# Internal Language of Finitely Complete $(\infty, 1)$-categories

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**Abstract**

We prove that the homotopy theory of Joyal's tribes is equivalent to that of fibration categories. As a consequence, we deduce a variant of the conjecture asserting that Martin-Löf Type Theory with dependent sums and intensional identity types is the internal language of $(\infty, 1)$-categories with finite limits.

## 1 Introduction

In recent years, two frameworks for abstract homotopy theory have emerged: higher category theory, developed extensively by Joyal [Joy08] and Lurie [Lur09], and Homotopy Type Theory (HoTT) initiated by Voevodsky [VAG$^+$] and further developed in [UF13]. The homotopy-theoretic theorems proven in the latter are often labeled as Synthetic Homotopy Theory, which is supposed to express two ideas. First, we can reason about objects of an abstract $(\infty, 1)$-category as if they were spaces. Second, a theorem proven in HoTT becomes true in a wide class of higher categories.

Although the connection between higher categories and HoTT seems intuitive to those familiar with both settings, a formal statement of equivalence between them was only conjectured in [KL16, Kap] in three variants depending on the choice of type constructors and the corresponding higher categorical structures. These conjectures have far reaching consequences. As mentioned above, they allow one to use HoTT to reason about sufficiently structured higher categories, for example, since the Blakers–Massey Theorem has been proven in HoTT [FFLL16], it is satisfied in an arbitrary $(\infty, 1)$-topos. Conversely, a type-theoretic statement true in every $(\infty, 1)$-topos must be provable in HoTT and consequently, results in higher category theory can suggest new principles of logic, such as the Univalence Axiom of Voevodsky.

In the present paper, we prove a version of the first of these conjectures, asserting that Martin-Löf Type Theory with dependent sums and intensional identity types is the internal language of finitely complete $(\infty, 1)$-categories. To make this result precise, we assemble the categorical models of type theory, given by comprehension categories [Jac99], into a category $\mathsf{CompCat}_{\mathsf{Id},\Sigma}$ and compare it with the category $\mathsf{Lex}_\infty$ of quasicategories with finite limits. Our main theorem (cf. Theorem 9.10) states:

**Theorem.** *The homotopy theory of categorical models of Martin-Löf Type Theory with dependent sums and intensional identity types is equivalent to the homotopy theory of finitely complete $(\infty, 1)$-categories.*

Ideally, one would like to establish an equivalence between suitable syntactically presented type theories and finitely complete $(\infty, 1)$-categories and the theorem above is an important step in this direction. However, a complete result along these lines would require a proof of the Initiality Conjecture[1] of Voevodsky and a comparison between contextual categories and comprehension categories, both currently being investigated: the former by Voevodsky, and the latter by Cho, Knapp, Newstead and Wong.

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[1] `https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2015_06_30_RDP.pdf`

Our approach builds on two recent developments. First, it was shown in [Szu17b] that the homotopical category $\mathsf{Lex}_\infty$ is DK-equivalent to that of fibration categories. Second, the connection between fibration categories and type theory was observed in [AKL15] and then explored in detail by Shulman [Shu15] and in the unpublished work of Joyal. Both Shulman and Joyal gave a categorical axiomatization of the properties of fibration categories arising from type theory, introducing the notions of a type-theoretic fibration category and a tribe, respectively. In particular, tribes are closely related to comprehension categories with dependent sums and intensional identity types, but their presentation is more convenient for comparison with other settings of abstract homotopy theory.

The equivalence between comprehension categories and tribes is fairly straightforward and thus the heart of the paper is the proof that the forgetful functor $\mathsf{Trb} \to \mathsf{FibCat}$ from the homotopical category of tribes to the homotopical category of fibration categories is a DK-equivalence. The most straightforward way of accomplishing that would be to construct its homotopy inverse. However, associating a tribe to a fibration category in a strictly functorial manner has proven difficult. Another approach would be to verify Waldhausen's approximation properties which requires constructing a fibration category structure on $\mathsf{Trb}$. While this idea does not appear to work directly, it can refined using semisimplicial methods.

To this end, in course of the proof, we will consider the following homotopical categories.

$$\begin{tikzcd}
 & \mathsf{sTrb} \arrow[r] \arrow[d] & \mathsf{sFibCat} \arrow[d] & \\
\mathsf{CompCat}_{\mathsf{Id},\Sigma} \arrow[r] & \mathsf{Trb} \arrow[r] & \mathsf{FibCat} \arrow[r] & \mathsf{Lex}_\infty
\end{tikzcd}$$

In the top row, $\mathsf{sTrb}$ and $\mathsf{sFibCat}$ denote the categories of semisimplicial tribes and semisimplicial fibration categories, respectively. The vertical forgetful functors are directly verified to be homotopy equivalences. Moreover, both $\mathsf{sTrb}$ and $\mathsf{sFibCat}$ are fibration categories allowing us to verify that the top map is a DK-equivalence by checking the approximation properties.

### Outline

In Section 2, we review background on fibration categories and tribes, and in Section 3, we introduce their semisimplicially enriched counterparts. Then in Section 4, we construct fibration categories of semisimplicial fibration categories and semisimplicial tribes, following [Szu16].

To associate a tribe to a fibration category, we use injective model structures on categories of simplicial presheaves, which we briefly recall in Section 5. In Section 6, we study the hammock localization of a fibration category and construct tribes of representable presheaves over such localizations.

In Section 7, we use them to verify the approximation properties for the forgetful functor $\mathsf{Trb} \to \mathsf{FibCat}$. As mentioned above, this argument is insufficient and we rectify it in Section 8 using semisimplicial enrichments. Finally in Section 9, we complete the proof by relating tribes to comprehension categories.

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## 2 Fibration categories and tribes

To start off, we discuss the basic theory of fibration categories and tribes. Fibration categories were first introduced by Brown [Bro73] as an abstract framework for homotopy theory alternative to Quillen's model categories.

**Definition 2.1.** A *fibration category* is a category $\mathcal{C}$ equipped with a subcategory of *weak equivalences* (denoted by $\xrightarrow{\sim}$) and a subcategory of *fibrations* (denoted by $\twoheadrightarrow$) subject to the following axioms.

(F1) $\mathcal{C}$ has a terminal object 1 and all objects are fibrant.

(F2) Pullbacks along fibrations exist in $\mathcal{C}$ and (acyclic) fibrations are stable under pullback.

(F3) Every morphism factors as a weak equivalence followed by a fibration.

(F4) Weak equivalences satisfy the 2-out-of-6 property.

We will need a few fundamental facts about fibration categories which we now recall. For a more thorough discussion, see [RB06].

**Definition 2.2.** Let $\mathcal{C}$ be a fibration category.

(1) A *path object* for an object $a \in \mathcal{C}$ is a factorization of the diagonal morphism $a \to a \times a$ as $(\pi_0, \pi_1)\sigma \colon a \xrightarrow{\sim} Pa \twoheadrightarrow a \times a$.

(2) A *homotopy* between morphisms $f, g \colon a \to b$ is a morphism $H \colon a \to Pb$ such that $(\pi_0, \pi_1)H = (f, g)$.

(3) A morphism $f \colon a \to b$ is a *homotopy equivalence* if there is a morphism $g \colon b \to a$ such that $gf$ is homotopic to $\mathrm{id}_a$ and $fg$ is homotopic to $\mathrm{id}_b$.

(4) An object $a$ is *cofibrant* if for every acyclic fibration $p \colon x \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow y$ there is a lift in every diagram of the form

$$\begin{tikzcd} & x \arrow[d, two heads, "p", "\sim"'] \\ a \arrow[r] \arrow[ur, dashed] & y. \end{tikzcd}$$

**Lemma 2.3.** *In a fibration category $\mathcal{C}$, a morphism $f \colon a \to b$ between cofibrant objects is a weak equivalence if and only if it is a homotopy equivalence.*

*Proof.* If $f$ is a weak equivalence, then it is a homotopy equivalence by [Bau89, Cor. 2.12].

Conversely, let $f$ be a homotopy equivalence and $g \colon b \to a$ its homotopy inverse. Moreover, let $H$ be a homotopy between $gf$ and $\mathrm{id}_a$ and let $G$ a homotopy between $fg$ and $\mathrm{id}_b$ which yield the following diagram.

$$\begin{tikzcd} & & & b \arrow[rr, "\mathrm{id}"] \arrow[dl, "g"'] \arrow[dr, "G"] \arrow[dd, "fg"] & & b \\ a & & a \arrow[ll, "\mathrm{id}"'] \arrow[dl, "H"'] \arrow[dr, "f"] \arrow[dd, "gf"] & & Pb \arrow[ur, "\sim"'] \arrow[dl, "\sim"] & \\ & Pa \arrow[ul, "\sim"] \arrow[dr, "\sim"'] & & b \arrow[dl, "g"] & & \\ & & a & & & \end{tikzcd}$$

By 2-out-of-6 it follows that $f$ is a weak equivalence. $\square$

**Definition 2.4.** A commutative square

$$\begin{tikzcd} a \arrow[r] \arrow[d] & c \arrow[d] \\ b \arrow[r] & d \end{tikzcd}$$

in a fibration category is a *homotopy pullback* if given a factorization of $c \xrightarrow{\sim} c' \twoheadrightarrow d$, the induced morphism $a \to b \times_d c'$ is a weak equivalence.

**Lemma 2.5.** *Let $\mathcal{C}$ be fibration category.*

*(1) If two squares in $\mathcal{C}$ are weakly equivalent, then one is a homotopy pullback if and only if the other one is.*

*(2) Every homotopy pullback in $\mathcal{C}$ is weakly equivalent to a strict pullback along a fibration.*

*(3) Every homotopy pullback in $\mathcal{C}$ is weakly equivalent to a strict pullback of two fibrations.*

*Proof.* These are all well-known; they follow directly from the Gluing Lemma [RB06, Lem. 1.4.1(2b)]. □

**Definition 2.6.**

(1) A *homotopical category* is a category equipped with a class of weak equivalences that satisfies the 2-out-of-6 property.

(2) A *homotopical functor* between homotopical categories is a functor that preserves weak equivalences.

(3) A *Dwyer–Kan equivalence* (or *DK-equivalence* for short) is a homotopical functor that induces an equivalence of homotopy categories and a weak homotopy equivalences on mapping spaces in the hammock localizations (cf. [DK80]).

Here, the homotopy category of a homotopical category $\mathcal{C}$ is its localization at the class of weak equivalences denoted by $\operatorname{Ho}\mathcal{C}$.

**Definition 2.7.**

(1) A functor between fibration categories is *exact* if it preserves weak equivalences, fibrations, terminal objects and pullbacks along fibrations.

(2) A *weak equivalence* between fibration categories is an exact functor that induces an equivalence of the homotopy categories.

The homotopical category of small fibration categories will be denoted by $\mathsf{FibCat}$.

A useful criterion for an exact functor to be a weak equivalence are the following approximation properties. They were originally introduced by Waldhausen [Wal85] in the context of algebraic K-theory and later adapted by Cisinski to the setting of abstract homotopy theory.

**Definition 2.8.** An exact functor $F : \mathcal{C} \to \mathcal{D}$ satisfies the *approximation properties* if:

(App1) $F$ reflects weak equivalences;

(App2) for every pair of objects $b \in \mathcal{C}$, $x \in \mathcal{D}$ and a morphism $x \to Fb$, there is a commutative square

$$\begin{tikzcd} x \arrow[r] & Fb \\ x' \arrow[u, "\sim"] \arrow[r, "\sim"'] & Fa \arrow[u] \end{tikzcd}$$

where the labeled morphisms are weak equivalences and the one on the right is the image of a morphism $a \to b$.

**Theorem 2.9** (Cisinski)**.** *If $F \colon \mathcal{C} \to \mathcal{D}$ is an exact functor between fibration categories, then the following are equivalent:*

*(1) $F$ is a weak equivalence;*

*(2) $F$ satisfies the approximation properties;*

*(3) $F$ is a DK-equivalence.*

*Proof.* The equivalence between conditions (1) and (2) was proven in [Cis10a, Thm. 3.19]. The equivalence between conditions (1) and (3) was proven in [Cis10b, Thm. 3.25]. □

Properties of fibration categories arising from type theory were axiomatized by Shulman [Shu15] as *type-theoretic fibration categories*. A similar notion of a tribe was introduced by Joyal and developed extensively in his unpublished manuscript. In the remainder of this section, we recall basic definitions and results of the theory of tribes, almost all of which are folklore.

**Definition 2.10.** A *tribe* is a category $\mathcal{T}$ equipped with a subcategory whose morphisms are called *fibrations* subject to the following axioms. (A morphism with the left lifting property with respect to all fibrations is called *anodyne* and denoted by $\overset{\sim}{\rightarrowtail}$.)

(T1) $\mathcal{T}$ has a terminal object 1 and all objects are fibrant.

(T2) Pullbacks along fibrations exist in $\mathcal{T}$ and fibrations are stable under pullback.

(T3) Every morphism factors as an anodyne morphism followed by a fibration.

(T4) Anodyne morphisms are stable under pullbacks along fibrations.

**Definition 2.11.** Let $\mathcal{T}$ be a tribe.

(1) A *path object* for an object $x \in \mathcal{T}$ is a factorization of the diagonal morphism $x \to x \times x$ as $(\pi_0, \pi_1)\sigma \colon x \overset{\sim}{\rightarrowtail} Px \twoheadrightarrow x \times x$.

(2) A *homotopy* between morphisms $f, g \colon x \to y$ is a morphism $H \colon x \to Py$ such that $(\pi_0, \pi_1)H = (f, g)$.

(3) A morphism $f \colon x \to y$ is a *homotopy equivalence* if there is a morphism $g \colon y \to x$ such that $gf$ is homotopic to $\mathrm{id}_x$ and $fg$ is homotopic to $\mathrm{id}_y$.

