Solutions for Problem Set 3 MATH 4122/9022

Octavian Mitrea

February 15, 2018

4.2 Let $A \subset \mathbb{R}^n$, $n \in \mathbb{Z}_+$ and let $\varepsilon > 0$. We show first that there exist $G, F \subset \mathbb{R}$, where G is open and F is closed, such that $F \subset A \subset G$ and

$$m(G \setminus A) < \varepsilon/2, \;\; m(A \setminus F) < \varepsilon/2. \tag{0.1}$$

We know that $m(A) = \inf\{m(U) \,|\, A \subset U, U \text{ open in } \mathbb{R}^n\}$, hence there exists an open set $G \subset \mathbb{R}^n$ such that $A \subset G$ and $m(G) < m(A) + \varepsilon/2$. Since $m(G) < \infty$, we have $m(G \setminus A) = m(G) - m(G \cap A) = m(G) - m(A) < \varepsilon/2$, which proves the first part of (0.1). Next, since A^c is also Lebesgue-measurable, by what we proved above, there exists an open set $G \subset \mathbb{R}^n$, $A^c \subset G$, such that $m(G \setminus A^c) < \varepsilon/2$. Then $F := G^c$ is closed, $F \subset A$ and the second part of (0.1) is satisfied. So, since $G \setminus F = (G \setminus A) \cup (A \setminus F)$ and $(G \setminus A) \cap (A \setminus F) = \varnothing$, we have $m(G \setminus F) = m(G \setminus A) + m(A \setminus F) < \varepsilon$.

4.5 Let $A \subset \mathbb{R}^n$ be Lebesgue-measurable. By definition, $m(A) = \inf\{\sum_{k=0}^{\infty} m(R_k) \,|\, A \subset \cup_{k=0}^{\infty} R_k,$ where all R_k are rectangles in $\mathbb{R}^n\}$. It is clear that $m(R_k)$ is invariant under translations, $m(R_k) = m(R_k + x), \forall k \geq 0$, therefore $m(A + x) = m(A)$. Next, note that if R is a rectangle in $\mathbb{R}^n$ then $m(cR) = c^n m(R)$, $\forall c > 0$, because we stretch each of the n sides of R by the same factor of c. This implies the result.

4.6 (1) It is straightforward to see that $B = \displaystyle\bigcap_{N \geq 1} \bigcup_{n \geq N} A_n$. Therefore, B is Lebesgue measurable as a countable intersection of countable unions of Lebesgue measurable sets.

(2) For every $k \in \mathbb{Z}_+$, let $B_k = \displaystyle\bigcap_{N=1}^{k} \bigcup_{n \geq N} A_n$. Then, $\{B_k\}_{k=1}^{\infty}$ forms a decreasing sequence such that $B_k \downarrow B$. Since $B_k \subset [0, 1]$, we have that $m(B_k) < \infty$ for all $k \geq 1$, and hence $m(B) = \lim_{k \to \infty} m(B_k)$. By construction, $A_k \subset B_k$ for all $k \geq 1$, hence $m(B_k) > \delta$, which implies that $m(B) \geq \delta$.

(3) Let $\varepsilon > 0$ be arbitrary. Since the series $\sum_{n=1}^{\infty} m(A_n)$ of non-negative real numbers is convergent, one can choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} m(A_n) < \varepsilon$.

Since every point $x \in B$ belongs to infinitely many of the A_n, there exists some $n(x) > N$ such that $x \in A_{n(x)}$, hence $B \subset \cup_{n \geq N} A_n$. It follows that $m(B) \leq m(\bigcup_{n \geq N} A_n) \leq \sum_{n=N}^{\infty} m(A_n) < \varepsilon$, which proves the statement, because the ε was arbitrary.

