# Solutions for Problem Set 5 MATH 4122/9022

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**1.** (a) Suppose that $f^{-1}(B) \in \mathcal{M}$ for all Borel measurable sets $B \subset \mathbb{R}$, in particular, $f^{-1}((a,\infty)) \subset \mathcal{M}$ for any $a \in \mathbb{R}$, since $(a,\infty)$ is Borel, hence $f$ s measurable. The converse is Proposition 5.11 in the textbook.

(b) For every $f = \sum_{j=1}^{m} c_j \chi_{F_j}$, $F_j \in \mathcal{M}$, there are pairwise disjoint $E_1, \ldots, E_n \in \mathcal{M}$ (for some $n \geq m$) and $a_1, \ldots, a_n$, such that $f = \sum_{i=1}^{n} a_i \chi_{E_i}$. If $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ and the $E_i \in \mathcal{M}$ are pairwise disjoint, clearly $f(X) = \{a_1, \ldots, a_n\}$ is finite. If $a \in \mathbb{R}$ then $f^{-1}((a,\infty)) = \cup_{a_i > a} E_i$, which is measurable. Conversely, let $f(X) = \{a_1, \ldots, a_n\}$. Then, $E_i := f^{-1}(\{a_i\}) = f^{-1}((-\infty, a] \cap [a,\infty)) = f^{-1}((-\infty, a]) \cap f^{-1}([a,\infty))$ is measurable by Proposition 5.5. By construction, $f = \sum_{i=1}^{n} a_i \chi_{E_i}$, hence $f$ is simple.

**5.1** Let $a \in \mathbb{R}$ and let $\{r_n\}$ be sequence of rational numbers converging to $a$ from the right, which we can always find since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, $f^{-1}((a,\infty)) = f^{-1}(\cup_{n=1}^{\infty}(r_n,\infty)) = \cup_{n=1}^{\infty} f^{-1}((r_n,\infty))$ which is a countable union of measurable sets, hence it is measurable.

**5.2** For each $x \in (0,1)$, denote by $r_x > 0$ and $g_x$ the positive constant and the Borel measurable function given by the hypothesis. It follows that $(0,1) = \cup_{x\in(0,1)}[(x - r_x, x + r_x) \cap (0,1)]$. As a topological space (with the subspace topology), $(0,1)$ is second countable, hence there exists a countable subcover $\{(x_n - r_n, x_n + r_n)\}$ such that $(0,1) = \cup_{n=1}^{\infty}[(x_n - r_n, x_n + r_n) \cap (0,1)]$, where we wrote $r_n$ instead of $r_{x_n}$. Then, for all $a \in \mathbb{R}$, we have $f^{-1}((a,\infty)) = f^{-1}((a,\infty)) \cap \cup_{n=1}^{\infty}[(x_n - r_n, x_n + r_n) \cap (0,1)] = \cup_{n=1}^{\infty}[f^{-1}((a,\infty)) \cap (x_n - r_n, x_n + r_n) \cap (0,1)] = \cup_{n=1}^{\infty}[g_n^{-1}((a,\infty)) \cap (x_n - r_n, x_n + r_n) \cap (0,1)]$ (again, here $g_n := g_{x_n}$) because $f = g_n$ on $(x_n - r_n, x_n + r_n) \cap (0,1)$ by hypothesis. Since the last set is the countable union of Borel measurable sets, it follows that $f^{-1}((a,\infty))$ is Borel measurable.

**5.3** Suppose $f : X \to (0,\infty)$. Since $f > 0$ implies $g > 0$, it suffices to show that $g^{-1}((a,\infty))$ is measurable for all $a > 0$. Indeed, for any $b \leq 0$, we have $g^{-1}((b,\infty)) = g^{-1}((0,\infty)) = f^{-1}((0,\infty))$, which is measurable. So, let $a > 0$. Then $g^{-1}((a,\infty)) = \{x : 1/f(x) > a\} = \{x : f(x) < 1/a\}$ which is measurable, and this proves the statement.

