Solutions for Problem Set 9 MATH 4122/9022

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12.1 Let μ be a signed measure on a measurable space $(X, \mathcal{A})$. Suppose first that A is a null set and let $P, N \in \mathcal{A}$ be a pair of sets given by the Hahn decomposition theorem (P is positive, N is negative, $X = P \cup N, P \cap N = \emptyset$). Then, by the Jordan decomposition theorem, $|\mu|(A) = \mu^+(A) + \mu^-(A) = \mu(A \cap P) - \mu(A \cap N) = 0$, because $A \cap P \subset A$, $A \cap N \subset A$ and A is a null set. Conversely, if $|\mu|(A) = 0$ then $\mu^+(A) = \mu^-(A) = 0$, since both μ^+ and μ^- are positive measures. If $B \subset A$ then $\mu^+(B) = \mu^-(B) = 0$, by monotonicity, hence $\mu(B) = 0$.

12.3 Let $s := \sup\left\{\left|\int_A f d\mu\right| : |f| \leq 1\right\}$. By Exercise 12.2 (not included in this problem set), $\left|\int_A f d\mu\right| \leq \int_A |f| \, d|\mu| \leq \int_A d|\mu| = |\mu|(A)$, for all $|f| \leq 1$. This implies $s \leq |\mu|(A)$. For the converse inequality, let $P, S \in \mathcal{A}$ be given by the Hahn decomposition theorem, with P is positive and N is negative. Then, $|\mu|(A) = \mu^+(A) + \mu^-(A) = \mu(A \cap P) - \mu(A \cap N) = |\mu(A \cap P) - \mu(A \cap N)| = \left|\int_{A \cap P} d\mu - \int_{A \cap P} d\mu\right| = \left|\int (\chi_{A \cap P} - \chi_{A \cap N}) d\mu\right| = \left|\int_A (\chi_P - \chi_N) d\mu\right|$. The function $f := \chi_P - \chi_N$ satisfies $|f| = 1$, which shows that $|\mu|(A) \in \left\{\left|\int_A f d\mu\right| : |f| \leq 1\right\}$, hence $|\mu|(A) \leq s$.

12.4 Let $P, S \in \mathcal{A}$ be given by the Hahn decomposition theorem, where P is positive and N is negative. The first inequality: $\lambda^+(A) = \lambda(A \cap P) = \mu(A \cap P) - \nu(A \cap P) \leq \mu(A \cap P) \leq \mu(A)$, where in the second last inequality we used the fact that both μ and ν are positive and finite. Similarly, the second inequality is derived as follows: $\lambda^-(A) = -\lambda(A \cap N) = -\mu(A \cap N) + \nu(A \cap N) \leq \nu(A \cap N) \leq \nu(A)$.

12.7 Let

$$a := \sup\left\{\left|\int_A f d\mu\right| : |f| \leq 1\right\}$$

and

$$b := \sup\left\{\sum_{j=1}^{n} |\mu(B_j)| : B_j \in \mathcal{A}, \{B_j\}_{j=1}^{n} \text{ is a partition of } A, n \in \mathbb{N}\right\}.$$

By Exercise 12.3, it suffices to show that $a = b$. As we have seen in the same exercise, $\left|\int_A f d\mu\right| \le |\mu|(A)$, for all $|f| \le 1$, so $a \le b$, since $|\mu|(A) = \mu(A \cap P) - \mu(A \cap N) = |\mu(A \cap P)| + |\mu(A \cap N)|$, which is an element of the set defined in the question (again, here P, N are given by the Hahn decomposition theorem). For the converse inequality, let $\{B_j\}_{j=1}^{n}$ be a partition of A, $B_j \in \mathcal{A}$ and define $f(x) := \dfrac{|\mu(B_j)|}{\mu(B_j)}$ for $x \in B_j$, $j = 1, \ldots, n$, and $f(x) = 0$ for all $x \in X \setminus A$ (note that $|f| \le 1$). Then, $\left|\int_A f d\mu\right| = \left|\sum_{j=1}^{n} \dfrac{|\mu(B_j)|}{\mu(B_j)} \cdot \int_{B_j} d\mu\right| = \sum_{j=1}^{n} |\mu(B_j)|$, which implies $b \le a$.

5. (a) Let $M(X)$ be the set of all complex measures on $(X, \mathcal{M})$ endowed with the two operations defined in the question. Then, $\mu_0 := 0$ (the zero-measure) is the additive identity, $(-\mu)(A) := -\mu(A)$ is the additive inverse of any $\mu \in M(X)$, $1 \in \mathbb{C}$ is the complex scalar satisfying $1 \cdot \mu = \mu$ and we let to the reader to prove the rest of the vector space axioms, which are straightforward to verify. For example, here is one of the distributivity axioms: if $a, b \in \mathbb{C}$, $\mu \in M(X)$ and $E \in \mathcal{M}$, then $[(a+b)\mu](E) = (a+b) \cdot \mu(E) = a \cdot \mu(E) + b \cdot \mu(E) = (a\mu)(E) + (b\mu)(E)$, hence $(a+b)\mu = a\mu + b\mu$.

