Problem Set 8
March 14, 2018.

1. Exercise 11.6.

2. Exercises 11.8–11.11.

3. Let $n \geq 2$ and let S be a standard n-simplex in $\mathbb{R}^n$ with base of length a, for some $a > 0$. That is,

$$S := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ \sum_{i=1}^{n} x_i \leq a\}.$$

Use Fubini Theorem (and induction) to find the Lebesgue integral $\int_{\mathbb{R}^n} \chi_S$.

4. Exercise 11.1.

5. Use Fubini Theorem from Problem 4, along with

$$\int_0^\infty e^{-tx} dt = \frac{1}{x} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(where i is the imaginary unit), to show that

$$\lim_{n \to \infty} \int_0^n \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

For Problems 6 and 7, let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be an $\mathcal{M}$-measurable function. Define the *distribution function* of f by

$$\mu_f(t) := \mu(\{x \in X : |f(x)| \geq t\}), \quad t > 0.$$

6. Show that $\mu_f : (0, \infty) \to [0, \mu(X)]$ is non-increasing and Borel measurable.

7. Prove that, for any $p \in [1, \infty)$,

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty \mu_f(t) p t^{p-1} dt.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p t^{p-1} dt$.