**5.5** It suffices to prove the statement for $f \geq 0$ since $f = f^+ - f^-$, where the functions $f^+(x) := \max\{0, f(x)\}$ and $f^-(x) := \max\{0, -f(x)\}$ are both non-negative and Lebesgue measurable. By Proposition 5.14, there exists a sequence of non-negative simple functions $s_n = \sum_{i=1}^{m_n} a_i^n \chi_{E_i^n}$ increasing to $f$ (where $m_n \in \mathbb{Z}_+$ depends on $n$), where for each $n$, the $E_i^n$ are pairwise disjoint. By Proposition 4.14(4) (which can be easily extended to $\mathbb{R}$), for all $n \in \mathbb{Z}_+$ and all $i = 1, \ldots, m_n$, there exists $D_i^n \subset E_i^n$ such that $D_i^n$ is Borel measurable and $m(E_i^n \setminus D_i^n) = 0$. Define $r_n := \sum_{i=1}^{m_n} a_i^n \chi_{D_i^n}$. Since $\chi_{D_i^n} = \chi_{E_i^n}$ a.e., we have $r_n = s_n$ a.e., for all $n \in \mathbb{Z}_+$. The sequence $(r_n)_{n=1}^{\infty}$ is pointwise increasing and bounded above by $f$, hence $\lim_{n\to\infty} r_n$ exists. Define $g(x) := \lim_{n\to\infty} r_n(x)$. Then, $g$ is Borel measurable and $g = \lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = f$ a.e..

**Remark 0.1.** This exercise can also be proved by making use of Lusin's theorem which is in fact valid on any closed interval of $\mathbb{R}$, in particular on any interval of the form $[n, n+1]$. For any such interval, choose $\varepsilon_m := 1/m$ and consider $F^n := \cup_m F_m^n$, where $F_m^n$ are the closed sets "produced" by Lusin's theorem applied to $[n, n+1]$ and $\varepsilon_m$. Then, define $g$ taking into consideration that $\mathbb{R} = \cup_{n=0}^{\infty}([n, n+1] \cup [-n-1, -n])$.

**5.7** Since $f$ is differentiable, $f$ is continuous on $\mathbb{R}$, hence Borel measurable. Regarding the derivative, for any $x \in \mathbb{R}$, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. In particular, $f'(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x) := \dfrac{f(x+1/n) - f(x)}{1/n}$, $n \in \mathbb{Z}_+$. But $f_n$ is continuous, hence Borel measurable for all $n \in \mathbb{Z}_+$. By Proposition 5.8, it follows that $f'$ is Borel measurable.

**5.8** Let $A \subset [0,1]$ be a Vitali set. For each $\alpha \in A$ define $f_\alpha := \chi_{\{\alpha\}}$ (the characteristic function of the set $\{\alpha\}$) which is non-negative and Lebesgue measurable (since $\{\alpha\}$ is). Then $g(x) := \sup_{\alpha\in A} f_\alpha(x)$ is equal to 1 if $x \in A$ and to 0, if $x \notin A$, i.e. $g = \chi_A$. Clearly $g$ is finite but it is not measurable, since the Vitali set $A$ is not measurable, therefore its characteristic function is not measurable.

**5.9** We can easily see that if $g$ is Borel measurable then $g \circ f$ is Lebesgue measurable, fact that addresses the first two questions (since any continuous function is Borel measurable). Indeed, for $a \in \mathbb{R}$, we have $(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty)))$ and this set is Lebesgue measurable since $g^{-1}((a, \infty))$ is Borel measurable and $f$ is Lebesgue measurable (see the first exercise of this probem set, point (a)). The answer to the last question is negative in general and the following is the standard example, given for $f$ defined on $[0, 1]$, but then the construction can be easily extended to the entire real line. Let $\varphi : [0,1] \to [0,1]$ be the *Cantor-Lebesgue* function defined in Example 4.11 in the textbook and let $h : [0,1] \to [0,2]$ be defined as $h(x) = x + \varphi(x)$. By the properties of $\varphi$, it follows that $h$ is continuous and bijective, with a continuous inverse $f := h^{-1}$. Let $C \subset [0,1]$ be the ternary Cantor set and $m$ the Lebesgue measure on $\mathbb{R}$. Then, $m(h(C)) > 0$ since $\varphi$ is locally constant in the complement of $C$ (constant on each interval in the complement of $C$) and $h$ will map every such interval to an interval of the same measure. By Exercise 4.14 (see Problem set 4), there exists a (Lebesgue) non-measurable subset $A \subset h(C)$. It follows that $f(A) = h^{-1}(A) \subset C$ and, since $m(C) = 0$, the set $f(A)$ is Lebesgue measurable, hence $g := \chi_{f(A)}$ is Lebesgue measurable. However, $(g \circ f)^{-1}(\{1\}) = f^{-1}(\chi_{f(A)}^{-1}(\{1\})) = f^{-1}(f(A)) = A$ which is not Lebesgue measurable, which proves that $g \circ f$ is not Lebesgue measurable (here we used the fact that $\{1\}$ is Borel measurable and applied again Exercise 1.(a)).