**Lemma 2.12** (cf. [Shu15, Lem. 3.6])**.** *An anodyne morphism $f \colon x \overset{\sim}{\rightarrowtail} y$ in a tribe is a homotopy equivalence.*

*Proof.* Using the lifting property of $f$ against $x \twoheadrightarrow 1$ we obtain a retraction $r \colon y \to x$. On the other hand, a lift in

$$\begin{tikzcd}
x \arrow[r, "\sigma f"] \arrow[d, tail, "f"', "\sim"] & Py \arrow[d, two heads, "(\pi_0, \pi_1)"] \\
y \arrow[r, "(\mathrm{id}, fr)"'] & y \times y
\end{tikzcd}$$

is a homotopy between $\mathrm{id}_y$ and $fr$. $\square$

**Lemma 2.13** (Joyal)**.** *Homotopy equivalences in a tribe $\mathcal{T}$ are saturated, i.e., a morphism is a homotopy equivalence if and only if it becomes an isomorphism in $\operatorname{Ho}\mathcal{T}$. In particular, homotopy equivalences satisfy 2-out-of-6.*

*Proof.* Let $\mathcal{B}$ be an arbitrary category and $F \colon \mathcal{T} \to \mathcal{B}$ be any functor. If $F$ identifies homotopic morphisms, then it carries homotopy equivalences to isomorphisms by definition. Conversely, if $F$ inverts homotopy equivalences, then it identifies homotopic morphisms. Indeed, this follows from the fact that for every object $x$, the morphism $\sigma \colon x \to Px$ is a homotopy equivalence by Lemma 2.12. Thus the localization of $\mathcal{T}$ at homotopy equivalences coincides with its quotient by the homotopy relation which implies saturation. Consequently, homotopy equivalences satisfy 2-out-of-6 since isomorphisms in $\operatorname{Ho}\mathcal{T}$ do. $\square$

**Lemma 2.14** (cf. [Shu15, Lem. 3.7])**.** *If $f \colon x \to y$ and $g \colon y \to z$ are morphisms such that $g$ and $gf$ are anodyne, then so is $f$.*

*Proof.* Since $g$ is anodyne, there is a lift in the square

$$\begin{tikzcd}
y \arrow[r, "\mathrm{id}"] \arrow[d, tail, "g"', "\sim"] & y \arrow[d, two heads] \\
z \arrow[r] \arrow[ur, dashed, "r"] & 1.
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
x \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & x \arrow[r, "\mathrm{id}"] \arrow[d, tail, "gf"', "\sim"] & x \arrow[d, "f"] \\
y \arrow[r, "g"'] & z \arrow[r, "r"'] & y
\end{tikzcd}$$

shows that $f$ is a retract of $gf$ and so it is anodyne. $\square$

**Lemma 2.15** (Joyal)**.** *Given a commutative diagram of the form*

$$\begin{tikzcd}
& x_0 \arrow[rr] \arrow[dl] \arrow[ddl] & & y_0 \arrow[dl, tail, "\sim"'] \arrow[ddl] \\
x_1 \arrow[rr] \arrow[d, two heads] & & y_1 \arrow[d, two heads] & \\
a \arrow[rr] & & b &
\end{tikzcd}$$

*where all squares are pullbacks, if $y_0 \to y_1$ is anodyne, then so is $x_0 \to x_1$.*

*Proof.* Pick a factorization $a \overset{\sim}{\rightarrowtail} a' \twoheadrightarrow b$ and form pullback squares

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d] & x_0' \arrow[r] \arrow[d] & y_0 \arrow[d, tail, "\sim"] \\
x_1 \arrow[r] \arrow[d] & x_1' \arrow[r] \arrow[d] & y_1 \arrow[d, two heads] \\
a \arrow[r, tail, "\sim"'] & a' \arrow[r, two heads] & b.
\end{tikzcd}$$

The morphism $x_1' \to a'$ is a fibration as a pullback of a fibration $y_1 \to b$ and thus the morphism $x_1 \to x_1'$ is anodyne as a pullback of an anodyne morphism $a \to a'$ along a fibration. Similarly, $x_0' \to x_1'$ is anodyne. Furthermore, since $y_0 \to b$ is also a fibration, the same argument implies that $x_0 \to x_0'$ is anodyne. Therefore, $x_0 \to x_1$ is also anodyne by Lemma 2.14. □

**Definition 2.16.**

(1) A functor between tribes is a *homomorphism* if it preserves fibrations, anodyne morphisms, terminal objects and pullbacks along fibrations.

(2) A homomorphism between tribes is a *weak equivalence* if it induces a weak equivalence of the homotopy categories.

The homotopical category of small tribes will be denoted by $\mathsf{Trb}$.

**Theorem 2.17.** *Every tribe with its subcategories of fibrations and homotopy equivalences is a fibration category. Moreover, every homomorphism of tribes is an exact functor. This yields a homotopical forgetful functor* $\mathsf{Trb} \to \mathsf{FibCat}$.

*Proof.* In [Shu15, Thm. 3.13], it is proven that every type-theoretic fibration category is a "category of fibrant objects". A type-theoretic fibration category is defined just like a tribe except that the statement of Lemma 2.15 is used instead of axiom (T4). (The definition also includes an additional axiom about dependent products which is not used in the proof of Thm. 3.13.) Similarly, a category of fibrant objects is defined just like a fibration category except that only 2-out-of-3 is assumed in the place of 2-out-of-6. However, the latter was verified in Lemma 2.13.

A homomorphism of tribes preserves fibrations, terminal objects and pullbacks along fibrations by definition. It also preserves anodyne morphism and hence path objects and, consequently, homotopies and homotopy equivalences. Thus it is an exact functor. □

For clarity of exposition, from this point on, we will refer to homotopy equivalences in a tribe as weak equivalences.

**Lemma 2.18** (cf. [Shu15, Lem. 3.11])**.** *Every acyclic fibration in a tribe admits a section. In particular, every object in a tribe is cofibrant.* □

**Lemma 2.19.** *Let*

$$\begin{tikzcd}
x_\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & x_1 \arrow[dr, tail, "\sim"] \arrow[dd, two heads] & \\
& y_\emptyset \arrow[rr] \arrow[dd] & & y_1 \arrow[dd, two heads] \\
x_0 \arrow[rr] \arrow[dr, tail, "\sim"] & & x_{01} \arrow[dr, tail, "\sim"] & \\
& y_0 \arrow[rr] & & y_{01}
\end{tikzcd}$$

*be a cube in a tribe where* $x_1 \twoheadrightarrow x_{01}$ *and* $y_1 \twoheadrightarrow y_{01}$ *are fibrations and the front and back squares are pullbacks. If all* $x_0 \overset{\sim}{\rightarrowtail} y_0$, $x_1 \overset{\sim}{\rightarrowtail} y_1$ *and* $x_{01} \overset{\sim}{\rightarrowtail} y_{01}$ *are anodyne, then so is* $x_\emptyset \to y_\emptyset$.

*Proof.* Taking pullbacks in the right and left faces, we obtain a diagram

$$\begin{tikzcd}
x_\emptyset \arrow[rr] \arrow[dr] \arrow[dddr, bend right] & & x_1 \arrow[dr] \arrow[dddr, bend right] & & \\
& z \arrow[rr] \arrow[dr] \arrow[dd] & & z' \arrow[dr] \arrow[dd] & \\
& & y_\emptyset \arrow[rr] \arrow[dd] & & y_1 \arrow[dd, two heads] \\
& x_0 \arrow[rr] \arrow[dr, tail, "\sim"] & & x_{01} \arrow[dr, tail, "\sim"] & \\
& & y_0 \arrow[rr] & & y_{01}
\end{tikzcd}$$

where all the downward arrows are fibrations by (T2). Moreover, $z \to y_\emptyset$ and $z' \to y_1$ are anodyne by (T4). Thus $x_1 \to z'$ is anodyne by Lemma 2.14 and Lemma 2.15 implies that $x_\emptyset \to z$ is anodyne. It follows that the composite $x_\emptyset \to z \to y_\emptyset$ is also anodyne as required. ☐

**Corollary 2.20** (Joyal)**.** *Product of anodyne morphisms is anodyne.* ☐

We conclude this section by constructing fibration categories and tribes of Reedy fibrant diagrams. The argument given in the proof of the lemma below is standard, but it will reappear in various modified forms throughout the paper.

**Definition 2.21.**

(1) A category $J$ is *inverse* if there is a function, called *degree*, $\deg\colon \mathrm{ob}(J) \to \mathbb{N}$ such that for every non-identity map $j \to j'$ in $J$ we have $\deg(j) > \deg(j')$.

Let $J$ be an inverse category.

(2) Let $j \in J$. The *matching category* $\partial(j \downarrow J)$ of $j$ is the full subcategory of the slice category $j \downarrow J$ consisting of all objects except $\mathrm{id}_j$. There is a canonical functor $\partial(j\downarrow J) \to J$, assigning to a morphism (regarded as an object of $\partial(j \downarrow J)$) its domain.

(3) Let $X \colon J \to \mathcal{C}$ and $j \in J$. The *matching object* of $X$ at $j$ is defined as a limit of the composite

$$M_j X := \lim \big(\partial(j \downarrow J) \longrightarrow J \xrightarrow{X} \mathcal{C}\big).$$

The canonical morphism $X_j \to M_j X$ is called the *matching morphism*.

(4) Let $\mathcal{C}$ be a fibration category. A diagram $X \colon J \to \mathcal{C}$ is called *Reedy fibrant*, if for all $j \in J$, the matching object $M_j X$ exists and the matching morphism $X_j \to M_j X$ is a fibration.

(5) Let $\mathcal{C}$ be a fibration category and let $X, Y \colon J \to \mathcal{C}$ be Reedy fibrant diagrams in $\mathcal{C}$. A morphism $f \colon X \to Y$ of diagrams is a *Reedy fibration* if for all $j \in J$ the induced morphism $X_j \to M_j X \times_{M_j Y} Y_j$ is a fibration.

(6) If $\mathcal{C}$ is a fibration category, then $\mathcal{C}^J_{\mathrm{R}}$ denotes the category of Reedy fibrant diagrams in $\mathcal{C}$ over $J$.

**Lemma 2.22.** *Let $J$ be a homotopical inverse category.*

*(1) If $\mathcal{C}$ is a fibration category, then so is $\mathcal{C}^J_{\mathrm{R}}$ with levelwise weak equivalences and Reedy fibrations.*

*(2) If $\mathcal{T}$ is a tribe, then so is $\mathcal{T}^J_{\mathrm{R}}$ with Reedy fibrations. Moreover, both anodyne morphisms and weak equivalences in $\mathcal{T}^J_{\mathrm{R}}$ are levelwise.*

*Proof.* Part (1) is [RB06, Thm. 9.3.8(1a)].

In part (2), (T1) is evident and (T2) is verified exactly as in part (1).

Every morphism in $\mathcal{T}^J_{\mathrm{R}}$ factors into a levelwise anodyne morphism followed by a Reedy fibration by the proof of [Hov99, Thm. 5.1.3]. Moreover, levelwise anodyne morphisms have the left lifting property with respect to Reedy fibrations by [Hov99, Prop. 5.1.4]. In particular, they are anodyne.

Thus for (T3) it suffices to verify that every anodyne morphism $x \overset{\sim}{\rightarrowtail} y$ is levelwise anodyne. Factor it into a levelwise anodyne morphism $x \to x'$ followed by a Reedy fibration $x' \to y$. Then there is a lift in

$$\begin{tikzcd} x \arrow[r] \arrow[d, tail, "\sim"'] & x' \arrow[d, two heads] \\ y \arrow[r] & y \end{tikzcd}$$

which exhibits $x \to y$ as a retract of $x \to x'$ and hence the former is levelwise anodyne. Since anodyne morphisms coincide with levelwise anodyne morphisms, they are stable under pullback along Reedy fibrations (which are, in particular, levelwise), which proves (T4).

Let $\mathcal{T}^J_{\mathrm{lvl}}$ be the fibration category of Reedy fibrant diagrams in the underlying fibration category of $\mathcal{T}$ with levelwise weak equivalences as constructed in part (1). We verify that a morphism $f$ is a weak equivalence in $\mathcal{T}^J_{\mathrm{R}}$ if and only if it is a weak equivalence in $\mathcal{T}^J_{\mathrm{lvl}}$. A path object in $\mathcal{T}^J_{\mathrm{R}}$ is also a path object in $\mathcal{T}^J_{\mathrm{lvl}}$ since anodyne morphisms in $\mathcal{T}^J_{\mathrm{R}}$ are levelwise. It follows that $f$ is a weak equivalence in $\mathcal{T}^J_{\mathrm{R}}$ if and only if it is a homotopy equivalence in $\mathcal{T}^J_{\mathrm{lvl}}$. By Lemma 2.3, it suffices to verify that all objects of $\mathcal{T}^J_{\mathrm{lvl}}$ are cofibrant. By [Hov99, Prop. 5.1.4], an object of $\mathcal{T}^J_{\mathrm{lvl}}$ is cofibrant if and only if it is levelwise cofibrant so the conclusion follows by Lemma 2.18. □

## 3 Semisimplicial fibration categories and tribes

As mentioned in the introduction, the homotopical category $\mathsf{Trb}$ does not appear to carry a structure of a fibration category for the reasons that will be explained in Section 4. To resolve this issue we introduce semisimplicial enrichments of both tribes and fibration categories, following Schwede [Sch13, Sec. 3]. The key ingredient of this solution is the notion of a frame. The category of frames in a fibration category (tribe) carries a natural structure of a semisimplicial fibration category (tribe). Although our approach is inspired by Joyal's theory of simplicial tribes, we were forced to modify the enriching category since the construction of the category of frames has no simplicial counterpart.

We begin this section by reviewing basics of semisimplicial sets. (For a more complete account, see [RS71].) Let $\Delta_\sharp$ denote the category whose objects are finite non-empty totally ordered sets of the form $[m] = \{0 < \ldots < m\}$ and morphisms are injective order preserving maps. A *semisimplicial set* is a presheaf over $\Delta_\sharp$. A representable semisimplicial set will be denoted by $\Delta_\sharp[m]$ and its boundary (obtained by removing the top-dimensional simplex) by $\partial\Delta_\sharp[m]$.

The inclusion $\Delta_\sharp \hookrightarrow \Delta$ induces a forgetful functor from simplicial sets to semisimplicial sets which admits a left adjoint. *Weak homotopy equivalences* of semisimplicial sets are created by this adjoint from weak homotopy equivalences of simplicial sets.

The *geometric product* of semisimplicial sets $K$ and $L$ is defined by the coend formula

$$K \boxtimes L = \int^{[m],[n]} K_m \times L_n \times \mathrm{N}_\sharp([m] \times [n])$$

where $\mathrm{N}_\sharp P$ is the semisimplicial nerve of a poset $P$, i.e., the semisimplicial set whose $k$-simplices are injective order preserving maps $[k] \hookrightarrow P$. This defines a symmetric monoidal structure on the category of semisimplicial sets with $\Delta_\sharp[0]$ as a unit.

**Definition 3.1.** A *semisimplicial fibration category* $\mathcal{C}$ is a fibration category that carries a semisimplicial enrichment with cotensors by finite semisimplicial sets satisfying the following *pullback-cotensor property.*

(SF) If $i\colon K \rightarrowtail L$ is a monomorphism between finite semisimplicial sets and $p\colon a \twoheadrightarrow b$ is a fibration in $\mathcal{C}$, then the induced morphism $(i^*, p_*)\colon a^L \to a^K \times_{b^K} b^L$ is a fibration. Moreover, if either $i$ or $p$ is acyclic, then so is $(i^*, p_*)$.

**Definition 3.2.** A *semisimplicial tribe* $\mathcal{T}$ is a tribe that carries a semisimplicial enrichment that makes the underlying fibration category semisimplicial and satisfies the following condition.

(ST) If $K$ is a finite semisimplicial set and $x \overset{\sim}{\rightarrowtail} y$ is anodyne, then so is $x^K \to y^K$.

**Lemma 3.3.** *Let $J$ be a homotopical inverse category.*

*(1) If $\mathcal{C}$ is a semisimplicial fibration category, then so is $\mathcal{C}^J_{\mathrm{R}}$.*

*(2) If $\mathcal{T}$ is a semisimplicial tribe, then so is $\mathcal{T}^J_{\mathrm{R}}$.*

*Proof.* This follows from Lemma 2.22 and the fact that cotensors are computed levelwise. □

**Definition 3.4.**

(1) A semisimplicial functor between semisimplicial fibration categories is *exact* if it is exact as a functor of the underlying fibration categories and it preserves cotensors by finite semisimplicial sets.

(2) A semisimplicial functor between semisimplicial tribes is a *homomorphism* if it is a homomorphism of the underlying tribes and it preserves cotensors by finite semisimplicial sets.

**Definition 3.5.**

(1) A *weak equivalence* of semisimplicial fibration categories is an exact functor that is a weak equivalence of their underlying fibration categories.

(2) A *weak equivalence* of semisimplicial tribes is a homomorphism that is a weak equivalence of their underlying tribes.

A *frame* in a fibration category $\mathcal{C}$ is a Reedy fibrant, homotopically constant semisimplicial object in $\mathcal{C}$. The category of frames in a fibration category $\mathcal{C}$ will be denoted by $\operatorname{Fr}\mathcal{C}$. If $\mathcal{C}$ is semisimplicial, then every object $a$ has the *canonical frame* $a^{\Delta_\sharp[-]}$.

