Solutions for Problem Set 6 MATH 4122/9022

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6.1 Let $f = \sum_{i=1}^m a_i \chi_{A_i}$. For each $S \subset \{1, \ldots, m\}$ define $\widetilde{A}_S := (\bigcap_{j \in S} A_j) \cap (\bigcap_{j \notin S} A_j^c)$. It is straightforward to verify that the sets $\{\widetilde{A}_S\}$ form a partition of $\bigcup_{i=1}^m A_i$ and $A_i = \bigcup_{i \in S} \widetilde{A}_S$ (the union here is over all subsets $S \subset \{1, \ldots, m\}$ that contain i). Therefore, $f = \sum_{i=1}^m a_i \chi_{A_i} = \sum_{i=1}^m a_i \sum_{i \in S} \chi_{\widetilde{A}_S} = \sum_S \widetilde{a}_S \chi_{\widetilde{A}_S}$, where $\widetilde{a}_S = \sum_{i \in S} a_i$. It follows that,

$$\int f d\mu = \sum_{i=1}^m a_i \mu(A_i) = \sum_{i=1}^m a_i \sum_{i \in S} \mu(\widetilde{A}_S) = \sum_S \widetilde{a}_S \mu(\widetilde{A}_S).$$

Let $\{c_1, \ldots, c_M\}$ be all the nonzero values taken by f and let $C_k := \{f = c_k\}, k = 1, \ldots, M$. Then $f = \sum_{k=1}^N c_k \chi_{C_k}$. Note that, by construction, each C_k is a union of sets $\widetilde{A}_S$ and each corresponding coefficient $\widetilde{a}_S$ is equal to c_k, so

$$\int f d\mu = \sum_S \widetilde{a}_S \mu(\widetilde{A}_S) = \sum_{k=1}^M c_k \mu(C_k). \tag{0.1}$$

The quantity at the end of equation (0.1) does not depend on the representation of f. Since the representation we worked with was arbitrarily chosen it follows that (0.1) is valid for any representation, which proves that the integral of non-negative simple functions is well defined.

6.2 This exercise can be solved by using the definition of the integral, starting with simple functions, etc.. Here is another solution. First note that, Proposition 6.3(4) implies that, if f, g are integrable such that $f = g$ a.e. with respect to some measure μ, then $\int f d\mu = \int g d\mu$. For our specific case, define the constant function $g(x) = f(y), \forall x \in X$, which satisfies $g = f$ a.e. with respect to δ_y, because $y \in \{f = g\}$. It follows that $\int f d\delta_y = \int g d\delta_y = \int f(y) d\delta_y = f(y) \int d\delta_y = f(y)$.

6.3 Let $\{s_n\}$ be a sequence of nonnegative simple functions such that $s_n \uparrow f$ (which always exists by Proposition 5.14). Let $s_n := \sum_{i=1}^{m_n} a_i^n \chi_{A_i^n}$, $X = \cup_{i=1}^{m_n} A_i^n, \forall n$. If there exists n such that $\mu(A_i^n) = \infty$ and $a_i^n \neq 0$ for some $1 \leq i \leq m_n$, then

$\int_X s_n d\mu = \infty$, hence $\int_X f d\mu = \infty$. Also, $\sum_k f(k) \geq \sum_k s_n(k) \geq \sum_k a_i^n \chi_{A_i^n}(k) = \infty$, so this proves the statement in this case.

Suppose now that for all n and $1 \leq i \leq m_n$, $\mu(A_i^n) < \infty$. For every n we have $\int s_n = \sum_{i=1}^{m_n} a_i^n \mu(A_i^n) = \sum_{i=1}^{m_n} a_i^n |A_i^n|$. On the other hand, $\sum_k s_n(k) = \sum_k \sum_{i=1}^{m_n} a_i^n \chi_{A_i^n}(k) = \sum_{i=1}^{m_n} a_i^n \sum_k \chi_{A_i^n}(k) = \sum_{i=1}^{m_n} a_i^n |A_i^n|$, because $\chi_{A_i^n}(k) = 1$ iff $k \in A_i^n$. Hence, the statement is true for every simple function s_n.

For any n, we have $\int_X f d\mu \geq \int_{\{1,\dots,n\}} f d\mu = \sum_{k=1}^n \int_{\{k\}} f d\mu = \sum_{k=1}^n f(k)\mu(\{k\}) = \sum_{k=1}^n f(k)$, which implies $\int_X f d\mu \geq \sum_k f(k)$. On the other and, $\sum_k s_n(k) \leq \sum_k f(k), \forall n$, hence $\int_X f d\mu = \sup\{\sum_k s_n(k) \,|\, 0 \leq s_n \leq f, s_n\text{-simple function}\} \leq \sum_k f(k)$, which proves the statement.

