Solutions for Problem Set 7 MATH 4122/9022

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7.11 First, note that $\log(1+x) \geq x - x^2/2$, for all $x \geq 0$. Indeed, if $g(x) := \log(1+x) - x + x^2/2$ for all $x \geq 0$, then $g(0) = 0$ and $g'(x) = \dfrac{x^2}{1+x} \geq 0$, hence $g(x) \geq g(0) = 0$ for all $x \geq 0$. Then, $\left(1+\dfrac{x}{n}\right)^{-n} = e^{-n\log\left(1+x/n\right)} \leq e^{-n\left(x/n - x^2/2n^2\right)} = e^{\left(-x+x^2/2n\right)} \leq e^{\left(-x+x^2/2x\right)} \leq e^{-x/2}$ for all $0 \leq x \leq n$. Let $f_n(x) := \left(1+\dfrac{x}{n}\right)^{-n} \log\left[2+\cos(x/n)\right]\chi_{[0,n]}(x)$, $x \geq 0$. It follows that $|f_n(x)| = f_n(x) \leq e^{-x/2}\log\left[2+\cos(x/n)\right]\chi_{[0,n]}(x) \leq e^{-x/2}\log 3$, for all $x \geq 0$, since $-1 \leq \cos(x/n) \leq 1$. The function $x \mapsto e^{-x/2}\log 3$ is non-negative and integrable on $[0,\infty)$. Also, $\lim\limits_{n\to\infty} f_n(x) = e^{-x}\log 3$ so, by the Dominant Convergence Theorem (DCT), we have that

$$\begin{aligned}
\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^{-n}\log(2+\cos(x/n))dx &= \lim_{n\to\infty}\int_0^\infty f_n(x)dx \\
&= \int_0^\infty \lim_{n\to\infty} f_n(x)dx \\
&= \int_0^\infty e^{-x}\log 3 dx = \log 3.
\end{aligned}$$

7.12 We have

$$\begin{aligned}
\int_0^n \left(1-\frac{x}{n}\right)^n \log(2+\cos(x/n))dx &= \int_0^\infty \left(1-\frac{x}{n}\right)^n \log(2+\cos(x/n))\chi_{[0,n]}(x)dx \\
&= \int_0^\infty \left(1-\frac{x}{n}\right)^n \log(2+\cos(x/n))\chi_{[0,n)}(x)dx,
\end{aligned}$$

the last equality following from the fact that the two functions under the last two integrals are equal everywhere except at $x = n$. Note that $\log(1-x) \leq -x$ for all $0 \leq x < 1$ (the proof is straightforward, similar to the one in Exercise

7.11). It follows that, for $0 \le x < n$,

$$
\begin{aligned}
\left| \left(1 - \frac{x}{n}\right)^n \log(2 + \cos(x/n))\chi_{[0,n)}(x) \right| &= \left| e^{n \log\left(1 - \frac{x}{n}\right)} \log(2 + \cos(x/n))\chi_{[0,n)}(x) \right| \\
&\le e^{n(-x/n)} \log 3 \\
&= e^{-x} \log 3.
\end{aligned}
$$

The function $x \mapsto e^{-x} \log 3$ is non-negative and integrable on $[0, \infty)$ so, by DCT, we have

$$
\begin{aligned}
\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \log(2 + \cos(x/n))dx &= \lim_{n\to\infty} \int_0^\infty \left(1 - \frac{x}{n}\right)^n \log(2 + \cos(x/n))\chi_{[0,n)}(x)dx \\
&= \int_0^\infty \lim_{n\to\infty} \left[\left(1 - \frac{x}{n}\right)^n \log(2 + \cos(x/n))\chi_{[0,n)}(x)dx\right] \\
&= \int_0^\infty e^{-x} \log 3 = \log 3.
\end{aligned}
$$

7.13 Let $f_n(x) := \dfrac{1 + nx^2}{(1 + x^2)^n}$, $0 \le x \le 1$, $n \in \mathbb{Z}^+$. It is an easy computation to show that $f_n(x) \ge f_{n+1}(x)$, for all $0 \le x \le 1$ (in fact, for all $x \in \mathbb{R}$) and that $f_n \downarrow f$ point-wise on $[0, 1]$, where

$$
f(x) = \begin{cases} 0, \text{if } 0 < x \le 1, \\ 1, \text{if } x = 0. \end{cases}
$$