**Lemma 3.6.** *If $x \to y$ is a levelwise anodyne extension between frames in a tribe and $K$ is a finite semisimplicial set, then $\int_{[n]} x_n^{K_n} \to \int_{[n]} y_n^{K_n}$ is anodyne (in particular, these ends exist).*

*Proof.* Let $\operatorname{Sk}^k K$ be the $k$-skeleton of $K$, i.e., a semisimplicial set consisting of simplices of $K$ of dimension at most $k$. The argument proceeds by induction with respect to $k$ for all finite dimensional semisimplicial sets $K$ simultaneously. In particular, the end $\int_{[n]} x_n^{K_n}$ is constructed as follows. For each $k$ we have a pushout

$$\begin{tikzcd}
\partial\Delta_\sharp[k] \times K_k \arrow[r] \arrow[d, hook] & \operatorname{Sk}^{k-1} K \arrow[d] \\
\Delta_\sharp[k] \times K_k \arrow[r] & \operatorname{Sk}^k K
\end{tikzcd}$$

which yields a pullback

$$\begin{tikzcd}
\int_{[n]} x_n^{(\operatorname{Sk}^k K)_n} \arrow[r] \arrow[d] & \int_{[n]} x_n^{(\Delta_\sharp[k] \times K_k)_n} \arrow[d, two heads] \\
\int_{[n]} x_n^{(\operatorname{Sk}^{k-1} K)_n} \arrow[r] & \int_{[n]} x_n^{(\partial\Delta_\sharp[k] \times K_k)_n} .
\end{tikzcd}$$

The right arrow coincides with $x_k^{K_k} \to (M_k x)^{K_k}$ and hence it is a fibration. The end $\int_{[n]} x_n^{(\operatorname{Sk}^{k-1} K)_n}$ exists by the inductive hypothesis and thus so does $\int_{[n]} x_n^{(\operatorname{Sk}^k K)_n}$.

For $k = 0$, the morphism

$$\int_{[n]} x_n^{(\operatorname{Sk}^0 K)_n} \longrightarrow \int_{[n]} y_n^{(\operatorname{Sk}^0 K)_n}$$

coincides with $x_0^{K_0} \to y_0^{K_0}$ and hence it is anodyne by Corollary 2.20. For $k > 0$ we note that

$$\int_{[n]} x_n^{(\Delta_\sharp[k] \times K_k)_n} \longrightarrow \int_{[n]} y_n^{(\Delta_\sharp[k] \times K_k)_n}$$

coincides with $x_k^{K_k} \to y_k^{K_k}$ so it is anodyne, again by Corollary 2.20. Moreover, the morphisms

$$\int_{[n]} x_n^{(\operatorname{Sk}^{k-1} K)_n} \longrightarrow \int_{[n]} y_n^{(\operatorname{Sk}^{k-1} K)_n} \qquad \text{and} \qquad \int_{[n]} x_n^{(\partial\Delta_\sharp[k] \times K_k)_n} \longrightarrow \int_{[n]} y_n^{(\partial\Delta_\sharp[k] \times K_k)_n}$$

are anodyne by the inductive hypothesis. Thus the assumptions of Lemma 2.19 are satisfied when we apply it to the map from the square above to the analogous square for $y$. It follows that

$$\int_{[n]} x_n^{(\operatorname{Sk}^k K)_n} \longrightarrow \int_{[n]} y_n^{(\operatorname{Sk}^k K)_n}$$

is also anodyne. $\square$

**Theorem 3.7.**

*(1) For a fibration category $\mathcal{C}$, the category of frames* $\operatorname{Fr}\mathcal{C}$ *is a semisimplicial fibration category and the evaluation at* $0$ *functor* $\operatorname{Fr}\mathcal{C} \to \mathcal{C}$ *is a weak equivalence.*

*(2) For a tribe $\mathcal{T}$, the category of frames* $\operatorname{Fr}\mathcal{T}$ *is a semisimplicial tribe and the evaluation at* $0$ *functor* $\operatorname{Fr}\mathcal{T} \to \mathcal{T}$ *is a weak equivalence.*

*Proof.* Part (1) is [Sch13, Thms. 3.10 and 3.17].

For part (2), $\operatorname{Fr}\mathcal{T}$ is a tribe by Lemma 2.22. The axiom (SF) follows from part (1) and (ST) follows from Lemma 3.6. □

We note two properties of the semisimplicial structure of $\operatorname{Fr}\mathcal{C}$. First, the object

$$(K \rhd a)_m = \int_{[n]\in\Delta_\sharp} a_n^{(\Delta_\sharp[m]\boxtimes K)_n}$$

is a cotensor of $a$ by $K$ by the proof of [Sch13, Thm. 3.17]. Moreover, for frames $a$ and $b$, the hom-object is a semisimplicial set whose $m$-simplices are maps of frames $a \to \Delta_\sharp[m] \rhd b$.

If $\mathcal{C}$ is semisimplicial fibration category, then Lemma 3.3 yields another structure of a semisimplicial fibration category on $\operatorname{Fr}\mathcal{C}$ with levelwise cotensors. In general, this structure differs from the one described in Theorem 3.7 and in the remainder of the paper we will always consider the latter. However, the two cotensor operations agree on canonical frames.

**Lemma 3.8.** *Let $\mathcal{C}$ be a semisimplicial fibration category and $a \in \mathcal{C}$. Then for every finite semisimplicial set $K$, we have $K \rhd (a^{\Delta_\sharp[-]}) \cong (a^K)^{\Delta_\sharp[-]}$.*

*Proof.* We have the following string of isomorphisms, natural in $m$:

$$(K \rhd (a^{\Delta_\sharp[-]}))_m \cong \int_{[n]} (a^{\Delta_\sharp[n]})^{(\Delta_\sharp[m]\boxtimes K)_n} \cong a^{\int^{[n]} \Delta_\sharp[n]\times(\Delta_\sharp[m]\boxtimes K)_n} \cong a^{K\boxtimes\Delta_\sharp[m]} \cong (a^K)^{\Delta_\sharp[m]}. \qquad \square$$

We now need to establish a semisimplicial equivalence between a semisimplicial fibration category $\mathcal{C}$ and $\operatorname{Fr}\mathcal{C}$ (note that the evaluation at 0 functor of Theorem 3.7 is not semisimplicial). By the preceding lemma, there is a semisimplicial exact functor $\mathcal{C} \to \operatorname{Fr}\mathcal{C}$ given by $a \mapsto a^{\Delta_\sharp[-]}$ which is in fact a weak equivalence. However, it is only pseudonatural in $\mathcal{C}$ which is insufficient for our purposes. We can correct this defect by introducing a modified version of $\operatorname{Fr}\mathcal{C}$.

Let $\widehat{\operatorname{Fr}}\mathcal{C}$ be the full subcategory of $(\operatorname{Fr}\mathcal{C})^{\widehat{[1]}}_{\mathrm{R}}$ spanned by objects $X$ such that $X_1$ is (isomorphic to) the canonical frame on $X_{1,0}$.

**Proposition 3.9.**

*(1) If $\mathcal{C}$ is a semisimplicial fibration category, then so is $\widehat{\operatorname{Fr}}\mathcal{C}$. Moreover, both the evaluation at* $0$ *functor $\widehat{\operatorname{Fr}}\mathcal{C} \to \operatorname{Fr}\mathcal{C}$ and the evaluation at $(1,0)$ functor $\widehat{\operatorname{Fr}}\mathcal{C} \to \mathcal{C}$ are semisimplicial exact.*

*(2) If $\mathcal{T}$ is a semisimplicial tribe, then so is $\widehat{\operatorname{Fr}}\mathcal{T}$. Moreover, both the evaluation at* $0$ *functor $\widehat{\operatorname{Fr}}\mathcal{T} \to \operatorname{Fr}\mathcal{T}$ and the evaluation at $(1,0)$ functor $\widehat{\operatorname{Fr}}\mathcal{T} \to \mathcal{T}$ are semisimplicial homomorphisms.*

*Proof.* If $\mathcal{C}$ is a semisimplicial fibration category, then so is $(\operatorname{Fr}\mathcal{C})^{\widehat{[1]}}_{\mathrm{R}}$ by Lemma 3.3. The subcategory $\widehat{\operatorname{Fr}}\mathcal{C}$ contains the terminal object and is closed under pullbacks along fibrations. Moreover, it is closed under powers by finite semisimplicial sets by Lemma 3.8. Therefore, it is enough to verify that it has factorizations. Given a morphism $a \to b$, we first factor $a_{1,0} \to b_{1,0}$ as $a_{1,0} \xrightarrow{\sim} \widehat{a}_{1,0} \twoheadrightarrow b_{1,0}$ in $\mathcal{C}$. If we set $\widehat{a}_1$ to the canonical frame on $\widehat{a}_{1,0}$, then $a_1 \to \widehat{a}_1$ is a levelwise weak equivalence and $\widehat{a}_1 \to b_1$ is a Reedy fibration by (SF). To complete the factorization, it suffices to factor $a_0 \to \widehat{a}_1 \times_{b_1} b_0$ into a levelwise weak equivalence and a Reedy fibration.

The two evaluation functors $\widehat{\mathrm{Fr}}\mathcal{C} \to \mathrm{Fr}\,\mathcal{C}$ and $\widehat{\mathrm{Fr}}\mathcal{C} \to \mathcal{C}$ are semisimplicial exact by construction. (In particular, preservation of powers by the latter follows from Lemma 3.8.)

In part (2) we proceed in similar manner, this time using (ST) to construct a factorization into a levelwise anodyne morphism followed by a Reedy fibration. Using this factorization and a retract argument as in the proof of Lemma 2.22 we verify that a morphism of $\widehat{\mathrm{Fr}}\mathcal{T}$ is anodyne if and only if it is levelwise anodyne which directly implies the remaining axioms. □

**Lemma 3.10.**

*(1) If $\mathcal{C}$ is a semisimplicial fibration category, then the evaluation at $(1,0)$ functor $\widehat{\mathrm{Fr}}\mathcal{C} \to \mathcal{C}$ is a weak equivalence.*

*(2) If $\mathcal{T}$ is a semisimplicial tribe, then the evaluation at $(1,0)$ functor $\widehat{\mathrm{Fr}}\mathcal{T} \to \mathcal{T}$ is a weak equivalence.*

*Proof.* Part (2) is a special case of part (1) which is verified as follows. Define a functor $F\colon \mathcal{C} \to \widehat{\mathrm{Fr}}\mathcal{C}$ so that $(Fa)_0 = (Fa)_1 = a^{\Delta_\sharp[-]}$ with the identity structure map. Then $F$ is a homotopical functor and $(Fa)_{1,0} = a$ for all $a \in \mathcal{C}$. Moreover, for any $b \in \widehat{\mathrm{Fr}}\mathcal{C}$ the structure map $b_0 \to b_1$ provides natural weak equivalence $b \xrightarrow{\sim} Fb_{1,0}$. Hence the evaluation functor is a homotopy equivalence. □

**Lemma 3.11.**

*(1) If $\mathcal{C}$ is a semisimplicial fibration category, then the evaluation at $0$ functor $\widehat{\mathrm{Fr}}\mathcal{C} \to \mathrm{Fr}\,\mathcal{C}$ is a weak equivalence.*

*(2) If $\mathcal{T}$ is a semisimplicial tribe, then the evaluation at $0$ functor $\widehat{\mathrm{Fr}}\mathcal{T} \to \mathrm{Fr}\,\mathcal{T}$ is a weak equivalence.*

*Proof.* Part (2) follows from part (1) which will be proven by verifying the approximation properties of Theorem 2.9.

(App1) is obvious. For (App2) consider an object $b \in \widehat{\mathrm{Fr}}\mathcal{C}$ and a morphism $a \to b_0$ in $\mathrm{Fr}\,\mathcal{C}$. First, factor $a \to a \times b_0$ as a weak equivalence $a \xrightarrow{\sim} \bar{a}$ followed by a Reedy fibration $\bar{a} \twoheadrightarrow a \times b_0$. Factor the resulting morphism $\bar{a} \to \bar{a} \times_{b_{1,0}} \bar{a}$ as a weak equivalence $\bar{a}_0 \xrightarrow{\sim} \widehat{a}_0$ followed by a fibration $\widehat{a}_0 \twoheadrightarrow \bar{a} \times_{b_{1,0}} \bar{a}$. The pullback $\bar{a} \times_{b_{1,0}} \bar{a}$ is the 0th level of $\bar{a} \times_{b_1} \bar{a}_0^{\Delta_\sharp[-]}$ and thus $\widehat{a}_0 \twoheadrightarrow \bar{a} \times_{b_{1,0}} \bar{a}$ lifts to a Reedy fibration $\widehat{a} \twoheadrightarrow \bar{a} \times_{b_1} \bar{a}_0^{\Delta_\sharp[-]}$ by [Szu17a, Lem. 1.10] and [Szu16, Prop. 2.5][2]. We define $\widetilde{a}$ by forming the pullback square on the left.

$$\begin{tikzcd}
a \arrow[r, "\sim"] & \bar{a} \\
\widetilde{a} \arrow[u, two heads, "\sim"] \arrow[r, "\sim"'] & \widehat{a} \arrow[u, two heads, "\sim"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] & b_0 \\
\widetilde{a} \arrow[u, "\sim"] \arrow[r, "\sim"'] & \widehat{a} \arrow[u]
\end{tikzcd}$$

The right square is obtained by composing with $\bar{a} \to b_0$ which verifies (App2) since $\widehat{a}$ lifts to an object of $\widehat{\mathrm{Fr}}\mathcal{C}$ given by the acyclic fibration $\widehat{a} \to \bar{a}_0^{\Delta_\sharp[-]}$. □

**Proposition 3.12.**

*(1) The functor $\mathsf{sFibCat} \to \mathsf{FibCat}$ is a DK-equivalence.*

*(2) The functor $\mathsf{sTrb} \to \mathsf{Trb}$ is a DK-equivalence.*

*Proof.* The proofs of two parts are parallel. The functor $\mathrm{Fr}\colon \mathsf{FibCat} \to \mathsf{sFibCat}$ is homotopical. By Theorem 3.7 and Lemmas 3.10 and 3.11, it is a homotopy inverse to the functor $\mathsf{sFibCat} \to \mathsf{FibCat}$. □

[2] These references use the notion of a fibration between fibration categories which will be studied in depth in Section 4.

## 4 Fibration categories of fibration categories and tribes

In this section, we construct the fibration categories of semisimplicial fibration categories and semisimplicial tribes. We begin by recalling the fibration category of fibration categories of [Szu16].

**Definition 4.1** ([Szu16, Definition 2.3])**.** An exact functor $P\colon \mathcal{E} \to \mathcal{D}$ between fibration categories is a *fibration* if it satisfies the following properties.

(1) It is an *isofibration*: for every object $a \in \mathcal{E}$ and an isomorphism $f'\colon Pa \to b'$ there is an isomorphism $f\colon a \to b'$ such that $Pf = f'$.

(2) It has the *lifting property for WF-factorizations*: for any morphism $f\colon a \to b$ of $\mathcal{E}$ and a factorization

$$\begin{tikzcd} Pa \arrow[rr, "Pf"] \arrow[dr, "\sim", "i'"'] & & Pb \\ & c' \arrow[ur, two heads, "p'"'] & \end{tikzcd}$$

there exists a factorization

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\sim", "i"'] & & b \\ & c \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

such that $Pi = i'$ and $Pp = p'$ (in particular, $Pc = c'$).

