Solutions for Problem Set 4 MATH 4122/9022

Octavian Mitrea

February 22, 2018

4.13 Note that N is a Vitali set, same as the set which is the subject of the last exercise in this problem set. Suppose $m(A) > 0$ and observe that for every $p \neq q \in \mathbb{Q} \cap [0,1]$ the sets $p + A \subset p + N$ and $q + A \subset q + N$ are disjoint because $p + N$ and $q + N$ are (recall the construction of a Vitali set). On the one hand, $r + A \subset [0,2]$ for all $r \in [0,1]$, hence $m(\bigcup_{p \in \mathbb{Q} \cap [0,1]} (p+A)) \leq 2$. On the other hand, $m(r + A) = m(A)$ for all $r \in [0,1]$ so $m(\bigcup_{p \in \mathbb{Q} \cap [0,1]} (p + A)) = \sum_{p \in \mathbb{Q} \cap [0,1]} m(p + A) = \sum_{p \in \mathbb{Q} \cap [0,1]} m(A) = \infty$ because all sets under the union are pairwise disjoint and we also assumed $m(A) > 0$. This leads to a contradiction.

4.14 Extend the construction of Vitali sets to the real line: for any $x, y \in \mathbb{R}$, let $x \sim y$ be the equivalence relation defined by the condition $x - y \in \mathbb{Q}$ and let V be a set given by selecting exactly one representative from every class of equivalence. Then, $\mathbb{R} = \cup_{q \in \mathbb{Q}} (q + V)$ where all the sets under the union symbol are pairwise disjoint. It follows that $A = \cup_{q \in \mathbb{Q}} [A \cap (q + V)]$. If there exists $q \in \mathbb{Q}$ such that $A \cap (q + A)$ is non-measurable then we are done. So, suppose that all $A_q := A \cap (q + V)$ are measurable, $q \in \mathbb{Q}$. Observe that $A_q - A_q \subset V - V$ and $(V - V) \cap (\mathbb{Q} \setminus \{0\}) = \varnothing$, because any two different elements of V are not equivalent. It follows that $A_q - A_q$ does not contain any open interval centered at the origin. Since A_q is measurable, by Steinhaus theorem (Exercise 4.11, Problem Set 3) we have $m(A_q) = 0$, for all $q \in \mathbb{Q}$. It follows that $0 < m(A) = m(\cup_{q \in \mathbb{Q}} A_q) = \sum_{q \in \mathbb{Q}} m(A_q) = 0$, which is a contradiction. This proves that there must be a $q \in \mathbb{Q}$ such that $A \cap (q + A)$ is non-measurable, which proves the statement.

Remark 0.1. The version of Steinhaus theorem presented in Exercise 4.11 in the textbook is stated for A being Borel measurable. In fact the theorem is true for any Lebesgue measurable set A. The proof included in the posted solutions for Problem Set 3 was done for this more general case.

4.15 Addressed in the solutions for Problem Set 2.

4.16 (1) Let $X = \mathbb{R}$ and define

$$\mu^*(A) = \begin{cases} 0, & A \text{ is countable,} \\ 1, & A \text{ and } A^c \text{ are uncountable,} \\ 2, & A^c \text{ is countable.} \end{cases}$$

It is easy to see that μ^* is well defined and the verification of μ^* being an outer measure is a just a routine exercise. For all $n \in \mathbb{Z}_+$ define $A_n := [-n, n], B := [n, \infty)$. Then $A_n \uparrow \mathbb{R}$ and $B_n \downarrow \varnothing$. Thus, $\mu^*(A_n) = 1, \forall n \in \mathbb{Z}_+$ but $\mu^*(\mathbb{R}) = 2$ because its complement is $\varnothing$ which is countable. Hence $\mu^*(A_n)$ does not converge to $\mu^*(\mathbb{R})$. Also, $\mu^*(B_n) = 1$ but $\mu^*(\varnothing) = 0$ which proves that $\mu^*(B_n)$ does not converge to $\mu^*(\varnothing)$.

(2) First we prove that μ^* is *regular*, i.e. for every $A \subset X$ there exists $B \in \mathcal{A}$ such that $A \subset B$ and $\mu^*(A) = \mu^*(B)$. By the definition of μ^*, for every $n > 0$, there exists $B_n \in \mathcal{A}$ s.t. $A \subset B_n$ and $\mu(B_n) \leq \mu^*(A) + 1/n$. Define $B := \cap_n B_n$, which is μ-measurable, so $\mu^*(B) \leq \mu^*(B_n) \leq \mu^*(A) + 1/n, \forall n > 0$, since μ^* restricts to μ on $\mathcal{A}$. This means that $\mu^*(B) \leq \mu^*(A)$. On the other hand, $A \subset B_n$ for all $n > 0$, so $A \subset B$, hence $\mu^*(A) \leq \mu^*(B)$ which proves that in fact $\mu^*(A) = \mu^*(B)$.