(4) Define $A_n := [0, 1/n]$, $n \in \mathbb{Z}_+$. Then all A_n are Lebesgue measurable subsets of $[0,1]$ of measure $m(A_n) = 1/n > 0$, so $\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} 1/n = \infty$. On the other hand, since $1/n \to 0$, it follows that the only point in $[0,1]$ that is in infinitely many sets A_n is 0, hence $B = \{0\}$ and $m(B) = 0$.

4.7 Let $\varepsilon \in (0,1)$ and define $E := (0,\varepsilon) \cup Q_{[\varepsilon,1]}$, where $Q_{[\varepsilon,1]}$ is the set of all rational numbers in $[\varepsilon, 1]$. Then, the closure of E is $[0,1]$, because $Q_{[\varepsilon,1]}$ is dense in $[\varepsilon,1]$ and $m(E) = m((0,\varepsilon)) + m(Q_{[\varepsilon,0]}) = \varepsilon$, because $(0,\varepsilon)$ and $Q_{[\varepsilon,0]}$ are disjoint and $Q_{[\varepsilon,0]}$ is a null set.

4.8 Let $F \subset [0,1]$ be closed and define $Q_F := \{q_1, q_2, \dots\}$ to be an enumeration of the rational numbers in F, which of course satisfies $\overline{Q_F} = F$. For every Borel set $B \subset [0,1]$, define

$$\mu(B) = \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{q_n}(B),$$

where δ_x represents the point mass measure corresponding to $x \in \mathbb{R}$. It is immediate to show that μ is a measure on $[0,1]$. For every Borel set $B \subset [0,1]$ we have $\mu(B) \leq \mu([0,1]) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, hence μ is finite. Also, $\mu(F^c) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{q_n}(F^c) = 0$. Because Q_F is dense in F, then F is the smallest set (with respect to inclusion) with the above properties. Indeed, if $F' \subset F$ is a subset of F satisfying the same properties, then there is an open subset $U \subset F \setminus F'$ and a rational number $q \in U$. But $\mu(\{q\}) > 0$, which is a contradiction.

4.9 Consider the elements of $[0,1]$ in their ternary expansion $a = 0.a_1a_2\dots$, where $a_j \in \{0,1,2\}, \forall j \in \mathbb{N}$. For all $k \in \mathbb{N}$ define $A_k := \{0.a_1a_2 \dots \mid a_k = 1\}$. For all $k \in \mathbb{N}$, A_k is a union of 3^{k-1} intervals of length $\dfrac{1}{3^k}$ each. Hence A_k is Lebesgue measurable and $m(A_k) = \dfrac{1}{3} > 0$, for all $k \in \mathbb{N}$. For any $j \neq k$, the symmetric difference of A_j and A_k is the union of some nontrivial intervals, so $m(A_j \Delta A_k) > 0$. Lastly, assuming w.l.o.g. that $j < k$, the intersection $A_j \cap A_k$ consists of disjoint intervals whose total length is $\dfrac{1}{3}$ of $m(A_j)$, hence $m(A_j \cap A_k) = \dfrac{1}{3} m(A_j) = m(A_k)m(A_j)$.

4.10 We prove this by contradiction and show that if $m(A) > 0$ then there exists an interval I for which $m(A \cap I) > (1-\varepsilon)m(I)$. Let us assume first that the result is true for Borel sets of finite measure. Then, if $A \subset \mathbb{R}$ is a Borel set such that $m(A) = \infty$, since m is σ-finite, there exists a Borel set $A' \subset A$ such that $m(A') < \infty$. By applying the result for the finite case, the statement follows. Suppose now that $m(A) < \infty$ and let $\varepsilon > 0$. By the first part of the proof of Exercise 4, there exists an open set $U \subset \mathbb{R}, A \subset U$, such that $m(U) < m(A) + \frac{\varepsilon}{1-\varepsilon} m(A) = \frac{1}{1-\varepsilon} m(A) < \infty$. Since U is open, there exists a family of pairwise disjoint open intervals $\{I_n\}_{n=1}^{\infty}$ such that $U = \cup_{n=1}^{\infty} I_n$, so $m(U) = \sum_{n=1}^{\infty} m(I_n)$. Since $A \subset U$, we have $m(A) = m(A \cap U) = m(A \cap \cup_{n=1}^{\infty} I_n) = \sum_{n=1}^{\infty} m(A \cap I_n)$. Using all of the above, we get that $\sum_{n=1}^{\infty} m(I_n) < \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} m(A \cap I_n)$. This means that there exists $n_0 \in \mathbb{N}$ such that $m(I_{n_0}) < \frac{1}{1-\varepsilon} m(A \cap I_{n_0})$, which is in contradiction with the hypothesis of our statement.