(3) It has the *lifting property for pseudofactorizations*: for any morphism $f\colon a \to b$ of $\mathcal{E}$ and a diagram

$$\begin{tikzcd} Pa \arrow[r, "Pf"] & Pb \\ c' \arrow[u, two heads, "i'", "\sim"'] \arrow[r, "\sim", "s'"'] & d' \arrow[u, two heads, "u'"'] \end{tikzcd}$$

there exists a diagram

$$\begin{tikzcd} a \arrow[r, "f"] & b \\ c \arrow[u, two heads, "i", "\sim"'] \arrow[r, "\sim", "s"'] & d \arrow[u, two heads, "u"'] \end{tikzcd}$$

such that $Pi = i'$, $Ps = s'$ and $Pu = u'$ (in particular, $Pc = c'$ and $Pd = d'$).

**Theorem 4.2** ([Szu16, Thm. 2.8])**.** *The category of fibration categories with weak equivalences and fibrations as defined above is a fibration category.* □

The key difficulty lies in the construction of path objects and it is addressed in [Szu16, Thm. 2.8] by a modified version of the Reedy structure. This modification is not available in the setting of tribes where for a tribe $\mathcal{T}$ it is difficult to ensure that both the path object $P\mathcal{T}$ is a tribe and a homomorphism $\mathcal{T} \to P\mathcal{T}$ exists.

For semisimplicial tribes, we can use the standard Reedy structure to construct $P\mathcal{T}$ and the homomorphism $\mathcal{T} \to P\mathcal{T}$ can be defined using cotensors.

We proceed to define fibrations of semisimplicial tribes. Since the definition does not depend on the enrichment, we first give it for ordinary tribes.

**Definition 4.3.** A homomorphism $P\colon \mathcal{S} \to \mathcal{T}$ between tribes is a *fibration* if it is a fibration of underlying fibration categories and satisfies the following properties.

(4) It has the *lifting property for AF-factorizations*: for any morphism $f\colon x \to y$ of $\mathcal{S}$ and a factorization

$$\begin{tikzcd} Px \arrow[rr, "Pf"] \arrow[dr, tail, "\sim", "i'"'] & & Py \\ & z' \arrow[ur, two heads, "p'"'] & \end{tikzcd}$$

there exists a factorization

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, tail, "\sim", "i"'] & & y \\ & z \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

such that $Pi = i'$ and $Pp = p'$ (in particular, $Pz = z'$).

(5) It has the *lifting property for lifts*: for a square

$$\begin{tikzcd} a \arrow[r] \arrow[d, tail, "\sim"'] & x \arrow[d, two heads] \\ b \arrow[r] & y \end{tikzcd}$$

in $\mathcal{S}$ and a lift for its image

$$\begin{tikzcd} Pa \arrow[r] \arrow[d, tail, "\sim"'] & Px \arrow[d, two heads] \\ Pb \arrow[r] \arrow[ur, dashed, "f'"] & Py \end{tikzcd}$$

in $\mathcal{T}$, there exists a lift $f$ in the original square such that $Pf = f'$.

(6) It has the *lifting property for cofibrancy lifts*: for any acyclic fibration $p\colon x \overset{\sim}{\twoheadrightarrow} y$, a morphism $f\colon a \to y$ in $\mathcal{S}$ and a lift

$$\begin{tikzcd} & Px \arrow[d, two heads, "\sim"', "Pp"] \\ Pa \arrow[r, "Pf"'] \arrow[ur, dashed, "g'"] & Py, \end{tikzcd}$$

there is a morphism $g\colon a \to x$ such that $Pg = g'$ and $pg = f$.

This definition is similar to Joyal's definition of a *meta-fibration*, i.e., a homomorphism of tribes satisfying conditions (1), (4) and (5) above as well as the *lifting property for sections of acyclic fibrations*: for any acyclic fibration $p$ in $\mathcal{S}$ and a section $s$ of $Pp$, there is a section $s'$ of $p$ such that $Ps' = s$. The latter can be shown to be equivalent to (6). Our definition also includes conditions (2) and (3) since we need the forgetful functor $\mathsf{sTrb} \to \mathsf{sFibCat}$ to preserve fibrations.

**Definition 4.4.**

(1) An exact functor of semisimplicial fibration categories is a *fibration* if it is a fibration of their underlying fibration categories.

(2) A homomorphism of semisimplicial tribes is a *fibration* if it is a fibration of their underlying tribes.

Recall that a functor $I \to J$ of small categories is a *cosieve* if it is injective on objects, fully faithful and if $i \to j$ is a morphism of $J$ such that $i \in I$, then $j \in I$. The following lemma gives a basic technique of constructing fibrations, using cosieves.

**Lemma 4.5.** *Let $I \to J$ be a cosieve between homotopical inverse categories.*

*(1) If $\mathcal{C}$ is semisimplicial fibration category, then the induced functor $\mathcal{C}_{\mathrm{R}}^J \to \mathcal{C}_{\mathrm{R}}^I$ is a fibration.*

*(2) If $\mathcal{T}$ is semisimplicial tribe, then the induced functor $\mathcal{T}_{\mathrm{R}}^J \to \mathcal{T}_{\mathrm{R}}^I$ is a fibration.*

*Proof.* Part (1) follows directly from [Szu16, Lemma 1.10].

In particular, the functor in part (2) is a fibration of underlying fibration categories. We check the remaining conditions of Definition 4.3.

Let $x \to y$ be a morphism in $\mathcal{T}_{\mathrm{R}}^J$ and a factorization $x|I \overset{\sim}{\rightarrowtail} x' \twoheadrightarrow y|I$ of its restriction to $I$. By induction, it suffices to extend it to the subcategory generated by $I$ and an object $j \in J$ of minimal degree among those not in $I$. The partial factorization above induces a morphism $x_j \to M_j x' \times_{M_j y} y_j$ which we factor as an anodyne morphism $x_j \overset{\sim}{\rightarrowtail} x'_j$ followed by a fibration $x'_j \twoheadrightarrow M_j x' \times_{M_j y} y_j$. This proves the lifting property for AF-factorizations.

Let

$$\begin{tikzcd} a \arrow[r] \arrow[d, tail, "\sim"'] & x \arrow[d, two heads] \\ b \arrow[r] & y \end{tikzcd}$$

be a lifting problem in $\mathcal{T}_{\mathrm{R}}^J$ and $b|I \to x|I$ a solution of its restriction to $I$. Again, it is enough to extend it to the subcategory generated by $I$ and an object $j \in J$ of minimal degree among those not in $I$. This extension can be chosen as a solution in the following lifting problem:

$$\begin{tikzcd} a_j \arrow[r] \arrow[d, tail, "\sim"'] & x_j \arrow[d, two heads] \\ b_j \arrow[r] & M_j x \times_{M_j y} y_j. \end{tikzcd}$$

This proves the lifting property for lifts.

The verification of the lifting property for cofibrancy lifts is analogous. □

In the next two lemmas, we construct path objects and pullbacks along fibrations in the categories $\mathsf{sFibCat}$ and $\mathsf{sTrb}$.

For a semisimplicial fibration category $\mathcal{C}$ let $P\mathcal{C}$ denote $\mathcal{C}_{\mathrm{R}}^{\mathrm{Sd}\,\widehat{[1]}^{\mathrm{op}}}$, where $\mathrm{Sd}\,\widehat{[1]}$ is the homotopical poset $\{0 \xrightarrow{\sim} 01 \xleftarrow{\sim} 1\}$. It comes with a functor $\mathcal{C} \to P\mathcal{C}$ that maps $x$ to the diagram $x \leftarrow x^{\Delta_\sharp[1]} \to x$ (which is semisimplicial exact by (SF)) and a functor $P\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ that evaluates at 0 and 1. For a semisimplicial tribe $\mathcal{T}$, we define an analogous factorization $\mathcal{T} \to P\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ in $\mathsf{sTrb}$ (where $\mathcal{T} \to P\mathcal{T}$ is a semisimplicial homomorphism by (SF) and (ST)).

**Lemma 4.6.**

*(1) The object $P\mathcal{C}$ with the factorization $\mathcal{C} \to P\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is a path object for $\mathcal{C}$ in* $\mathsf{sFibCat}$.

*(2) The object $P\mathcal{T}$ with the factorization $\mathcal{T} \to P\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is a path object for $\mathcal{T}$ in* $\mathsf{sTrb}$.

*Proof.* The proofs of both parts are analogous so we only prove part (2). $P\mathcal{T}$ is a semisimplicial tribe by Lemma 3.3. The evaluation $P\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is a fibration by Lemma 4.5. The functor $\mathcal{T} \to P\mathcal{T}$ has a retraction given by evaluation at 0 and thus it suffices to check that this evaluation is a weak equivalence. It is induced by a homotopy equivalence $[0] \to \mathrm{Sd}\,\widehat{[1]}$ and hence it is a weak equivalence by [Szu16, Lem. 1.8(3)]. □

**Lemma 4.7.**

*(1) Pullbacks along fibrations exist in* $\mathsf{sFibCat}$.

*(2) Pullbacks along fibrations exist in* $\mathsf{sTrb}$.

*Proof.* For part (1), let $F\colon \mathcal{C} \to \mathcal{E}$ and $P\colon \mathcal{D} \to \mathcal{E}$ be exact functors between semisimplicial fibration categories with $P$ a fibration. Form a pullback of $P$ along $F$ in the category of semisimplicial categories.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{D} \arrow[d, two heads, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}$$

Define a morphism $f$ of $\mathcal{P}$ to be a weak equivalence (a fibration) if both $Gf$ and $Qf$ are weak equivalences (fibrations). By [Szu16, Prop. 2.4], $\mathcal{P}$ is a pullback of the underlying fibration categories. It remains to verify that it is a semisimplicial fibration category.

Let $(x, y)$ be an object of $\mathcal{P}$ and $K$ a finite semisimplicial set. We form a cotensor $x^K$ in $\mathcal{C}$ and lift its image $F(x^K)$ in $\mathcal{E}$ to a cotensor $y^K$ in $\mathcal{D}$ using the fact that $P$ is an isofibration. Then $(x^K, y^K)$ is a cotensor of $(x, y)$ by $K$ in $\mathcal{P}$. The pullback-cotensor property (SF) is satisfied in $\mathcal{P}$ since it is satisfied in both $\mathcal{C}$ and $\mathcal{D}$.

The proof of part (2) is very similar. However, there are a few differences, so we spell it out.

Let $F\colon \mathcal{R} \to \mathcal{T}$ and $P\colon \mathcal{S} \to \mathcal{T}$ be homomorphisms between semisimplicial tribes with $P$ a fibration. Form a pullback of $P$ along $F$ in the category of semisimplicial categories.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "G"] \arrow[d, "Q"'] & \mathcal{S} \arrow[d, two heads, "P"] \\
\mathcal{R} \arrow[r, "F"'] & \mathcal{T}
\end{tikzcd}$$

Define a morphism $f$ of $\mathcal{P}$ to be a fibration if both $Gf$ and $Qf$ are fibrations.

First, let $1_{\mathcal{R}}$ be a terminal object of $\mathcal{R}$. Since $P$ is an isofibration there is a terminal object $1_{\mathcal{S}}$ of $\mathcal{S}$ such that $P1_{\mathcal{S}} = F1_{\mathcal{R}}$. Then $(1_{\mathcal{R}}, 1_{\mathcal{S}})$ is a terminal object of $\mathcal{P}$. Moreover, all objects are fibrant since all objects of $\mathcal{R}$ and $\mathcal{S}$ are fibrant. This proves (T1).

Similarly, to construct a pullback in $\mathcal{P}$, we first construct it in $\mathcal{R}$ and then lift its image from $\mathcal{T}$ to $\mathcal{S}$. Fibrations are stable under pullback since they are stable in both $\mathcal{R}$ and $\mathcal{S}$. This proves (T2).

To factor a morphism as a levelwise anodyne morphism followed by a fibration in $\mathcal{P}$, we first factor it in $\mathcal{R}$ and then lift the image of this factorization from $\mathcal{T}$ to $\mathcal{S}$. A retract argument as in the proof of Lemma 2.22 shows that every anodyne morphism is levelwise anodyne.

Conversely, to solve a lifting problem between a levelwise anodyne morphism and a fibration, we first solve it in $\mathcal{R}$ and then lift the image of this solution from $\mathcal{T}$ to $\mathcal{S}$. It follows that every morphism factors as an anodyne morphism followed by a fibration, proving (T3).

Levelwise anodyne morphisms are stable under pullbacks along fibrations and thus so are the anodyne morphisms. This proves (T4).

Before we can conclude the proof, we need to verify that a morphism $f$ in $\mathcal{P}$ is a weak equivalence if and only if $Qf$ an $Gf$ are weak equivalences in $\mathcal{R}$ and $\mathcal{S}$, respectively. To this end, we consider the fibration category $\mathcal{P}_{\mathrm{lvl}}$ arising as the pullback of the underlying fibration categories of $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. Note that $\mathcal{P}$ and $\mathcal{P}_{\mathrm{lvl}}$ have the same underlying category and the same fibrations while weak equivalences in $\mathcal{P}_{\mathrm{lvl}}$ are levelwise. Moreover, every object of $\mathcal{P}_{\mathrm{lvl}}$ is cofibrant. Indeed, to find a lift against an acyclic fibration $p$ in $\mathcal{P}_{\mathrm{lvl}}$, using Lemma 2.18, we first pick a lift against $Qp$ in $\mathcal{R}$ and then lift its image in $\mathcal{T}$ to a lift against $Gp$ in $\mathcal{S}$. A path object in $\mathcal{P}$ is also a path object in $\mathcal{P}_{\mathrm{lvl}}$ since anodyne morphisms in $\mathcal{P}$ are levelwise. The conclusion follows by the same argument as in the proof of Lemma 2.22.

Finally, since weak equivalences in $\mathcal{P}$ are levelwise, the proof that it is a semisimplicial tribe is the same as in the case of fibration categories above. □

In the construction of the fibration categories $\mathsf{sFibCat}$ and $\mathsf{sTrb}$, we will need the following characterization of acyclic fibrations.

**Lemma 4.8.**

*(1) An exact functor $P \colon \mathcal{C} \to \mathcal{D}$ of semisimplicial fibrations categories is an acyclic fibration if and only if it is a fibration, satisfies (App1) and for every fibration $q \colon x' \twoheadrightarrow Py$ in $\mathcal{D}$, there is a fibration $p \colon x \twoheadrightarrow y$ such that $Pp = q$.*

*(2) A homomorphism $P \colon \mathcal{S} \to \mathcal{T}$ of semisimplicial tribes is an acyclic fibration if and only if it is a fibration, satisfies (App1) and for every fibration $q \colon x' \twoheadrightarrow Py$ in $\mathcal{T}$, there is a fibration $p \colon x \twoheadrightarrow y$ such that $Pp = q$.*

*Proof.* This follows directly from [Szu16, Prop. 2.5]. □

**Theorem 4.9.**

*(1) The category of semisimplicial fibration categories with weak equivalences and fibrations as defined above is a fibration category.*

*(2) The category of semisimplicial tribes with weak equivalences and fibrations as defined above is a fibration category.*

*Proof.* The proofs of both parts are parallel. Axioms (F1) and (F4) are immediate. (F2) follows from Lemmas 4.7 and 4.8 while (F3) follows from Lemma 4.6 and [Bro73, Factorization lemma, p. 421]. □

## 5 Presheaves over simplicial categories

In this section we introduce a tribe of injectively fibrant simplicial presheaves over a simplicial category, which will be the starting point of constructions of tribes in Sections 6 and 7. We begin by recalling model structures on the categories of simplicial presheaves.

**Theorem 5.1** ([Lur09, Prop. A.3.3.2])**.** *The category of simplicial (enriched) presheaves over a small simplicial category carries two cofibrantly generated proper model structures:*

*(1) the* injective *model structure where weak equivalences are levelwise weak homotopy equivalences and cofibrations are monomorphisms;*

*(2) the* projective *model structure where weak equivalences are levelwise weak homotopy equivalences and fibrations are levelwise Kan fibrations.* □

The fibrations of these model structures are usually called *injective fibrations* and *projective fibrations*, respectively. We will almost always use injective fibrations, so we will call them *fibrations* for brevity. Similarly, *injectively fibrant* presheaves will be referred to as *fibrant* presheaves.