The function f is integrable and $\int_0^1 f = 0$, since $f = 0$ a.e. on $[0, 1]$. Clearly, $|f_n(x) \log(2 + \cos(x/n))\chi_{[0,1]}| \le |f_n(x)| \log 3$. Since the sequence $\{f_n(x)\}$ is decreasing for all $0 \le x \le 1$ and both $f_1 \equiv 1$ and f (the point-wise limit) are bounded, it follows that there exists $M > 0$ such that $|f_n(x)| < M$, for all $0 \le x \le 1$ and $n \in \mathbb{Z}^+$. By DCT we get that the required limit exists and is equal to 0.

7.14 Define $f_n(x) := ne^{-nx} \ge 0$, for all $x \ge 0$. The integral in question becomes $\int_0^\infty f_n(x) \sin(1/x)dx = \int f_n(x) \sin(1/x)\chi_{(0,\infty)}(x)dx$. It is straightforward to show that, for all $0 < x < \infty$, $\{f_n(x)\}$ is decreasing and $f_n \downarrow 0$. Since $|f_n(x) \sin(1/x)\chi_{(0,\infty)}(x)| \le f_n(x) \le f_1(x)$ and $f_1(x) = e^{-x}$ is bounded on $(0, \infty)$, it follows that there exists $M > 0$ such that $|f_n(x) \sin(1/x)\chi_{(0,\infty)}(x)| < M$. Since e^{-x} is integrable on $(0, \infty)$, by DCT,

$$
\lim_{n\to\infty} \int f_n(x) \sin(1/x)\chi_{(0,\infty)}(x)dx = \int \lim_{n\to\infty} [f_n(x) \sin(1/x)\chi_{(0,\infty)}(x)]dx = 0.
$$

7.15 Since f is continuous at 1, it follows that $\lim_{n\to\infty} f(1+x/n^2)g(x)\chi_{[-n,n]}(x) = f(1)g(x)$, for all $x \in \mathbb{R}$. For every $x \in \mathbb{R}$ we have

$$\left|f(1+x/n^2)g(x)\chi_{[-n,n]}(x)\right| \leq M\left|g(x)\chi_{[-n,n]}(x)\right| \leq M\left|g(x)\right|,$$

where M is a bound for f (i.e. $|f| < M$). By hypothesis, g is integrable, so we can apply DCT. It follows that

$$\begin{aligned}\lim_{n\to\infty}\int_{-n}^{n} f(1+x/n^2)g(x)dx &= \lim_{n\to\infty}\int f(1+x/n^2)g(x)\chi_{[-n,n]}(x)dx \\ &= \int \lim_{n\to\infty}[f(1+x/n^2)g(x)\chi_{[-n,n]}(x)]dx \\ &= f(1)\int g(x)dx.\end{aligned}$$

7.25 (1) $\nu(\emptyset) = \int_\emptyset f d\mu = 0$, because $\mu(\emptyset) = 0$. Let $\{A_n\} \subset \mathcal{A}$ be a pairwise disjoint, countable family of measurable sets and let $A := \bigcup_{n=1}^{\infty} A_n$. Then,

$$\begin{aligned}\nu(A) = \int_A f d\mu = \int f\chi_{\left(\cup_{n=1}^{\infty}A_n\right)}d\mu = \int \sum_{n=1}^{\infty} f\chi_{A_n}d\mu \\ = \sum_{n=1}^{\infty}\int f\chi_{A_n}d\mu = \sum_{n=1}^{\infty}\int_{A_n} f d\mu = \sum_{n=1}^{\infty}\nu(A_n),\end{aligned}$$

where we used the fact that $\{A_n\}$ are pairwise disjoint and Proposition 7.6.

(2) It is enough to prove the required identity for the case where g is a simple function. Indeed, under that assumption, let g be a nonnegative integrable function. By Proposition 5.14, there exists a sequence $\{s_n\}$ of nonnegative measurable simple functions such that $s_n \uparrow g$, which implies $s_n f \to gf$. In fact, since $\{s_n\}$ is increasing and $f \geq 0$, it follows that $s_n f \uparrow gf$. So, on one hand, by the Monotone Convergence Theorem (MTC), $\int s_n f d\mu \to \int fg d\mu$. On the other, by the assumption we made that the formula is true for simple functions, $\int s_n f d\mu = \int s_n d\nu$ which again by MTC, converges to $\int g d\nu$, so $\int g d\nu = \int fg d\mu$. The identity for the the general case follows immediately by applying it to g^+, g^-.