To prove the statement, first note that $\mu^*(A) \geq \mu^*(A_n)$, hence $\lim\limits_{n\to\infty} \mu^*(A_n)$ exists and $\mu^*(A) \geq \lim\limits_{n\to\infty} \mu^*(A_n)$. Therefore, it suffices to show the converse inequality. Let $B_n \in \mathcal{A}$ be the sets that result from the regularity of μ^*: $A_n \subset B_n$, $\mu(B_n) = \mu^*(A_n)$, for all $n > 0$. Define the sets $C_n := \bigcap\limits_{k=n}^{\infty} B_k$, for all $n > 0$, which satisfy $A_n \subset C_n$ and $C_n \subset C_{n+1}$, for all $n > 0$. Let $C := \bigcup\limits_{n=1}^{\infty} C_n$. Clearly, $A \subset C$. It follows that, for every $n > 0$, $\mu^*(A_n) = \mu(B_n) \geq \mu(C_n)$ and by taking the limit, $\lim\limits_{n\to\infty} \mu^*(A_n) \geq \lim\limits_{n\to\infty} \mu(C_n) = \mu(C) \geq \mu^*(A)$, which proves the statement.

4.17 We have $B = \bigcup\limits_{x \in A} [x-1, x+1] = \left(\bigcup\limits_{x \in A} (x-1, x+1) \right) \cup \left(\bigcup\limits_{x \in A} \{x-1\} \right) \cup \left(\bigcup\limits_{x \in A} \{x+1\} \right)$. The set $\bigcup\limits_{x \in A} (x-1, x+1)$ is open, hence Lebesgue-measurable. Also, it is easy to check that $\bigcup\limits_{x \in A} \{x-1\} = -1 + A$ and $\bigcup\limits_{x \in A} \{x+1\} = 1 + A$, which are both Lebesgue-measurable. It follows that B is measurable as the union of three measurable sets.

4.18 To prove the statement is the same as proving that, if $m(A) = 0$, then there exists $c \in \mathbb{R}$ such that $(A + c) \cap \mathbb{Q} = \varnothing$. Let $B := \bigcup_{q\in\mathbb{Q}} (q + A)$, which has measure 0, since $m(q + A) = m(A) = 0$, for all $q \in \mathbb{Q}$. Then, by construction, B is invariant with respect to translations by rationals, i.e. $B = p + B$ for all $p \in \mathbb{Q}$.

Next, we show that, for any $r \in \mathbb{R}$, $(r + B) \cap \mathbb{Q} \neq \varnothing$ iff $\mathbb{Q} \subset r + B$. Again, one direction is trivial, so we prove the other. By hypothesis, there exists $b \in B$ s.t. $q := r + b \in \mathbb{Q}$. Let $p \in \mathbb{Q}$ be arbitrarily fixed. Then, $p = (p - q) + q = (p-q)+r+b = r+(p-q)+b \in r+(p-q)+B = r+B$. Since p was arbitrarily fixed, it follows that $\mathbb{Q} \subset r + B$.

To end the proof, note that there exists $r_0 \in \mathbb{R}$ s.t. $0 \notin r_0+B$, becuse otherwise, for every $r \in \mathbb{R}$ we would have $-r \in B$, i.e. $\mathbb{R} \subset B$ which is impossible, since $m(B) = 0$. So, $\mathbb{Q} \not\subset r_0 + B$, hence $(r_0 + B) \cap \mathbb{Q} = \varnothing$, which proves the result.

2. (*a*) Let V be a Vitali set and suppose that $m^*(V) = 0$. By the construction of V, as a Vitali set, for every $r \in [0, 1]$ there exists $v \in V$ s.t. $r \in v+(\mathbb{Q}\cap[0, 1])$, i.e. $r = v + q$ for some $q \in \mathbb{Q} \cap [0, 1]$, hence $r \in q + V$. It follows that $[0, 1] \subset \bigcup_{q\in\mathbb{Q}\cap[0,1]} (q + V)$. So,

$$1 = m([0, 1]) = m^*([0, 1]) \leq m^*\Big(\bigcup_{q\in\mathbb{Q}\cap[0,1]} (q + V)\Big) \leq \sum_{q\in\mathbb{Q}\cap[0,1]} m^*(q + V) = \sum_{q\in\mathbb{Q}\cap[0,1]} m^*(V) = 0\,,$$

which is a contradiction.

(*b*) If $\varepsilon > 1$ then, trivially $m^*(V) \leq m^*([0, 1]) = m([0, 1]) = 1 < \varepsilon$, for any Vitali set $V \subset [0, 1]$. Suppose that $0 < \varepsilon \leq 1$. Let $W \subset [0, 1]$ be a Vitali set. For each $w \in W$ choose $q \in \mathbb{Q}$ s.t. $v := w - q < \varepsilon$, which is possible since $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$. The set V of all such v's is again a Vitali set: $w - v = q \in \mathbb{Q}$, hence $v \sim w$ and, $w_1 \not\sim w_2$ implies $v_1 \not\sim v_2$, where $v_i := w_i - q_i, i = 1, 2$. Since $v < \varepsilon$ for all $v \in V$, we have $V \subset [0, \varepsilon]$, hence $m^*(V) < \varepsilon$.