The cofibrations of the injective model structure are closed under pullback since they are exactly monomorphisms. This yields the following corollary.

**Corollary 5.2.** *The category of fibrant presheaves over a small simplicial category is a tribe.* □

In the remainder of this section, we review a few standard facts about homotopy theory of presheaves.

We will use the notion of a homotopy pullback as in Definition 2.4 in the fibration category underlying the tribe above. However, since it arises from a right proper model structure, Definition 2.4 applies verbatim even to non-fibrant presheaves. In particular, Lemma 2.5 holds for non-fibrant presheaves as well.

**Lemma 5.3.** *A square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

*of presheaves over $\mathcal{A}$ is a homotopy pullback if and only if for every object $a \in \mathcal{A}$ and every point of $B_a$, the induced map from the homotopy fiber of $A \to B$ to the homotopy fiber of $C \to D$ is a weak homotopy equivalence.*

*Proof.* Since every injective fibration is in particular a projective fibration, the square above is a homotopy pullback if and only if it is a levelwise homotopy pullback. Moreover, by [MV15, Prop. 3.3.18] a square of simplicial sets is a homotopy pullback if and only if the induced maps on all homotopy fibers are weak homotopy equivalences. □

**Definition 5.4.** If $\mathcal{A}$ is a simplicial category, then a presheaf $A$ over $\mathcal{A}$ is *homotopy representable* if there exist $a \in \mathcal{A}$ and a weak equivalence $\mathcal{A}(-, a) \xrightarrow{\sim} A$ (called a *representation* of $A$).

For brevity, homotopy representable presheaves will be called representable.

**Lemma 5.5.** *A presheaf weakly equivalent to a representable one is also representable.*

*Proof.* If $A \xrightarrow{\sim} B$ is a weak equivalence and $A$ is representable, then so is $B$. Thus it is enough to check that if $B$ is representable, then so is $A$. Let $r_b \colon \mathcal{A}(-, b) \xrightarrow{\sim} B$ be a representation. Since $A_b \to B_b$ is a weak homotopy equivalence, we can pick a vertex in $A_b$ and a path connecting its image in $B_b$ to $r_b(\mathrm{id}_b)$ which yields a homotopy commutative triangle

$$\begin{tikzcd} & A \arrow[d, "\sim"'] \\ \mathcal{A}(-, b) \arrow[ur, dashed] \arrow[r, "r_b", "\sim"'] & B \end{tikzcd}$$

and thus the dashed arrow provides a representation of $A$. ☐

**Lemma 5.6** ([Hir03, Cor. 7.3.12(2)])**.** *If $X \twoheadrightarrow Y$ is a fibration between presheaves and*

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr] & \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

*is a homotopy commutative triangle, then there is a map $g \colon A \to X$ homotopic to $f$ making the triangle commute strictly.* ☐

## 6 Hammock localization of a fibration category

The goal of the present section is the construction of the tribe of representable fibrant presheaves over the hammock localization $\mathrm{L^H}\,\mathcal{C}$ [DK80] of a fibration category $\mathcal{C}$. In the context of fibration categories, the mapping spaces of the hammock localization can be approximated by the categories of fractions.

In a fibration category $\mathcal{C}$, a *right fraction* from $a$ to $b$ is a diagram of the form

$$a \xleftarrow[\sim]{v} a' \xrightarrow{s} b$$

which we denote by $s\bar{v} \colon a \rightsquigarrow b$. Such a fraction is *Reedy fibrant* if the morphism $e \to a \times b$ is a fibration. We will write $\mathrm{Frac}_{\mathcal{C}}(a, b)$ for the category of right fractions from $a$ to $b$ and $\mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(a, b)$ for the category of Reedy fibrant right fractions from $a$ to $b$.

**Theorem 6.1.** *For all $a, b \in \mathcal{C}$ the canonical maps*

$$\mathrm{N}\,\mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(a, b) \xrightarrow{\iota} \mathrm{N}\,\mathrm{Frac}_{\mathcal{C}}(a, b) \longrightarrow \mathrm{L^H}\,\mathcal{C}(a, b)$$

*are weak homotopy equivalences.*

This theorem is a variant of a classical result of Dwyer and Kan [DK80, Prop. 6.2]. The proof of the first part uses semisimplicial methods which we summarize in the following lemmas.

**Lemma 6.2.** *The forgetful functor from simplicial sets to semisimplicial sets is a homotopy equivalence.*

*Proof.* We will show that the unit and the counit of the free/forgetful adjunction $F \dashv U$ are both weak homotopy equivalences. First, note that $FU\Delta[m]$ is the nerve of the category $[m]'$ whose morphisms are exactly the morphisms of $[m]$ along with one idempotent endomorphism of each object. There is a natural transformation connecting $\mathrm{id}\colon [m]' \to [m]'$ with the constant functor at 0 and hence $FU\Delta[m]$ is contractible. Thus $\eta_{\Delta[m]}\colon \Delta[m] \to FU\Delta[m]$ is a weak homotopy equivalence for all $m$. A standard inductive argument over the skeleta of a simplicial set $K$ (using the Gluing Lemma in the inductive step) shows that $\eta$ is a weak homotopy equivalence everywhere. The weak homotopy equivalences of semisimplicial sets are created by $F$ and thus the triangle identity $F\epsilon \cdot \eta F = \mathrm{id}_F$ implies that $\epsilon$ is also a weak homotopy equivalence. $\square$

A *homotopy* between semisimplicial maps $f, g\colon K \to L$ is a semisimplicial map $K \boxslash \Delta_\sharp[1] \to L$ that extends $[f, g]\colon K \boxslash \partial\Delta_\sharp[1] \to L$. The free simplicial set functor is monoidal and hence it carries such semisimplicial homotopies to simplicial homotopies.

**Lemma 6.3.** *Let $K \to L$ be a semisimplicial map such that for every $m$, every square of the form*

$$\begin{tikzcd} \partial\Delta_\sharp[m] \arrow[r] \arrow[d, hook] & K \arrow[d] \\ \Delta_\sharp[m] \arrow[r] & L \end{tikzcd}$$

*admits a lift up to homotopy, i.e., there is a map $\Delta_\sharp[m] \to K$ making the upper triangle commute strictly and the lower triangle commute up to homotopy relative to $\partial\Delta_\sharp[m]$. Then $K \to L$ is a weak homotopy equivalence.*

*Proof.* The argument of [KS17, Lem. 5.3] shows that $K \to L$ is a semisimplicial homotopy equivalence. The preceding remark implies that the free functor carries it to a simplicial homotopy equivalence and thus it is a weak homotopy equivalence. $\square$

*Proof of Theorem 6.1.* The second morphism is a weak homotopy equivalence by [NSS15, Thm. 3.61].

To verify that $\iota$ is a weak homotopy equivalence, by Quillen's Theorem A [Qui73, p. 85] and [LTW79, Thm. 4.1 and Rmk. 5.6] it is enough to show that for every fraction $s\bar{v} \in \mathrm{Frac}_{\mathcal{C}}(a, b)$ the slice $\mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)$ is contractible. Here, Ex denotes Kan's classical functor [Kan57].

We define a semisimplicial set $\mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}}$ as follows. An $m$-simplex in $\mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}}$ is an $m$-simplex $(\mathrm{Sd}[m])^{\mathrm{op}} \to s\bar{v}\downarrow\iota$ of $\mathrm{Ex}\,\mathrm{N}(s\bar{v}\downarrow\iota)$ that is Reedy fibrant. Here for a poset $P$, $\mathrm{Sd}\,P$ denotes the poset of chains (i.e., finite non-empty totally ordered subsets) in $P$, ordered by inclusion. Explicitly, such an $m$-simplex consists of a functor $t\bar{w}\colon (\mathrm{Sd}[m])^{\mathrm{op}} \to \mathrm{Frac}_{\mathcal{C}}(a, b)$ Reedy fibrant as a diagram in $\mathcal{C} \downarrow a \times b$ together with a morphism $\lambda\colon s\bar{v} \to t_{[m]}\bar{w}_{[m]}$.

We will check that the inclusion $\mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}} \hookrightarrow \mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)$ is a weak homotopy equivalence. By Lemma 6.3, it is enough to find a lift up to homotopy in every diagram of the form

$$\begin{tikzcd} \partial\Delta_\sharp[m] \arrow[r] \arrow[d, hook] & \mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}} \arrow[d] \\ \Delta_\sharp[m] \arrow[r] & \mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota). \end{tikzcd}$$

The data of the diagram corresponds to a simplex $(t\bar{w}, \lambda)$ as above such that $t\bar{w}$ is Reedy fibrant only over $\mathrm{Sd}\,\partial\Delta_\sharp[m]$, i.e., the subposet of $\mathrm{Sd}[m]$ obtained by removing the top element. By [Szu17a, Lem. 1.9(1)] we can find a Reedy fibrant $t'\bar{w}'$ together with a weak equivalence $u\colon t\bar{w} \xrightarrow{\sim} t'\bar{w}'$ that restricts to the identity over $\mathrm{Sd}\,\partial\Delta_\sharp[m]$. Then $(t'\bar{w}', u_{[m]}\lambda)$ is a simplex $\Delta_\sharp[m] \to \mathrm{Ex}\,\mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}}$ which makes the upper triangle commute while $u$ yields a homotopy in the lower one by taking the composite

$$(\mathrm{Sd}([m] \times [1]))^{\mathrm{op}} \longrightarrow (\mathrm{Sd}[m])^{\mathrm{op}} \times [1] \xrightarrow{u} s\bar{v} \downarrow \iota$$

where the first map takes a chain $A \subseteq [m] \times [1]$ to $(\mathrm{proj}_0 A, \min \mathrm{proj}_1 A)$.

By Lemmas 6.2 and 6.3, it suffices to show that $\operatorname{Ex} \mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}}$ is a contractible Kan complex. To extend a map $(t\bar{w}, \lambda) \colon \partial\Delta_\sharp[m] \to \operatorname{Ex} \mathrm{N}(s\bar{v} \downarrow \iota)^{\mathrm{R}}$ to $\Delta_\sharp[m]$ we factor $s\bar{v} \to \lim_{(\operatorname{Sd} \partial\Delta[m])^{\mathrm{op}}} t\bar{w}$ as a weak equivalence $s\bar{v} \to t_{[m]}\bar{w}_{[m]}$ followed by a fibration. $\square$

As a consequence we obtain the following corollary.

**Corollary 6.4.** *If $\mathcal{T}$ is a tribe, then every zig-zag in $\mathrm{L^H}\,\mathcal{T}$ is homotopic to a single morphism in $\mathcal{T}$.*

*Proof.* By Theorem 6.1, it suffices to show that every fraction is homotopic to a single arrow. Let

$$x \xleftarrow[\sim]{v} x' \xrightarrow{s} y$$

be such a fraction. Since $v$ is a weak equivalence, we can choose a homotopy inverse $w \colon a \to x'$. Let $H \colon x' \to Px'$ be a homotopy from $wv$ to $\mathrm{id}_{x'}$, where $(\pi_0, \pi_1)\sigma \colon x' \to Px' \to x' \times x'$ is a path object. Then

$$\begin{tikzcd}
 & x \arrow[d, "w"] & x' \arrow[l, "v"'] \arrow[r] \arrow[d, "H"] & x' \arrow[d] \arrow[dr, "s"] & \\
x \arrow[ur] \arrow[r, "w"] \arrow[dr, "w"'] & x' & Px' \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & x' \arrow[r, "s"] & y \\
 & x' \arrow[u] & x' \arrow[l] \arrow[r] \arrow[u, "\sigma"] & x' \arrow[u] \arrow[ur, "s"'] &
\end{tikzcd}$$

(where the unlabeled arrows are identities) is a homotopy between the original fraction and the composite $sw$ in $\mathrm{L^H}\,\mathcal{T}$. $\square$

We next turn our attention to homotopy limits in the category of representable fibrant presheaves over $\mathrm{L^H}\,\mathcal{C}$. In Lemma 6.6 we characterize homotopy terminal objects and in Proposition 6.9 homotopy pullbacks.

**Lemma 6.5.** *A morphism $f \colon a \to b$ in $\mathcal{C}$ is a weak equivalence if and only if the induced map*

$$\mathrm{L^H}\,\mathcal{C}(-, a) \xrightarrow{f_*} \mathrm{L^H}\,\mathcal{C}(-, b)$$

*is a weak equivalence.*

*Proof.* If $f$ is a weak equivalence, then so is $f_*$ by [DK80, Prop. 3.3].

Conversely, the map $\pi_0\,\mathrm{L^H}\,\mathcal{C}(-, a) \to \pi_0\,\mathrm{L^H}\,\mathcal{C}(-, b)$ coincides with $\operatorname{Ho}\mathcal{C}(-, a) \to \operatorname{Ho}\mathcal{C}(-, b)$ which is therefore an isomorphism. Thus $f$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$ and it follows from [RB06, Thm. 7.2.7] that it is a weak equivalence. $\square$

**Lemma 6.6.** *For every $a \in \mathcal{C}$ the morphism $a \to 1$ is a weak equivalence if and only if $\mathrm{L^H}\,\mathcal{C}(-, a) \to 1$ is a weak equivalence of presheaves.*

*Proof.* First assume that $a \to 1$ is a weak equivalence. For every $e \in \mathcal{C}$ we have weak homotopy equivalences

$$\mathrm{L^H}\,\mathcal{C}(e,a) \xrightarrow{\sim} \mathrm{L^H}\,\mathcal{C}(e,1) \xleftarrow{\sim} \mathrm{N}\,\mathrm{Frac}_{\mathcal{C}}(e,1)$$

by Theorem 6.1 and Lemma 6.5. Thus it is enough to check that $\mathrm{Frac}_{\mathcal{C}}(e,1)$ is contractible which is the case since it has a terminal object $e \leftarrow e \to 1$.

