Problem Set 9
March 21, 2018.

1. Exercise 12.1.

2. Exercise 12.3.

3. Exercise 12.4.

4. Exercise 12.7.

5. Let $(X, \mathcal{M})$ be a measurable space.

(a) Prove that the collection of all complex measures on $(X, \mathcal{M})$ is a complex vector space (with addition and scalar multiplication defined as $(\mu + \lambda)(E) := \mu(E) + \lambda(E)$ and $(c \cdot \mu)(E) := c \cdot \mu(E)$, for $E \in \mathcal{M}$).

(b) Let $M(X)$ denote the complex vector space of all complex measures on $(X, \mathcal{M})$. Prove that the function defined as $\|\mu\| := |\mu|(X)$ is a norm on $M(X)$.

6. Suppose $\lambda, \lambda_1, \lambda_2$ are measures on a σ-algebra $\mathcal{M}$, and μ is a positive measure on $\mathcal{M}$. Prove the following statements:

(a) If λ is concentrated on a set $A \in \mathcal{M}$, then so is $|\lambda|$.

(b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(c) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.

(d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.

(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda \equiv 0$.

7. Use Problem 6 to prove the *uniqueness* of Lebesgue decomposition (of a complex measure λ relative to a positive σ-finite measure μ) in part (1) of the Radon-Nikodym Theorem stated in class.