6.4 Since μ is σ-finite, there exist μ-measurable sets $\{E_i\}$ such that $X = \cup_{i=1}^\infty E_i$ and $\mu(E_i) < \infty$. Let $A_n := \cup_{i=1}^n E_i$. Then $A_n \subset A_{n+1}$ and $\mu(A_n) < \infty$ for all n. Let σ_n be nonnegative simple functions such that $\sigma_n \uparrow f$. Define $s_n := \sigma_n \chi_{A_n}$, which are also nonnegative simple functions and satisfy $\mu(\{s_n \neq 0\}) < \infty$, because $\mu(A_n) < \infty$. Also, $s_n \leq s_{n+1}$ because $\sigma_n \leq \sigma_{n+1}$ and $\chi_{A_n} \leq \chi_{A_{n+1}}$. Since $\sigma_n \uparrow f$ and $\chi_{A_n} \uparrow 1$ it follows that $s_n \uparrow f$.

6.5 Recall that $x \wedge y := \min\{x, y\}$. Let $\{s_k\}$ be a sequence of nonnegative simple functions such that $s_k \uparrow f$. Since every s_k is bounded, it is clear that there exists n_k such that $s_k \wedge n = s_k, \forall n > n_k$. It means that, for every k, $\lim\limits_{n\to\infty} \int (s_k \wedge n) = \lim\limits_{n>n_k, n\to\infty} \int (s_k \wedge n) = \int s_k$. Since $\int f \wedge n \geq \int s_k \wedge n, \forall k$, we have $\lim\limits_{n\to\infty} \int f \wedge n \geq \lim\limits_{n\to\infty} \int s_k \wedge n = \int s_k, \forall k$, which implies $\lim\limits_{n\to\infty} \int f \wedge n \geq \int f$. Clearly $f \wedge n \leq f$, hence $\int (f \wedge n) \leq \int f$, which implies $\lim\limits_{n\to\infty} \int (f \wedge n) \leq \int f$ and this ends the proof.

6.6 It suffices to prove the statement for $f \geq 0$, since otherwise, we can apply the result to $|f|$. For $\varepsilon > 0$ there exists a nonnegative simple function $s := \sum_{i=1}^m a_i \chi_{A_i}$ such that $\int_X f \leq \int_X s + \varepsilon/2$. For any $A \in \mathcal{A}$, $\int_A s = \sum_{i=1}^m a_i \mu(A_i \cap A) \leq \sum_{i=1}^m a_i \mu(A) = \mu(A) \sum_{i=1}^m a_i$. Let $\delta := \varepsilon/(2\sum_{i=1}^m a_i)$. Then, for any $A \in \mathcal{A}$ with $\mu(A) < \delta$ we have $\int_A f = \int_A f - \int_A s + \int_A s \leq (\int_X f - \int_X s) + \int_A s < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

6.7 By hypothesis, for each n there exists $M_n > 0$ such that $|f_n| < M_n$. So, $\int |f_n| d\mu < M_n \int d\mu = M_n \mu(X) < \infty$ so all f_n are integrable. Also, for an arbitrary $\varepsilon > 0$, by the uniform convergence of f_n to f, we have $|f| - |f_n| \leq |f_n - f| < \varepsilon$ for all $n > n_\varepsilon$, for some $n_\varepsilon > 0$. So, $|f| < |f_n| + \varepsilon < M_n + \varepsilon$, hence f is also integrable. Lastly $|\int f_n d\mu - \int f d\mu| \leq \int |f_n - f| d\mu < \varepsilon \int d\mu = \varepsilon \mu(X)$, for all $n > n_\varepsilon$, which proves the statement.

7.3 By the integrability assumption, it suffices to consider the case $f \geq 0$. Suppose first that $A_n \uparrow A$, hence $A = \cup_n A_n$. Then $f_n := f\chi_{A_n}$ form an increasing sequence of non-negative measurable functions, and $f_n \uparrow f$. Thus, by the Monotone Convergence Theorem (MCT), $\int_{A_n} f = \int_A f_n \longrightarrow \int_A f$. For the second part, suppose $A_n \downarrow A$, hence $A = \cap_n A_n$. Then $|\int_{A_n} f - \int_A f| = |\int f\chi_{A_n} - \int f\chi_A| \leq \int |f||\chi_{A_n} - \chi_A|$. We cannot apply MTC directly, because the functions $g_n := |\chi_{A_n} - \chi_A|$, although nonnegative, form a decreasing sequence. However, note that $\int g_1 < \infty$, so if we define $h_n := g_1 - g_n$, then each h_n is nonegative and integrable. Also, the sequence $\{h_n\}$ is increasing and $h_n \uparrow g_1$, because $g_n \downarrow 0$. So, by MTC, $\lim_{n\to\infty} \int h_n = \int \lim_{n\to\infty} h_n$ i.e. $\lim_{n\to\infty} \int (g_1 - g_n) = \int \lim_{n\to\infty}(g_1 - g_n)$, which is the same as $\int g_1 - \lim_{n\to\infty} \int g_n = \int g_1$, hence $\lim_{n\to\infty} \int g_n = 0$. This implies the result.