Conversely, assume that $\mathrm{L^H}\,\mathcal{C}(-,a) \to 1$ is a weak equivalence. Then in the diagram

$$\begin{tikzcd} \mathrm{L^H}\,\mathcal{C}(-,a) \arrow[rr] \arrow[dr, "\sim"'] & & \mathrm{L^H}\,\mathcal{C}(-,1) \arrow[dl, "\sim"] \\ & 1 & \end{tikzcd}$$

both downward maps are weak equivalences by the first part of the proof and hence so is the horizontal one. It follows that $a \to 1$ is a weak equivalence by Lemma 6.5. $\square$

**Lemma 6.7.** *If $q\colon A \twoheadrightarrow B$ is a fibration between representable presheaves over $\mathrm{L^H}\,\mathcal{C}$, then for every fixed representation $r_b\colon \mathrm{L^H}\,\mathcal{C}(-,b) \xrightarrow{\sim} B$ there are a representation $r_a\colon \mathrm{L^H}\,\mathcal{C}(-,a) \xrightarrow{\sim} A$ and a fibration $p\colon a \twoheadrightarrow b$ such that the square*

$$\begin{tikzcd} \mathrm{L^H}\,\mathcal{C}(-,a) \arrow[r, "r_a"] \arrow[d, "p_*"'] & A \arrow[d, "q"] \\ \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[r, "r_b"'] & B \end{tikzcd}$$

*commutes.*

*Proof.* Pick some representation $r_{\widetilde{a}}\colon \mathrm{L^H}\,\mathcal{C}(-,\widetilde{a}) \xrightarrow{\sim} A$ and consider $qr_{\widetilde{a}}(\mathrm{id}_{\widetilde{a}}) \in B_{\widetilde{a}}$. Since both

$$\mathrm{N}\,\mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(\widetilde{a},b) \longrightarrow \mathrm{L^H}\,\mathcal{C}(\widetilde{a},b) \xrightarrow{r_b} B_{\widetilde{a}}$$

are weak homotopy equivalences (see Theorem 6.1) and $B_{\widetilde{a}}$ is a Kan complex, there are a fraction $p\overline{w}\colon \widetilde{a} \rightsquigarrow b$ and an edge in $B_{\widetilde{a}}$ connecting $r_b(p\overline{w})$ to $qr_{\widetilde{a}}(\mathrm{id}_{\widetilde{a}})$. This yields a diagram

$$\begin{tikzcd} \mathrm{L^H}\,\mathcal{C}(-,a) \arrow[r, "r_{\widetilde{a}}w_*"] \arrow[d, "p_*"'] & A \arrow[d, "q"] \\ \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[r, "r_b"'] & B \end{tikzcd}$$

(where $a$ is the domain of $p$ and $\eta$) which does not commute strictly but only up to homotopy. However, since $q$ is a fibration this square can be strictified for the price of replacing $r_{\widetilde{a}}w_*$ by homotopic $r_a$ by Lemma 5.6. Since both $w_*$ and $r_{\widetilde{a}}$ are weak equivalences (by Lemma 6.5) so is $r_a$ which completes the proof. $\square$

**Lemma 6.8.** *Let $p\colon a \twoheadrightarrow b$ be a fibration in $\mathcal{C}$, then for every $e \in \mathcal{C}$ and a Reedy fibrant fraction $s\overline{v}\colon e \rightsquigarrow b$ the slice $p_* \downarrow s\overline{v}$ is the homotopy fiber of $p_*\colon \mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(e,a) \to \mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(e,b)$ over $s\overline{v}$.*

*Proof.* By Quillen's Theorem B [Qui73, p. 89] it is enough to check that for every morphism $u\colon s_0\bar{v}_0 \to s_1\bar{v}_1$ in $\mathrm{Frac}_{\mathcal{C}}(e,b)$, the induced functor $u_*\colon p_* \downarrow s_0\bar{v}_0 \to p_* \downarrow s_1\bar{v}_1$ is a weak homotopy equivalence. Indeed, $u_*$ has a right adjoint given by pullback provided that $u$ is a fibration.

On the other hand, given a general $u\colon e'_0 \to e'_1$, we take a factorization $e'_0 \xrightarrow{\sim} \bullet \twoheadrightarrow e'_0 \times_{e \times b} e'_1$ of $(\mathrm{id}, u)$ in the slice $\mathcal{C} \downarrow e \times b$ which yields a diagram

$$\begin{tikzcd}
 & e'_0 \arrow[dl, "u"'] \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
e'_1 & \bullet \arrow[l, two heads, "\sim"] \arrow[r, two heads, "\sim"'] & e'_0.
\end{tikzcd}$$

By the previous part of the argument, the two horizontal fibrations induce weak homotopy equivalences of the respective slices and so does the identity. Therefore, $u_*$ is a weak homotopy equivalence by 2-out-of-3. □

**Proposition 6.9.** *A square*

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "p"'] & c \arrow[d, "q"] \\
b \arrow[r, "g"'] & d
\end{tikzcd}$$

*in $\mathcal{C}$ is a homotopy pullback if and only if the associated square*

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-,a) \arrow[r] \arrow[d] & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-,c) \arrow[d] \\
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-,b) \arrow[r] & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-,d)
\end{tikzcd}$$

*is a homotopy pullback of presheaves.*

*Proof.* The statement is invariant under weak equivalences of squares in $\mathcal{C}$ so we can assume that both $p$ and $q$ are fibrations. Then the square in $\mathcal{C}$ is a homotopy pullback if and only if the morphism $(p,f)\colon a \to b \times_d c$ is a weak equivalence while Lemmas 5.3 and 6.8 imply that the square of presheaves is a homotopy pullback if and only if for all $e \in \mathcal{C}$ and $s\bar{v}\colon e \rightsquigarrow b$ the induced functor $g_*\colon p_* \downarrow s\bar{v} \to q_* \downarrow gs\bar{v}$ is a weak homotopy equivalence. We need to verify that these conditions are equivalent.

If the square in $\mathcal{C}$ is a homotopy pullback we can further assume that it is also a strict pullback. In this case $p_* \downarrow s\bar{v} \to q_* \downarrow gs\bar{v}$ turns out to be an isomorphism of categories. We construct its inverse as follows. Fix an object of $q_* \downarrow gs\bar{v}$, i.e.,

$$e \xleftarrow[\sim]{w} e'' \xrightarrow{t} c \quad \text{in } \mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(e,c) \text{ together with } \begin{tikzcd}
 & e'' \arrow[dl, "w"'] \arrow[d, "u"] \arrow[dr, "qt"] & \\
e & e' \arrow[l, "v"] \arrow[r, "gs"'] & d.
\end{tikzcd}$$

By the universal property of the original pullback we obtain a morphism $(su,t)\colon e'' \to a$. This yields

$$e \xleftarrow[\sim]{w} e'' \xrightarrow{(su,t)} a \quad \text{in } \mathrm{Frac}^{\mathrm{R}}_{\mathcal{C}}(e,a) \text{ together with } \begin{tikzcd} & e'' \arrow[dl, "w"'] \arrow[dr, "p(su,t)"] \arrow[d, "u"] & \\ e & e' \arrow[l, "v"] \arrow[r, "s"'] & b \end{tikzcd}$$

which is an object of $p_* \downarrow s\bar{v}$. This defines the inverse of $g_*$.

Conversely, assume that the square of presheaves is a homotopy pullback. The argument above shows that so is

$$\begin{tikzcd} \mathrm{L^H}\,\mathcal{C}(-, b \times_d c) \arrow[r] \arrow[d] & \mathrm{L^H}\,\mathcal{C}(-,c) \arrow[d] \\ \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[r] & \mathrm{L^H}\,\mathcal{C}(-,d) \end{tikzcd}$$

and hence the map $(p,f)_*\colon \mathrm{L^H}\,\mathcal{C}(-,a) \to \mathrm{L^H}\,\mathcal{C}(-,b\times_d c)$ is a weak equivalence. Thus so is $(p,f)$ by Lemma 6.5, i.e., the original square is a homotopy pullback. $\square$

**Theorem 6.10.** *The category $R\mathcal{C}$ of representable fibrant presheaves over the hammock localization of a fibration category is a tribe.*

*Proof.* By a retract argument as in the proof of Lemma 2.22, a map is anodyne in the category of representable fibrant presheaves if and only if it is anodyne in the category of all fibrant presheaves. Thus in light of Corollary 5.2 and Lemma 5.5, it suffices to verify that the terminal presheaf is representable and that representable presheaves are closed under homotopy pullbacks.

The first claim follows directly from Lemma 6.6. For the second one, by Lemma 2.5, it is enough to consider a strict pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

where both $C \to D$ and $B \to D$ are fibrations. We assume that $B$, $C$ and $D$ are representable and by Lemma 6.7 we can pick compatible representing objects $b$, $c$ and $d$ and fibrations $c \to d$ and $b \to d$. We denote the resulting pullback in $\mathcal{C}$ by $a$ and obtain a cube

$$\begin{tikzcd} \mathrm{L^H}\,\mathcal{C}(-,a) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{L^H}\,\mathcal{C}(-,c) \arrow[dd] \arrow[dr] & \\ & A \arrow[rr] \arrow[dd] & & C \arrow[dd] \\ \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[rr] \arrow[dr] & & \mathrm{L^H}\,\mathcal{C}(-,d) \arrow[dr] & \\ & B \arrow[rr] & & D \end{tikzcd}$$

where both the front and the back face are homotopy pullbacks (by Proposition 6.9). It follows from the Gluing Lemma that $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a) \to A$ is a weak equivalence, i.e., $A$ is representable. $\square$

The tribe $R\mathcal{C}$ is in fact (semi)simplicial, but this enrichment will play no role in our arguments.

We conclude the section with a technical result which will be needed in Section 7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of fibration categories and let $A$ be a presheaf over $\mathcal{C}$. A *left homotopy Kan extension* of $A$ along $F$ is a presheaf $X$ over $\mathcal{D}$ together with a map $A \to F^*X$ such that there is a projectively cofibrant replacement $A' \xrightarrow{\sim} A$ such that the adjoint transpose $\mathrm{Lan}_F\, A' \to X$ of the composite $A' \to A \to F^*X$ is a weak equivalence. (Note that a representation of $A$ is in particular a projectively cofibrant replacement of $A$.) Here, we write $\mathrm{Lan}_F$ for the strict left Kan extension along $F$.

**Lemma 6.11.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of fibration categories and let*

$$\begin{tikzcd}
X_\emptyset \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_0 \arrow[r] & X_{01}
\end{tikzcd}$$

*be a homotopy pullback of presheaves over $\mathcal{D}$ and let*

$$\begin{tikzcd}
A_\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & A_1 \arrow[dr] \arrow[dd] & \\
& F^*X_\emptyset \arrow[rr] \arrow[dd] & & F^*X_1 \arrow[dd] \\
A_0 \arrow[rr] \arrow[dr] & & A_{01} \arrow[dr] & \\
& F^*X_0 \arrow[rr] & & F^*X_{01}
\end{tikzcd}$$

*be a diagram of presheaves over $\mathcal{C}$ where the back face is also a homotopy pullback and $A_0$, $A_1$ and $A_{01}$ are all representable. If three of the diagonal arrows exhibit $X_0$, $X_1$ and $X_{01}$ as left homotopy Kan extensions of $A_0$, $A_1$ and $A_{01}$, respectively, then the fourth one exhibits $X_\emptyset$ as a left homotopy Kan extension of $A_\emptyset$.*

*Proof.* Without loss of generality we may assume that the back face is a strict pullback of two projective fibrations. To see that we view the cube as a map of squares $A \to F^*X$ and we choose a weak equivalence $X \xrightarrow{\sim} X'$ where $X'$ is a pullback of two projective fibrations. We then factor the composite $A \to F^*X \to F^*X'$ as $A \xrightarrow{\sim} A' \to F^*X'$ with $A'$ also a pullback of two projective fibrations. We do so by first factoring the map of underlying cospans as a weak equivalence followed by a Reedy fibration and completing the resulting cospan to a pullback.

We pick a square $a'$ in $\mathcal{C}$ together with compatible representations $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a') \xrightarrow{\sim} A'$ as in the proof of Theorem 6.10. By the assumption all the adjoint transposes $\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa_0) \to X'_0$, $\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa_1) \to X'_1$ and $\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa_{01}) \to X'_{01}$ are weak equivalences. Thus, by the Gluing Lemma, so is $\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa_\emptyset) \to X'_\emptyset$ which concludes the proof. $\square$

## 7 Approximating fibration categories by tribes

In this section we prove the following result.

**Theorem 7.1.** *The forgetful functor* $\mathsf{Trb} \to \mathsf{FibCat}$ *of Theorem 2.17 satisfies the approximation property (App2).*

Later, we will employ semisimplicial fibration categories and semisimplicial tribes to prove that this functor is a DK-equivalence. Note that this does not follow directly from the theorem above since $\mathsf{Trb}$ is not known to be a fibration category.

Throughout this section, we fix a fibration category $\mathcal{C}$, a tribe $\mathcal{T}$ and an exact functor $F\colon \mathcal{C} \to \mathcal{T}$. We will prove Theorem 7.1 by constructing a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{T} \\ \widehat{\mathcal{C}} \arrow[u, "\sim"] \arrow[r, "\sim"'] & \bar{\mathcal{C}} \arrow[u] \end{tikzcd}$$

where $\widehat{\mathcal{C}}$ is a fibration category and $\bar{\mathcal{C}}$ is a tribe. Both of these categories are variations of $R\mathcal{C}$ of Theorem 6.10. Their objects are representable fibrant presheaves over $\mathrm{L^H}\,\mathcal{C}$ equipped with additional structure that ensures the existence of the functors in the diagram above, in particular, that $\bar{\mathcal{C}} \to \mathcal{T}$ is a homomorphism.

We impose certain cardinality restriction to ensure that the categories $\bar{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ are small. We fix a cardinal number $\kappa$ such that

(1) $\mathrm{L^H}\,\mathcal{C}(-, a)$ is $\kappa$-small for all $a \in \mathcal{C}$;

(2) $F^* \,\mathrm{L^H}\,\mathcal{T}(-, x)$ is $\kappa$-small for all $x \in \mathcal{T}$;

(3) there is an (acyclic cofibration, fibration) factorization functor in the category of simplicial presheaves over $\mathrm{L^H}\,\mathcal{C}$ such that for every map of $\kappa$-small presheaves, the presheaf resulting from the factorization is also $\kappa$-small.

Throughout this section all presheaves will be implicitly assumed to be $\kappa$-small. Thanks to this assumption, the categories $\bar{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ constructed below will be essentially small and so can be replaced by equivalent small ones.

First, we construct a category $\bar{\mathcal{C}}$ as follows. An object is a tuple consisting of

- a fibrant presheaf $A$ over $\mathrm{L^H}\,\mathcal{C}$;
- an object $x$ of $\mathcal{T}$;
- a fibration $\bar{A} \twoheadrightarrow A \times F^* \,\mathrm{L^H}\,\mathcal{T}(-, x)$

subject to the following conditions:

- the map $\bar{A} \to A$ is a weak equivalence;
- there is a representation $\mathrm{L^H}\,\mathcal{C}(-, a) \xrightarrow{\sim} \bar{A}$ such that the composite $\mathrm{L^H}\,\mathcal{C}(-, a) \to \bar{A} \to F^* \,\mathrm{L^H}\,\mathcal{T}(-, x)$ corresponds to a weak equivalence $\mathrm{L^H}\,\mathcal{D}(-, Fa) \to \mathrm{L^H}\,\mathcal{T}(-, x)$.

Note that these conditions imply that $A$ and $\bar{A}$ are representable, but not that $\bar{A} \to A$ is a fibration nor that $\bar{A}$ is fibrant. We will denote such an object by $(A, \bar{A}, x)$ suppressing the structure map.

A morphism $(A, \bar{A}, x) \to (B, \bar{B}, y)$ consists of maps of presheaves $A \to B$ and $\bar{A} \to \bar{B}$ and a morphism $x \to y$ in $\mathcal{T}$ that are compatible in the sense that the square

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d] & \bar{B} \arrow[d] \\
A \times F^* \mathrm{L}^{\mathrm{H}} \mathcal{T}(-, x) \arrow[r] & B \times F^* \mathrm{L}^{\mathrm{H}} \mathcal{T}(-, y)
\end{tikzcd}$$

commutes. We call such a morphism a *fibration* if

- $A \to B$ is a fibration of presheaves;
- $x \to y$ is a fibration in $\mathcal{T}$;
- the map $\bar{A} \to \bar{B} \times_{B \times F^* \mathrm{L}^{\mathrm{H}} \mathcal{T}(-,y)} (A \times F^* \mathrm{L}^{\mathrm{H}} \mathcal{T}(-, x))$ induced from the square above is a fibration of presheaves.

Note that this definition does not imply that $\bar{A} \to \bar{B}$ is a fibration.

We proceed to prove that $\bar{\mathcal{C}}$ is a tribe and to characterize its anodyne morphisms (Lemma 7.3) and weak equivalences (Lemma 7.4).

**Lemma 7.2.** *In the category $\bar{\mathcal{C}}$:*

*(1) the object $(1, F^* \mathrm{L}^{\mathrm{H}} \mathcal{T}(-, 1), 1)$ with the identity structure map is terminal;*

*(2) pullbacks along fibrations exist.*

*Proof.* Part (1) is immediate.

