))] + \sum_{i=1}^{\infty} [\mu^*(A_i \cap (C \setminus B)) + \underbrace{\mu^*(A_i \cap (B \cap C))}_{=0}]$$

$$= \sum_{i=1}^{\infty} [\mu^*(A_i \cap (B \setminus C)) + \mu^*(A_i \cap (B \cap C)) + \mu^*(A_i \cap (C \setminus B))] = \sum_{i=1}^{\infty} l(A_i) \leq \mu^*(E) + \varepsilon.$$

Since ε was arbitrary, it follows that $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$.

Problem 4

(a) ($\Rightarrow$) The Caratheodory construction implies that $(\alpha_c)^0$ is regular, hence α is regular. ($\Leftarrow$) Let $\mathcal{M}$ denote the σ-algebra of α-measurable sets. For $Y \in \mathcal{P}(X)$, define $S_Y = \{A \in \mathcal{M} : Y \subset A\}$. Then, $(\alpha_c)^0(Y) = \inf\{\alpha(A) | A \in S_Y\}$. By monotonicity, $\alpha(Y) \leq \alpha(A)$, for every $A \supset Y$, so $\alpha(Y) \leq (\alpha_c)^0(Y)$. On the other hand, since α is regular, there exists $B \in S_Y$ such that $\alpha(Y) = \alpha(B)$, hence $\alpha(Y) \geq \inf\{\alpha(A) | A \in S_Y\}$. The result follows.

(b) ($\Rightarrow$) Trivial, because μ^0 is regular. ($\Leftarrow$) We have $\mu^0 = (\gamma_c)^0$ and by (a), since γ is regular, it follows that $(\gamma_c)^0 = \gamma$, hence the result.

(c) Let $\mathcal{M}$ be the σ-algebra on which μ is defined, and let $\mathcal{M}^0$ be the σ-algebra of μ^0-measurable sets. By the Caratheodory Extension Theorem, μ^0 is a regular outer measure which coincides with μ on $\mathcal{M}$, and $\mathcal{M}^0 \supset \mathcal{M}$. To complete the proof, it thus remains to show that $\mathcal{M}^0 \subset \mathcal{M}$ (i.e., that the domains of μ and $(\mu^0)_c$ coincide).

Pick $A \in \mathcal{M}^0$. Suppose first that $\mu^0(X) = \mu(X) < \infty$. As usual, for every $k \in \mathbb{Z}_+$, we can choose a collection $\{A_{ik}\}_{i=1}^{\infty} \subset \mathcal{M}$ such that $A \subset \bigcup_i A_{ik}$ and $\sum_i \mu(A_{ik}) \leq \mu^0(A) + \frac{1}{k}$. Then, setting $B_k = \bigcup_k A_{ik}$ for all k, and $B = \bigcap_k B_k$, we get that $B \in \mathcal{M}$, $B \supset A$ and $\mu(B) = \lim_{k \to \infty} \mu(B_k) = \mu^0(A)$, where the first equality follows from $\mu(B_1) \leq \mu^0(A) \leq \mu^0(X) < \infty$. Now, the set $C := B \setminus A$ is μ^0-measurable and a μ^0-null set, hence also a μ-null set (by definition of μ^0). By completeness of μ, we have $C \in \mathcal{M}$, and hence $A = B \cup C \in \mathcal{M}$.

In the general case, by σ-finiteness of μ, we can write $X = \bigcup_k X_k$ for some $\{X_k\}_k \subset \mathcal{M}$ with $\mu(X_k) < \infty$ for each k. Set $A_k := A \cap X_k$. Then, for all k, $A_k \in \mathcal{M}$ by the above argument, and hence $A = \bigcup_k A_k \in \mathcal{M}$.

(d) A direct consequence of (a) and the fact that μ^0 is regular.