**5.** (a) For a given subset $Y \subset X$ and $\delta > 0$, we shall refer to a cover $\{Y_n\}_n$ as a $\delta$-*cover* if $\operatorname{diam}(Y_n) < \delta, \forall n$. Now, let $Y \subset X$ and $0 < \delta_1 < \delta_2$. Then, any $\delta_1$-cover of $Y$ is also a $\delta_2$-cover, which implies that $h^p_{\delta_1}(Y) \geq h^p_{\delta_2}(Y)$, which proves the first statement. Next, since, by Caratheodory construction (see Proposition 4.2 in the textbook) $h^p_\delta$ is an outer measure it is straightforward to show that $h^{p*} := \sup\limits_{\delta>0} h^p_\delta$ is also an outer measure.

(b) This solution follows Example 2.7 in *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed, by *Kenneth Falconer*. The proof is independent of the scale factor $\dfrac{\alpha(p)}{2^p}$ used in the definition of the Hausdorff measure, so we may assume it is equal to 1 for all $p$. Let $C \subset [0,1]$ be the ternary Cantor set. At each step $k \geq 1$ in the construction of $C$ we obtain a set $E_k$ consisting of $2^k$ closed intervals, each of length $1/3^k$, which we shall refer to as $k$-*intervals*. Let $s := \log_3 2$. We show that

$$1/2 \leq h^s(C) \leq 1 \tag{0.1}$$

which implies that $\dim_{\mathcal{H}} C = s = \log_3 2$. The family of intervals $\{E_k\}_k$ is a $1/3^k$-cover of $C$, hence $h^s_{(1/3^k)}(C) \leq 2^k/3^{ks} = 1$. So, if $k \to \infty$ (i.e. $1/3^k \to 0$) we get that $h^s(C) \leq 1$ (which implies that $h^p(C) = 0, \forall p > s$). Next, we show that any cover $\{A_i\}_i$ of $C$ satisfies

$$\sum_i \operatorname{diam}(A_i)^s \geq \frac{1}{2}, \tag{0.2}$$

which would imply the result. In fact, it is enough to assume that $A_i$ are closed intervals and, by the compactness of $C$, that the cover is finite. Since $A_i \subset [0,1], \forall i$, for each $i$, there exists a $k$ such that

$$1/3^{k+1} \leq \operatorname{diam}(A_i) \leq 1/3^k. \tag{0.3}$$

It follows that $A_i$ can intersect at most one $k$-interval, since the distance between any two $k$-intervals is at least $1/3^k$. Also, if $j \geq k$ then $A_i$ intersects at most $2^{j-k} = 2^j(1/3^{sk}) \leq 2^j 3^s \operatorname{diam}(A_i)^s$ $j$-intervals of $E_j$, by (0.3). For $j$ large enough, we have $1/3^{j+1} \leq \operatorname{diam}(A_i)$ for all $i$. Therefore, since the intervals $\{A_i\}_i$ intersect all $2^j$ basic intervals of length $1/3^j$, we get (by counting the intervals) $\sum_i 2^j 3^s \operatorname{diam}(A_i)^s \geq 2^j$, i.e. $\sum_i \operatorname{diam}(A_i)^s \geq 1/3^s = 1/2$, which proves (0.2), hence proving (0.1).

(c) At each step $k \geq 1$ in the construction of $S$ we obtain a set $E_k \supset S$ consisting of $8^k$ closed (solid) squares, each of them having the area equal to $1/9^k$. Therefore, $m(S) = \lim_{k\to\infty} \dfrac{8^k}{9^k} = 0$. It is straightforward to adapt the proof done for the Cantor set to prove that $\dim_{\mathcal{H}} S = \log_3 8$. Indeed, at each step we obtain a cover $\{E_k\}_k$ of $S$ consisting of $8^k$ closed squares (as mentioned before) each with $\operatorname{diam}(E_k) = \sqrt{2}/3^k$. So, by putting $s := \log_3 8$, it follows that $h^s_{\sqrt{2}/3^k}(S) \leq 8^k \cdot \sqrt{2}/3^{ks} = 2^{s/2}$ for all $k$, which implies $h^s(S) \leq 2^{s/2} < \infty$. Also, following steps similar to those in the proof for the Cantor set, one can show that $h^s(S) \geq 2^{\frac{s-2}{2}}$, which proves the statement.