For part (2), given a fibration $(A, \bar{A}, x) \twoheadrightarrow (B, \bar{B}, y)$ and any morphism $(C, \bar{C}, z) \to (B, \bar{B}, y)$ we construct a pullback as follows. First, form pullbacks

$$\begin{tikzcd}
z \times_y x \arrow[r] \arrow[d] & x \arrow[d, two heads] \\
z \arrow[r] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
C \times_B A \arrow[r] \arrow[d] & A \arrow[d, two heads] \\
C \arrow[r] & B
\end{tikzcd}$$

and then construct a diagram

$$
\begin{tikzcd}
\overline{C \times_B A} \arrow[rr] \arrow[d] & & \bar{A} \arrow[d, two heads] & \\
\bullet \arrow[rr] \arrow[dr] \arrow[dd] & & \bullet \arrow[dr] \arrow[dd] & \\
& \bar{C} \arrow[rr] \arrow[dd, two heads] & & \bar{B} \arrow[dd, two heads] \\
(C \times_B A) \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, z \times_y x) \arrow[rr] \arrow[dr] & & A \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, x) \arrow[dr] & \\
& C \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, z) \arrow[rr] & & B \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, y)
\end{tikzcd}
$$

as follows. The bottom face of the cube is obtained by combining the two squares above. Then the left and right face are formed by taking pullbacks which gives rise to the two objects denoted by bullets. The right one has an induced map from $\bar{A}$ and the square at the very top is constructed by taking a pullback again.

Next, we show that $(C \times_B A, \overline{C \times_B A}, z \times_y x)$ is an object of $\bar{\mathcal{C}}$. By construction the bottom, left and right faces of the cube above as well as the square at the very top are homotopy pullbacks. Thus so is the top face of the cube below.

$$
\begin{tikzcd}
\overline{C \times_B A} \arrow[rr] \arrow[dr] \arrow[dd] & & \bar{A} \arrow[dr] \arrow[dd] & \\
& \bar{C} \arrow[rr] \arrow[dd] & & \bar{B} \arrow[dd] \\
C \times_B A \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
& C \arrow[rr] & & B
\end{tikzcd}
$$

The bottom face is also a homotopy pullback. Since three of the vertical arrows are weak equivalences, so is $\overline{C \times_B A} \to C \times_B A$. Moreover, by Lemma 6.11, $\overline{C \times_B A}$ has a representation $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, d) \xrightarrow{\sim} \overline{C \times_B A}$ such that the adjoint transpose of the composite $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, d) \xrightarrow{\sim} \overline{C \times_B A} \to F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, z \times_y x)$ is a weak equivalence.

The universal properties of the pullbacks constructed above imply that $(C \times_B A, \overline{C \times_B A}, z \times_y x)$ is a pullback of the two original morphisms in $\bar{\mathcal{C}}$. (Even though $\overline{C \times_B A}$ is not the same as $\bar{C} \times_{\bar{B}} \bar{A}$.) $\square$

**Lemma 7.3.** *The category $\bar{\mathcal{C}}$ with Reedy fibrations as defined above is a tribe. A morphism $(A, \bar{A}, x) \to (B, \bar{B}, y)$ is anodyne if and only if all $A \to B$, $\bar{A} \to \bar{B}$ and $x \to y$ are. Moreover, the forgetful functor $\bar{\mathcal{C}} \to \mathcal{T}$ is a homomorphism of tribes.*

In the statement of this proposition (as well as in the proof below), it is a slight abuse of language to call the map $\bar{A} \to \bar{B}$ "anodyne" since it is not a morphism of a tribe ($\bar{A}$ and $\bar{B}$ are not fibrant). However, acyclic cofibrations of the injective model structure on simplicial presheaves enjoy all the necessary properties of anodyne morphisms in a tribe and the proof applies as written.

*Proof.* (T1) and (T2) follow by Lemma 7.2.

(T3) is proven by an argument similar to the proof of Lemma 2.22. That is, every morphism can be factored as a levelwise anodyne morphism followed by a fibration, and then a retract argument shows that every anodyne morphism is levelwise anodyne. Conversely, every levelwise anodyne morphism is anodyne. Thus every morphism factors as an anodyne morphism followed by a fibration.

Every anodyne morphism $z \overset{\sim}{\rightarrowtail} y$ in $\mathcal{T}$ has a retraction and so it induces an acyclic cofibration $\mathrm{L^H}\,\mathcal{T}(-,y) \to \mathrm{L^H}\,\mathcal{T}(-,z)$ by Lemmas 2.12 and 6.5. Thus by the construction of pullbacks in $\bar{\mathcal{C}}$, levelwise anodyne morphisms are stable under pullbacks along fibrations and hence so are the anodyne morphisms which proves (T4).

The functor $\bar{\mathcal{C}} \to \mathcal{T}$ is a homomorphism since anodyne morphisms in $\bar{\mathcal{C}}$ are levelwise. $\square$

**Lemma 7.4.** *A morphism $(A, \bar{A}, x) \to (B, \bar{B}, y)$ in $\bar{\mathcal{C}}$ is a weak equivalence if and only if all $A \to B$ and $\bar{A} \to \bar{B}$ and $x \to y$ are weak equivalences.*

*Proof.* By an argument similar to the proof of the previous proposition, there is a fibration category $\bar{\mathcal{C}}_{\mathrm{lvl}}$ with the same underlying category and fibrations as $\bar{\mathcal{C}}$ and with levelwise weak equivalences. We can prove that every object of $\bar{\mathcal{C}}_{\mathrm{lvl}}$ is cofibrant and that path objects in $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}_{\mathrm{lvl}}$ agree using the same reasoning as in the proof of Lemma 4.7. Hence the conclusion follows as in the proof of Lemma 2.22. $\square$

Next, we construct the fibration category $\widehat{\mathcal{C}}$. An object is a tuple consisting of

- a fibrant presheaf $A$ over $\mathrm{L^H}\,\mathcal{C}$;
- an object $a$ of $\mathcal{C}$;
- a fibration $\bar{A} \twoheadrightarrow A \times F^* \,\mathrm{L^H}\,\mathcal{T}(-, Fa)$;
- a representation $\mathrm{L^H}\,\mathcal{C}(-,a) \xrightarrow{\sim} \bar{A}$

subject to the following conditions:

- the map $\bar{A} \to A$ is a weak equivalence;
- the composite $\mathrm{L^H}\,\mathcal{C}(-,a) \to \bar{A} \to F^*\,\mathrm{L^H}\,\mathcal{T}(-,Fa)$ is the unit of the adjunction $\mathrm{Lan}_F \dashv F^*$.

Such an object will be denoted by $(A, \bar{A}, a)$.

A morphism $(A, \bar{A}, a) \to (B, \bar{B}, b)$ consists of maps of presheaves $A \to B$ and $\bar{A} \to \bar{B}$ and a morphism $a \to b$ in $\mathcal{C}$ that are compatible in the sense that the diagram

$$\begin{tikzcd}
\mathrm{L^H}\,\mathcal{C}(-,a) \arrow[r] \arrow[d] & \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[d] \\
\bar{A} \arrow[r] \arrow[d] & \bar{B} \arrow[d] \\
A \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,Fa) \arrow[r] & B \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,Fb)
\end{tikzcd}$$

commutes. We call such a morphism a weak equivalence if all $A \to B$, $\bar{A} \to \bar{B}$ and $a \to b$ are weak equivalences. We call it a *fibration* if

- $A \to B$ is a fibration of presheaves;
- $a \to b$ is a fibration in $\mathcal{C}$;
- the induced map $\bar{A} \to \bar{B} \times_{B \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-,Fb)} (A \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa))$ induced from the square above is a fibration of presheaves.

**Lemma 7.5.** *The category $\widehat{\mathcal{C}}$ with weak equivalences and fibrations as defined above is a fibration category. Moreover, the forgetful functors $\widehat{\mathcal{C}} \to \mathcal{C}$ and $\widehat{\mathcal{C}} \to \bar{\mathcal{C}}$ are exact.*

*Proof.* Axioms (F1) and (F2) follow by an argument similar to the one in the proof of Lemma 7.3 and (F4) is immediate. Every morphism can be factored as a levelwise weak equivalence followed by a fibration by an argument similar to the one in the proof of Lemma 4.5.

The functors $\widehat{\mathcal{C}} \to \mathcal{C}$ and $\widehat{\mathcal{C}} \to \bar{\mathcal{C}}$ are exact by construction, with the latter preserving weak equivalences by Lemma 7.4. $\square$

Finally, in the three remaining lemmas we prove that the functors $\widehat{\mathcal{C}} \to \mathcal{C}$ and $\bar{\mathcal{C}} \to \mathcal{T}$ are weak equivalences.

**Lemma 7.6.** *The functor $\widehat{\mathcal{C}} \to \mathcal{C}$ is a weak equivalence.*

*Proof.* We construct a functor $\mathcal{C} \to \widehat{\mathcal{C}}$ sending $a$ to $(\Phi a, \bar{\Phi} a, a)$ as follows. Given $a \in \mathcal{C}$, take $\Phi a$ to be the functorial fibrant replacement of $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, a)$. Then (functorially) factor $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, a) \to \Phi a \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa)$ as a weak equivalence $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, a) \xrightarrow{\sim} \bar{\Phi} a$ followed by a fibration $\bar{\Phi} a \twoheadrightarrow \Phi a \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa)$. Clearly, the composite $\mathcal{C} \to \widehat{\mathcal{C}} \to \mathcal{C}$ is the identity.

We will show that the composite $\widehat{\mathcal{C}} \to \mathcal{C} \to \widehat{\mathcal{C}}$ is weakly equivalent to the identity, i.e., that every object $(A, \bar{A}, a) \in \widehat{\mathcal{C}}$ can be connected to $(\Phi a, \bar{\Phi} a, a)$ by a zig-zag of natural weak equivalences. First, factor $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, a) \to A \times \Phi a$ as a weak equivalence $\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-, a) \xrightarrow{\sim} \Psi(A, \bar{A}, a)$ followed by a fibration $\Psi(A, \bar{A}, a) \twoheadrightarrow A \times \Phi a$. Next, form a diagram

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\, \mathcal{C}(-,a) \arrow[d, "\sim"] \arrow[ddr, bend left] \arrow[ddd, bend right] & \\
\bar{\Psi}(A, \bar{A}, a) \arrow[d, two heads] & \\
\bullet \arrow[r, two heads] \arrow[d] & \Psi(A, \bar{A}, a) \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa) \arrow[d] \\
\bar{A} \times \bar{\Phi} a \arrow[r, two heads] & A \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa) \times \Phi a \times F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa)
\end{tikzcd}$$

as follows. The right vertical arrow is obtained by combining the fibration from the factorization above with the diagonal map of $F^* \mathrm{L}^{\mathrm{H}}\, \mathcal{T}(-, Fa)$. The object denoted by a bullet arises by taking a pullback and $\bar{\Psi}(A, \bar{A}, a)$ comes from factoring the resulting map as a weak equivalence followed by a fibration. Then $(\Psi(A, \bar{A}, a), \bar{\Psi}(A, \bar{A}, a), a)$ is an object of $\widehat{\mathcal{C}}$ and the vertical morphisms assemble into weak equivalences

$$(A, \bar{A}, a) \xleftarrow{\sim} (\Psi(A, \bar{A}, a), \bar{\Psi}(A, \bar{A}, a), a) \xrightarrow{\sim} (\Phi a, \bar{\Phi} a, a). \qquad \square$$

**Lemma 7.7.** *The forgetful functor $\widehat{\mathcal{C}} \to R\mathcal{C}$ is a weak equivalence.*

*Proof.* By Theorem 2.9, it suffices to verify the approximation properties. (App1) is immediate. For (App2) consider $(B, \bar{B}, b)$ in $\widehat{\mathcal{C}}$ and a map $A \to B$. Factor it as a weak equivalence $A \xrightarrow{\sim} A'$ followed by a fibration $A' \twoheadrightarrow B$. Since $A'$ is representable we can pick a morphism $a' \to b$ in $\mathcal{C}$ and a square

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a') \arrow[r] \arrow[d, "\sim"'] & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \arrow[d, "\sim"] \\
A' \arrow[r, two heads] & B
\end{tikzcd}$$

by Lemma 6.7. By the naturality of the unit we obtain a square

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a') \arrow[r] \arrow[d] & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \arrow[d] \\
A' \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa') \arrow[r] & B \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fb)
\end{tikzcd}$$

This gives a map from $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a')$ to the pullback denoted by the bullet in the diagram

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a') \arrow[r] \arrow[d, "\sim"] \arrow[ddd, bend right] & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \arrow[d, "\sim"] \\
\bar{A}' \arrow[d, two heads] & \bar{B} \arrow[dd, two heads] \\
\bullet \arrow[ur] \arrow[d, two heads] & \\
A' \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa') \arrow[r] & B \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fb)
\end{tikzcd}$$

which we then factor as a weak equivalence followed by a fibration. Altogether we obtain an object $(A', \bar{A}', a')$ and a morphism $(A', \bar{A}', a') \to (B, \bar{B}, b)$ thus completing the proof. $\square$

**Lemma 7.8.** *The forgetful functor $\bar{\mathcal{C}} \to R\mathcal{C}$ is a weak equivalence.*

*Proof.* By Theorem 2.9, it is enough to verify the approximation properties. (App1) is immediate. For (App2) consider $(B, \bar{B}, y)$ in $\bar{\mathcal{C}}$ and a map $A \to B$. By the definition, there is an object $b$ and a representation $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \xrightarrow{\sim} \bar{B}$. The map $\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fb) \to \mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, y)$, the adjoint transpose of the composite $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \to \bar{B} \to F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, y)$, is a weak equivalence. It is induced by a zig-zag $Fb \rightsquigarrow y$ homotopic to a morphism $w\colon Fb \to y$ by Corollary 6.4 which is a weak equivalence by Lemma 6.5. This homotopy induces a homotopy commutative triangle

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \arrow[d, "\sim"'] \arrow[dr] & \\
\bar{B} \arrow[r] & F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, y)
\end{tikzcd}$$

in which the diagonal arrow is induced by $w$. By Lemma 5.6, the map $\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, b) \xrightarrow{\sim} \bar{B}$ can be replaced by a homotopic one making the triangle commute strictly.

Factor the map $A \to B$ as a weak equivalence $A \xrightarrow{\sim} A'$ followed by a fibration $A' \twoheadrightarrow B$. Since $A'$ is representable we can pick a morphism $a' \to b$ in $\mathcal{C}$ and a square

$$\begin{tikzcd}
\mathrm{L^H}\,\mathcal{C}(-,a') \arrow[r] \arrow[d, "\sim"'] & \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[d, "\sim"] \\
A' \arrow[r, two heads] & B
\end{tikzcd}$$

by Lemma 6.7 where the map $\mathrm{L^H}\,\mathcal{C}(-,b) \xrightarrow{\sim} \bar{B}$ is the one constructed in the preceding paragraph. Thus we obtain a square

$$\begin{tikzcd}
\mathrm{L^H}\,\mathcal{C}(-,a') \arrow[r] \arrow[d] & \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[d] \\
A' \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,Fa') \arrow[r] & B \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,y)
\end{tikzcd}$$

where the bottom map is induced by the composite $Fa' \to Fb \to y$. This gives a map from $\mathrm{L^H}\,\mathcal{C}(-,a')$ to the pullback denoted by the bullet in the diagram

$$\begin{tikzcd}
\mathrm{L^H}\,\mathcal{C}(-,a') \arrow[r] \arrow[d, "\sim"] \arrow[ddd, bend right] & \mathrm{L^H}\,\mathcal{C}(-,b) \arrow[d, "\sim"] \\
\bar{A}' \arrow[d, two heads] & \bar{B} \arrow[dd, two heads] \\
\bullet \arrow[ur] \arrow[d, two heads] & \\
A' \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,Fa') \arrow[r] & B \times F^*\,\mathrm{L^H}\,\mathcal{T}(-,y)
\end{tikzcd}$$

which we then factor as a weak equivalence followed by a fibration. Altogether we obtain an object $(A', \bar{A}', Fa')$ and a morphism $(A', \bar{A}', Fa') \to (B, \bar{B}, y)$ as required. $\square$

*Proof of Theorem 7.1.* By Lemmas 7.7 and 7.8 the diagonal morphisms in the triangle

$$\begin{tikzcd}
\widehat{\mathcal{C}} \arrow[rr] \arrow[dr, "\sim"'] & & \bar{\mathcal{C}} \arrow[dl, "\sim"] \\
& R\mathcal{C} &
\end{tikzcd}$$

are weak equivalences and hence so is $\widehat{\mathcal{C}} \to \bar{\mathcal{C}}$. This along with Lemma 7.6 shows that both labeled arrows in the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{T} \\
\widehat{\mathcal{C}} \arrow[u, "\sim"] \arrow[r, "\sim"'] & \bar{\mathcal{C}} \arrow[u]
\end{tikzcd}$$

are weak equivalences thus completing the proof. $\square$

# 8 Equivalence between tribes and fibration categories

We are now ready to prove our key theorem.