7.4 (This solution follows the proof of Theorem 1.38 in Rudin's *Real and Complex Analysis*) Let $f := \sum_{n=1}^{\infty} f_n$ and $\varphi := \sum_{n=1}^{\infty} |f_n|$. By Proposition 7.4 (in Bass' textbook) and from the condition $\sum_{n=1}^{\infty} \int |f_n| < \infty$ it follows that $\int \varphi < \infty$. This implies that the set $E := \{\varphi = \infty\}$ has measure zero, hence the given series absolutely converges a.e.. Since $|f| = |\sum_n f_n| \leq \sum_n |f_n| = \varphi$ and $\int \varphi < \infty$, it follows that f is also integrable. Lastly, put $g_n := \sum_{i=1}^{n} f_i$. Then, $|g_n| \leq \varphi$, $g_n \to f$ pointwise and, by the Dominated Convergence Theorem, we get that $\int f = \sum_{n=1}^{\infty} \int f_n$.

7.5 Since $|f_n| \leq g_n$, we also have $|f| \leq g$ a.e., hence $|f_n - f| \leq g_n + g$ a.e.. It follows that the functions $h_n := g_n + g - |f_n - f|$ are non-negative a.e.. By Fatou's lemma, $\liminf_{n\to\infty} \int h_n \geq \int \liminf_{n\to\infty} h_n = 2\int g$ (because $g_n \to g, f_n \to f$ a.e.). On the other hand, $\liminf_{n\to\infty} \int h_n = \liminf_{n\to\infty} \left(\int (g_n + g) - \int |f_n - f| \right) = 2\int g - \limsup_{n\to\infty} \int |f_n - f|$, where we used the condition $\int g_n \to \int g$. It follows that $\limsup_{n\to\infty} \int |f_n - f| \leq 0$. But $\limsup_{n\to\infty} \int |f_n - f| \geq \liminf_{n\to\infty} \int |f_n - f| \geq \liminf_{n\to\infty} \left| \int (f_n - f) \right| \geq 0$, hence $\lim_{n\to\infty} \left| \int (f_n - f) \right|$ exists and it equals 0. Therefore $\lim_{n\to\infty} \int (f_n - f)$ exists and it is equal to 0, which proves the statement.

7.7 The solution makes use of Exercise 7.8 which we prove below. For any $A \in \mathcal{A}$ we have $\left| \int_A f_n - \int_A f \right| = \left| \int_A (f_n - f) \right| = \left| \int_X (f_n - f)\chi_A \right| \leq \int_X |f_n - f||\chi_A| \leq \int_X |f_n - f|$. We are exactly under the conditions of Exercise 7.8 (because f_n, f

are non-negative), hence $\int_X |f_n - f| \to 0$, which implies the result.

7.8 For each n define $g_n := |f_n| + |f| - |f_n - f|$, so $g_n \geq 0$ and $g_n \to 2|f|$. By Fatou's lemma, $\liminf_{n\to\infty} \displaystyle\int g_n \geq 2 \int |f|$. Following similar steps as in the solution for Exercise 7.5, applied to our g_n, and using the fact that by hypothesis $\int |f_n| \to \int |f|$, we prove that $\limsup_{n\to\infty} \displaystyle\int |f_n - f| \leq 0$. So, $0 \geq \limsup_{n\to\infty} \displaystyle\int |f_n - f| \geq \liminf_{n\to\infty} \int |f_n - f| \geq \liminf_{n\to\infty} \left| \int (f_n - f) \right| \geq 0$ and the result follows.

7.9 Fix $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary. By Problem 6.6 above, one can choose $\delta > 0$ such that $\int_I |f| < \varepsilon$ for every interval $I \subset \mathbb{R}$ of length less than δ. Then, for any x with $|x - x_0| < \delta$, we have $|F(x) - F(x_0)| = \left| \int_a^x f - \int_a^{x_0} f \right| = \left| \int_{x_0}^x f \right| \leq \int_{I_{x,x_0}} |f| < \varepsilon$, where I_{x,x_0} denotes the interval between x and x_0. This proves that F is continuous at x_0.