4.11 Let $f : \mathbb{R} \to \mathbb{R}$ be given as $f(x) = m((A + x) \cap A)$. If $x_n \to x$ then clearly $A + x_n \to A + x$, hence $A_n := (A + x_n) \cap A \to (A + x) \cap A$. Suppose first that $x > 0$. It is easy to see that, if $x_n \to x^-$ we have $A_n \downarrow (A + x) \cap A$ and if $x_n \to x^+$, then $A_n \uparrow (A + x) \cap A$. By Proposition 3.5, it follows that $m(A_n) \to m((A + x) \cap A)$ which proves that f is continuous in $(0, \infty)$. Similar steps show that f is continuous in $(-\infty, 0)$ and at 0, hence f is continuous in $\mathbb{R}$. Since $f(0) = m(A) > 0$, by continuity, there exists an interval $0 \in I \subset \mathbb{R}$ such that $f(x) = m((A + x) \cap A) > 0$, for all $x \in I$. It follows that for every $x \in I$, the set $(A + x) \cap A$ is nonempty. If, for a fixed $x \in I$, we choose an element $a \in (A + x) \cap A$, then $a = b + x$ for some $b \in A$ and, since also $a \in A$, it follows that $x \in B$. This shows that $0 \in I \subset B$, which proves the statement.

4.12 (This follows Rudin's proof published in *The American Mathematical Monthly*, Vol 90 No.1, 1983) Let $I = [0, 1]$ and let CTDP stand for a compact, totally disconnected subset of I of positive measure. Let $\{I_n\}$ be an enumeration of all closed intervals in I with rational endpoints.

Lemma 0.1. *Every closed interval I contains a CTDP.*

Proof of Lemma. We do it for $I = [0, 1]$; the procedure can be adapted to any closed interval I. Let $Q_{[0,1]} := \{q_1, q_2, \dots\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. For each $q_k \in Q_{[0,1]}$, put $U_k := \left(q_k - \frac{1}{2^{k+2}}, q_k + \frac{1}{2^{k+2}} \right)$ and let $U := \cup_{k=1}^{\infty} U_k$, $K := [0, 1] \setminus U$. Since U is open and $[0, 1]$ is compact, K is compact. Also, $m(U) \leq \sum_{k=1}^{\infty} m(U_k) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}$. This means that $m(K) = m([0, 1]) - m(U \cap [0, 1]) \geq \frac{1}{2} > 0$. ∎

Continuing with the solution, construct sequences $\{A_n\}, \{B_n\}$ of CTDP's by using the above lemma, as follows: start with disjoint CTDP's $A_1, B_1 \subset I_1$ and let $C_2 := A_1 \cup B_2$, which is again CTDP. Then, $I_2 \setminus C_2$ contains a nonempty closed interval (because C_2 is totally disconnected), say J, which again contains two disjoint CTDP's, A_2, B_2. Continue the process for all $n \in \mathbb{N}$ and let $A := \cup_{n=1}^{\infty} A_n$. If $\emptyset \neq U \subset I$ is open, then there exists $n \in \mathbb{N}$ such that $I_n \subset U$, hence $A_n, B_n \subset U$. So, $0 < m(A_n) \leq m(A \cap U) < m(A \cap U) + m(B_n) \leq m(U)$, where the last inequality follows from the fact that $A \cap B_n = \emptyset$, and this concludes the proof.