(b) We use the following definition for the total variation measure $|\mu|$ of a complex measure μ (see for example Rudin's *Real and Complex Analysis*, 3rd edition, p. 116):

$$|\mu|(A) = \sup\left\{\sum_{j=1}^{\infty} |\mu(B_j)| : B_j \in \mathcal{M}, \{B_j\}_j \text{ is a partition of A}\right\} \tag{0.1}$$

For the rest of the proof, we shall write $\sup \sum_{j=1}^{\infty} |\mu(B_j)|$ instead of the full expression (0.1), where the supremum is taken over all partitions $\{B_j\}_j$ of X with $B_j \in \mathcal{M}$. For $a \in \mathbb{C}$ we have $\|a\mu\| = |a\mu|(X) = \sup \sum_{j=1}^{\infty} |a\mu(B_j)| = |a| \sup \sum_{j=1}^{\infty} |\mu(B_j)| = |a| \, \mu(X) = |a| \, \|\mu\|$. Next, suppose that $\|\mu\| := |\mu|(X) = 0$ and let $A \in \mathcal{M}$. Then, $|\mu(A)| \le |\mu(A)| + |\mu(X \setminus A)| \le$

$\sup \sum_{j=1}^{\infty} |\mu(B_j)| = |\mu|(X) = 0$. This proves that $\|\mu\| = 0$ implies $\mu \equiv 0$. Lastly, for the triangle inequality, we have

$$\begin{aligned}
\|\mu + \nu\| &= |\mu + \nu|(X) \\
&= \sup \sum_{j=1}^{\infty} |\mu(B_j) + \nu(B_j)| \\
&\leq \sup \sum_{j=1}^{\infty} |\mu(B_j)| + |\nu(B_j)| \\
&\leq \sup \sum_{j=1}^{\infty} |\mu(B_j)| + \sup \sum_{j=1}^{\infty} |\nu(C_j)| \\
&= |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|.
\end{aligned}$$

The last inequality involves all possible partitions of X, separately for μ and ν, respectively. That is why we used different notations (B_j and C_j).

6. This is Proposition 6.8 in Rudin's *Real and Complex Analysis*, 3rd edition, p. 120., whose solution we present in here.

(a) First note that λ is concentrated on A iff $\lambda(E) = 0$ for all $E \in \mathcal{M}$ that do not intersect A. Indeed, one direction is immediate and, to prove the other one, just write $E = (E \setminus A) \cup (E \cap A)$, as a disjoint union, and note that, by hypothesis, $\lambda(E \setminus A) = 0$. Now, let $E \in \mathcal{M}$ such that $E \cap A = \emptyset$. Then, $\lambda(E) = 0$ so, for any partition $\{E_j\}_j$ of E we have $\lambda(E_j) = 0$ for all j, since none of the E_j intersect A. By the definition used in Question 5 above (for complex measures) and also, by Exercise 12.7 (for signed measures), it follows that $|\lambda|(E) = 0$.

(b) For two measures λ_1, λ_2 to be mutually singular (as per the definition in the textbook) is the same as saying that λ_1 is concentrated on A and λ_2 is concentrated on B, for some measurable sets $A, B \in \mathcal{M}$, $A \cap B = \emptyset$, $A \cup B = X$ (we leave the straightforward proof of this statement to the reader). Then, point (b) is an immediate consequence of point (a).

(c) $\lambda_1 \perp \mu$ implies the existence of two measurable sets $A_1, B_1 \in \mathcal{M}$, disjoint, $A_1 \cup B_1 = X$, such that λ_1 is concentrated on A_1 and μ on B_1. Similarly, there exists disjoint measurable sets $A_2, B_2 \in \mathcal{M}$ such that λ_2 is concentrated on A_2 and μ on B_2. Then, $\lambda_1 + \lambda_2$ is concentrated on $A := A_1 \cup A_2$, μ is concentrated on $B := B_1 \cap B_2$, $X = A \cup B$ and $A \cap B = \emptyset$.

(d) This point is immediate.

(e) Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$ and let $\{E_j\}_j$ be a partition of E. Since μ is positive, it follows that $\mu(E_j) = 0$ for all j, so $\lambda(E_j) = 0$ for all j, hence $\sum_j |\lambda(E_j)| = 0$, which implies $|\lambda|(E) = 0$.

(f) $\lambda_2 \perp \mu$ implies that there exists a set A such that $\mu(A) = 0$ and on which λ_2 is concentrated. Since $\lambda_1 \ll \mu$, $\lambda_1(B) = 0$ for all $B \subset A$, hence λ_1 is concentrated on the complement of A, which proves the statement.

(g) By point (f), it follows that $\lambda \perp \lambda$ which clearly implies $\lambda \equiv 0$.

7. Suppose there exist two pairs λ_a, λ_s and λ'_a, λ'_s such that $\lambda = \lambda_a + \lambda_s = \lambda'_a + \lambda'_s$ such that $\lambda_a \ll \mu$, $\lambda_s \perp \mu$ and $\lambda'_a \ll \mu$, $\lambda'_s \perp \mu$. Then

$$\lambda'_a - \lambda_a = \lambda_s - \lambda'_s \tag{0.2}$$

Then, by point (c) in the previous exercise, we have $\lambda_s - \lambda'_s \perp \mu$. By point (d), we have $\lambda'_a - \lambda_a \ll \mu$. By (0.2), $\lambda'_a - \lambda_a$ and $\lambda_s - \lambda'_s$ are the same measure, so by point (g) we must have $\lambda'_a - \lambda_a = \lambda_s - \lambda'_s = 0$, which proves the statement.