Problem Set 5
February 14, 2018.

0. Read (and comprehend) the proof of *Lusin's theorem* (Thm. 5.15 in the text).

1. Let $(X, \mathcal{M})$ be a measurable space and let $f : X \to \mathbb{R}$.

(a) Prove that f is a measurable function if and only if $f^{-1}(B) \in \mathcal{M}$ for every Borel measurable $B \subset \mathbb{R}$.

(b) Prove that f is a simple function if and only if $f(X)$ is a finite set and f is measurable.

2. Exercises 5.1–5.3.

3. Exercise 5.5.

4. Exercises 5.7–5.9.

5. Recall that, for $p \in [0, \infty)$, the *p-dimensional Hausdorff outer measure* on a metric space (X, d) is defined by

$$h^{p*}(Y) = \sup_{\delta>0} \inf \left\{ \sum_{k=1}^{\infty} h^p(Y_k) \mid \{Y_k\}_{k=1}^{\infty} \subset \mathcal{P}(X), Y \subset \bigcup_k Y_k, \operatorname{diam}(Y_k) < \delta, \forall k \right\} ,$$

for $Y \subset X$, where $h^p(Z) = \frac{\alpha(p)}{2^p} \operatorname{diam}(Z)^p$ for $Z \neq \varnothing$, $h^p(\varnothing) = 0$, and $\alpha(p) = \frac{(\Gamma(\frac{1}{2}))^p}{\Gamma(\frac{p}{2}+1)}$.

Let h^p denote the measure obtained from h^{p*} by restricting to the σ-algebra of h^{p*}-measurable sets. One can show (the Key Lemma in class) that if $s \in (0, \infty)$ is such that $h^s(X) < \infty$ then $h^p(X) = 0$ for all $p > s$, and define the *Hausdorff dimension* of X by

$$\dim_{\mathcal{H}}(X) = \inf\{p > 0 : h^p(X) = 0\} .$$

(a) For $\delta > 0$, denote by h^p_δ the function on $\mathcal{P}(X)$ defined by

$$h^p_\delta(Y) = \inf \left\{ \sum_{k=1}^{\infty} h^p(Y_k) \mid \{Y_k\}_{k=1}^{\infty} \subset \mathcal{P}(X), Y \subset \bigcup_k Y_k, \operatorname{diam}(Y_k) < \delta, \forall k \right\} .$$

By the Caratheodory Extension Theorem, h^p_δ is an outer measure on X.
Prove that, for every $Y \subset X$, the function $(0, \infty) \ni \delta \mapsto h^p_\delta(Y) \in [0, \infty]$ is decreasing.
Prove that h^{p*} is an outer measure on X.

(b) Let $C \subset [0, 1]$ be the ternary Cantor set. Complete the proof (sketched in class) of the fact that

$$\dim_{\mathcal{H}}(C) = \log_3 2.$$

(c) Let $S \subset [0, 1]^2 \subset \mathbb{R}^2$ denote the *Sierpiński carpet*; i.e., $S = \bigcap_{i=1}^{\infty} S_i$, where

$$S_1 = [0, 1]^2 \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^2 ,$$

and S_{i+1} is obtained from S_i by removing the open middle ninth square from each of the 8^i congruent squares of area $\frac{1}{9^i}$ that S_i is composed of.
Let m denote the Lebesgue measure in $\mathbb{R}^2$. Find $m(S)$. Find $\dim_{\mathcal{H}}(S)$. Justify your answers.