It remains to prove the formula for the case when $g = \sum_{k=1}^{n} a_k\chi_{A_k}$ is a nonnegative measurable simple function:

$$\begin{aligned}\int g d\nu &= \sum_{k=1}^{n} a_k \nu(A_k) = \sum_{k=1}^{n} a_k \int_{A_k} f d\mu \\ &= \sum_{k=1}^{n} \int_{A_k} a_k f d\mu = \sum_{k=1}^{n} \int a_k f \chi_{A_k} d\mu \\ &= \int f \sum_{k=1}^{n} a_k \chi_{A_k} d\mu = \int f g d\mu.\end{aligned}$$

8.5 It suffices to show that the limit is 0 for t taking only natural values, in which case we shall denote it as $n := t$. Define $A_n := \{x \in X : f(x) \geq n\}$, $n \in \mathbb{Z}^+$. Note that, by the definition of the integral of a nonnegative integrable function, $n\mu(A_n) \leq \int f\chi_{A_n}$, since $s_n := n\chi_{A_n}$ is a simple function satisfying $0 \leq s_n \leq f$. Also, clearly $f\chi_{A_n} \leq f$ and, by hypothesis, f is nonnegative and integrable. Lastly, we show that $\lim_{n\to\infty} f\chi_{A_n} = 0$. First note that $A_{n+1} \subset A_n$ for all n, so $A_n \downarrow A := \cap_{k=1}^{\infty} A_k$. Moreover, since f is a real-valued function, $A = \emptyset$: if $\exists a \in \cap_{k=1}^{\infty} A_k$ then $f(a) \geq n$ for all n, which is not possible for any real number $f(a) \in \mathbb{R}$. It follows that $\lim_{n\to\infty} f\chi_{A_n} = f\chi_A = 0$. By applying DCT, we obtain the result.

8.7 Let $A_n := \{x \in X : f(x) \geq n\}$. We have

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \int \chi_{A_n} = \int \sum_{n=1}^{\infty} \chi_{A_n},$$

by Proposition 7.6. Let $x \in X$. For all $n > f(x)$ we have $\chi_{A_n}(x) = 0$, so $\sum_{n=1}^{\infty} \chi_{A_n}(x) = \sum_{n=1}^{[f(x)]} \chi_{A_n}(x) = [f(x)]$, where $[f(x)]$ is the greatest integer less than or equal to $f(x)$. But $f(x) - 1 \leq [f(x)] \leq f(x)$, hence $f(x) - 1 \leq \sum_{n=1}^{\infty} \chi_{A_n}(x) \leq f(x)$. Since $x \in X$ was arbitrarily fixed, the latter double inequality is true for any $x \in X$. If f is integrable, then $\sum_{n=1}^{\infty} \mu(A_n) = \int \sum_{n=1}^{\infty} \chi_{A_n} \leq \int f < \infty$. Conversely, $0 \leq f \leq 1 + \sum_{n=1}^{\infty} \chi_{A_n}$ which implies that f is integrable: $\int f \leq \int \left(1 + \sum_{n=1}^{\infty} \chi_{A_n}\right) = \mu(X) + \sum_{n=1}^{\infty} \mu(A_n) < \infty$, since μ is finite.

9.1 Define

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 2, & \text{if } x \in [0,1] \cap (\mathbb{Q} + \pi), \\ 1, & \text{otherwise,} \end{cases}$$

where $\mathbb{Q}+\pi = \{r+\pi : r \in \mathbb{Q}\}$. Then, $f = \chi_{[0,1]\setminus\mathbb{Q}} + \chi_{[0,1]\cap(\mathbb{Q}+\pi)}$ is Lebesque measurable, and by density of $\mathbb{Q}$ and $\mathbb{Q}+\pi$ we have $\underline{R}(f) = 0$ and $\overline{R}(f) = 2$. On the other hand, as countable sets, both $[0,1]\cap\mathbb{Q}$ and $[0,1]\cap(\mathbb{Q}+\pi)$ have Lebesque measure zero and $f = 1$ on their complement, so $\displaystyle\int_{[0,1]} f = m([0,1]) = 1$.