**Theorem 8.1.** *The forgetful functor* $\mathsf{Trb} \to \mathsf{FibCat}$ *of Theorem 2.17 is a DK-equivalence.*

By Proposition 3.12, it suffices to show that the forgetful functor $\mathsf{sTrb} \to \mathsf{sFibCat}$ is a DK-equivalence, which we will do by verifying the approximation properties. This can be accomplished by refining the result of the previous section for which we will need the following two lemmas.

**Lemma 8.2.**

*(1) The functor* $\widehat{\mathcal{C}} \to \mathcal{C}$ *is a fibration of fibration categories.*

*(2) The functor* $\bar{\mathcal{C}} \to \mathcal{T}$ *is a fibration of tribes.*

Note that these are fibrations between non-semisimplicial fibration categories (tribes) as defined in Definitions 4.1 and 4.3. Using the next lemma we will promote them to fibrations in $\mathsf{sFibCat}$ and $\mathsf{sTrb}$.

*Proof.* For part (1), the isofibration condition is immediate. The lifting property for WF-factorizations is verified just like axiom (F3) in the proof of Lemma 7.5 except that a part of the factorization is given in advance. It remains to check the lifting property for pseudofactorizations. Let $(A, \bar{A}, a) \to (B, \bar{B}, b)$ be a morphism in $\widehat{\mathcal{C}}$ and let

$$\begin{tikzcd}
a \arrow[r] & b \\
a' \arrow[u, two heads, "\sim"] \arrow[r, "\sim"'] & a'' \arrow[u, two heads]
\end{tikzcd}$$

be a pseudofactorization of its image in $\mathcal{C}$. We form a diagram

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a') \arrow[rr] \arrow[d, "\sim"] \arrow[ddd, bend right] & & \mathrm{L}^{\mathrm{H}}\,\mathcal{C}(-, a) \arrow[d, "\sim"] \\
\bar{A}' \arrow[d, two heads] & & \bar{A} \arrow[dd, two heads] \\
\bullet \arrow[d, two heads] \arrow[urr] & & \\
A \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa') \arrow[rr] & & A \times F^* \,\mathrm{L}^{\mathrm{H}}\,\mathcal{T}(-, Fa)
\end{tikzcd}$$

by first taking a pullback, denoted by a bullet, and then factoring the resulting map as a weak equivalence followed by a fibration. This way we obtain an acyclic fibration $(A', \bar{A}', a') \overset{\sim}{\twoheadrightarrow} (A, \bar{A}, a)$. To construct the remaining part we lift the factorization of the composite $(A', \bar{A}', a') \overset{\sim}{\twoheadrightarrow} (A, \bar{A}, a) \to (B, \bar{B}, b)$.

For part (2), the verification of the first four properties is analogous to the proof of part (1). Next, we verify the lifting property for lifts. Let

$$\begin{tikzcd}
(A, \bar{A}, u) \arrow[r] \arrow[d, tail, "\sim"'] & (C, \bar{C}, x) \arrow[d, two heads] \\
(B, \bar{B}, v) \arrow[r] & (D, \bar{D}, y)
\end{tikzcd}$$

be a lifting problem in $\widehat{\mathcal{C}}$ and fix a solution $v \to x$ of its image in $\mathcal{C}$. Pick any solution $B \to C$ of its image in $R\mathcal{C}$. Since $\bar{A} \to \bar{B}$ is an acyclic cofibration by Lemma 7.3, there is a lift in

$$\begin{tikzcd}
\bar{A} \arrow[rr, two heads] \arrow[d, tail, "\sim"'] & & \bar{C} \arrow[d, two heads] \\
\bar{B} \arrow[r] \arrow[urr, dashed] & \bar{D} \times_{D \times F^* \mathrm{L^H}\, \mathcal{T}(-,y)} (C \times F^* \mathrm{L^H}\, \mathcal{T}(-, Fx)) &
\end{tikzcd}$$

which completes a lift in the original square. The proof of the lifting property for cofibrancy lifts is analogous. $\square$

**Lemma 8.3.**

*(1)* $\mathrm{Fr}\colon \mathsf{FibCat} \to \mathsf{sFibCat}$ *preserves fibrations.*

*(2)* $\mathrm{Fr}\colon \mathsf{Trb} \to \mathsf{sTrb}$ *preserves fibrations.*

*Proof.* Part (1) follows from [Szu17a, Lem. 1.11(1)].

For part (2), consider a fibration $P\colon \mathcal{S} \to \mathcal{T}$ of semisimplicial tribes. By part (1), $\mathrm{Fr}\, P$ is a fibration of underlying fibration categories.

Let $a \to b$ be a morphism in $\mathrm{Fr}\,\mathcal{S}$ and consider a factorization $Pa \overset{\sim}{\rightarrowtail} x \twoheadrightarrow Pb$. We lift it to a factorization $a \overset{\sim}{\rightarrowtail} a' \twoheadrightarrow b$ inductively. First, the factorization $Pa_0 \overset{\sim}{\rightarrowtail} x_0 \twoheadrightarrow Pb_0$ lifts to a factorization $a_0 \overset{\sim}{\rightarrowtail} a'_0 \twoheadrightarrow b_0$ since $P$ is a fibration. For the inductive step, the partial factorization below dimension $m$ induces a morphism $a_m \to M_m a' \times_{M_m b} b_m$ which we factor as an anodyne morphism $a_m \overset{\sim}{\rightarrowtail} a'_m$ followed by a fibration $a'_m \twoheadrightarrow M_m a' \times_{M_m b} b_m$. This proves the lifting property for AF-factorizations.

The other two conditions are verified in a similar manner. $\square$

*Proof of Theorem 8.1.* Consider the square

$$\begin{tikzcd}
\mathsf{sTrb} \arrow[r] \arrow[d, "\sim"'] & \mathsf{sFibCat} \arrow[d, "\sim"] \\
\mathsf{Trb} \arrow[r] & \mathsf{FibCat}
\end{tikzcd}$$

where the vertical functors are DK-equivalences by Proposition 3.12. The categories $\mathsf{sTrb}$ and $\mathsf{sFibCat}$ are fibration categories by Theorem 4.9 and the functor $\mathsf{sTrb} \to \mathsf{sFibCat}$ is exact by Definitions 4.1 and 4.3. It suffices to verify that this functor is a DK-equivalence and, in light of Theorem 2.9, we can do so by checking that it satisfies the approximation properties.

(App1) is immediate. For (App2) consider a semisimplicial fibration category $\mathcal{C}$, a semisimplicial tribe $\mathcal{T}$ and a semisimplicial exact functor $F\colon \mathcal{C} \to \mathcal{T}$. By Theorem 7.1 there are a fibration category $\widehat{\mathcal{C}}$, a tribe $\bar{\mathcal{C}}$ (not necessarily semisimplicial) and a square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{T} \\
\widehat{\mathcal{C}} \arrow[r, "\sim"'] \arrow[u, "\sim"] & \bar{\mathcal{C}} \arrow[u]
\end{tikzcd}$$

in FibCat. We form a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[rrr, "F"] & & & \mathcal{T} & & \\
& \widehat{\mathrm{Fr}}\mathcal{C} \arrow[ul, "\sim"'] \arrow[rrr] \arrow[dr, "\sim"] & & & \widehat{\mathrm{Fr}}\mathcal{T} \arrow[ul, "\sim"'] \arrow[dr, "\sim"] & \\
& & \mathrm{Fr}\,\mathcal{C} \arrow[rrr] & & & \mathrm{Fr}\,\mathcal{T} \\
& \widehat{\mathcal{C}}' \arrow[uuul, bend left] \arrow[uu] \arrow[rrr, "\sim"] \arrow[dr] & & & \bar{\mathcal{C}}' \arrow[uuul, bend left] \arrow[uu] \arrow[dr] & \\
& & \mathrm{Fr}\,\widehat{\mathcal{C}} \arrow[uu, two heads] \arrow[rrr, "\sim"'] & & & \mathrm{Fr}\,\bar{\mathcal{C}} \arrow[uu, two heads]
\end{tikzcd}
$$

as follows.

- The front square is obtained by applying Fr to the square above.
- The top two squares are naturality squares of transformations $\widehat{\mathrm{Fr}} \to \mathrm{Fr}$ and $\widehat{\mathrm{Fr}} \to \mathrm{id}$.
- The category $\widehat{\mathcal{C}}'$ is defined as the pullback $\widehat{\mathrm{Fr}}\mathcal{C} \times_{\mathrm{Fr}\,\mathcal{C}} \mathrm{Fr}\,\widehat{\mathcal{C}}$ which can be constructed since $\mathrm{Fr}\,\widehat{\mathcal{C}} \to \mathrm{Fr}\,\mathcal{C}$ is a fibration by Lemmas 8.2 and 8.3.
- The category $\bar{\mathcal{C}}'$ is defined as the pullback $\widehat{\mathrm{Fr}}\mathcal{T} \times_{\mathrm{Fr}\,\mathcal{T}} \mathrm{Fr}\,\bar{\mathcal{C}}$ which can be constructed since $\mathrm{Fr}\,\bar{\mathcal{C}} \to \mathrm{Fr}\,\mathcal{T}$ is a fibration by Lemmas 8.2 and 8.3.

The top diagonal arrows are weak equivalences by Lemmas 3.10 and 3.11. The functors $\mathrm{Fr}\,\widehat{\mathcal{C}} \to \mathrm{Fr}\,\mathcal{C}$ and $\mathrm{Fr}\,\widehat{\mathcal{C}} \to \mathrm{Fr}\,\bar{\mathcal{C}}$ are weak equivalences since Fr is homotopical.

Since the left and right squares are homotopy pullbacks, the functors $\widehat{\mathcal{C}}' \to \widehat{\mathrm{Fr}}\mathcal{C}$, $\widehat{\mathcal{C}}' \to \mathrm{Fr}\,\widehat{\mathcal{C}}$ and $\bar{\mathcal{C}}' \to \mathrm{Fr}\,\bar{\mathcal{C}}$ are weak equivalences. Therefore, by 2-out-of-3 in the square

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{T} \\
\widehat{\mathcal{C}}' \arrow[u, "\sim"] \arrow[r, "\sim"'] & \bar{\mathcal{C}}' \arrow[u]
\end{tikzcd}
$$

the labeled arrows are weak equivalences which completes the proof. $\square$

## 9 Application to internal languages

In the final section, we apply our results to establish an equivalence between categorical models of Martin-Löf Type Theory with dependent sums and intensional identity types and finitely complete $(\infty, 1)$-categories. We will use comprehension categories as our notion of categorical models. They were introduced by Jacobs [Jac93] and developed extensively in [Jac99].

**Definition 9.1.** A *comprehension category* is a category $\mathsf{C}$ with a terminal object equipped with a Grothendieck fibration $p\colon \mathsf{T} \to \mathsf{C}$ and a fully faithful *comprehension* functor $\chi\colon \mathsf{T} \to \mathsf{C}^{[1]}$ such that the triangle

$$\begin{tikzcd} \mathsf{T} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathsf{C}^{[1]} \arrow[dl, "\mathrm{cod}"] \\ & \mathsf{C} & \end{tikzcd}$$

commutes and $\chi$ carries cartesian morphisms to pullback squares.

**Definition 9.2.** A *fibration* in a comprehension category $\mathsf{C}$ is a morphism isomorphic to a composite of morphisms in the image of $\chi$.

Bare comprehension categories only model the structural rules of Martin-Löf Type Theory. Thus in the definition of a categorical model, we make additional assumptions on the comprehension category $\mathsf{C}$ to ensure that it carries an interpretation of the type constructors $\Sigma$ and $\mathsf{Id}$.

**Definition 9.3.** A *categorical model of type theory* is a comprehension category $\mathsf{C}$ that has

(1) *strong* $\Sigma$*-types* in the sense of [LW15, Def. 3.4.4.1];

(2) *weakly stable* $\mathsf{Id}$*-types* in the sense of [LW15, Def. 2.3.6];

such that all objects are fibrant.

Given $A \in \mathsf{T}(\Gamma)$, we will write $\Gamma.A$ for the domain of $\chi(A)$. This operation can be extended to dependent contexts as follows. Given a context $\Delta = (A_1, \ldots, A_m)$ where $A_1 \in \mathsf{T}(\Gamma)$, $A_2 \in \mathsf{T}(\Gamma.A_1)$, ..., $A_m \in \mathsf{T}(\Gamma.A_1.\cdots.A_{m-1})$, we will write $\Gamma.\Delta$ for the domain of $\chi(A_m)$. We will also use Garner's identity contexts [Gar09] which allows us to form $\mathsf{Id}_\Gamma \in \mathsf{T}(\Gamma.\Gamma)$.

**Definition 9.4.** Let $\mathsf{C}$ be a categorical model.

(1) A *homotopy* between morphisms $f, g\colon \Gamma \to \Delta$ is a morphism $H\colon \Gamma \to \Delta.\Delta.\mathsf{Id}_\Delta$ such that the triangle

$$\begin{tikzcd} & \Delta.\Delta.\mathsf{Id}_\Delta \arrow[d, "\chi(\mathsf{Id}_\Delta)"] \\ \Gamma \arrow[ur, "H"] \arrow[r, "f.g"'] & \Delta.\Delta \end{tikzcd}$$

commutes.

(2) A morphism $f\colon \Gamma \to \Delta$ is a *homotopy equivalence* if there is a morphism $g\colon \Delta \to \Gamma$ such that $fg$ is homotopic to $\mathrm{id}_\Delta$ and $gf$ is homotopic to $\mathrm{id}_\Gamma$.

**Remark 9.5.** Given $\Gamma \in \mathsf{C}$, $A \in \mathsf{T}(\Gamma)$ and $B \in \mathsf{T}(\Gamma.A)$, let $\Sigma_A B \in \mathsf{T}(\Gamma)$ denote the strong $\Sigma$-type of $A$ and $B$. For fixed $\Gamma$ and $A$ as above, the assignment $B \mapsto \Sigma_A B$ is a left adjoint of the pullback functor $\chi(A)^*\colon \mathsf{T}(\Gamma) \to \mathsf{T}(\Gamma.A)$. Conversely, if such a left adjoint exists, then its values are strong $\Sigma$-types.

**Definition 9.6.**

(1) A *morphism between categorical models* $\mathsf{C}$ and $\mathsf{C}'$ is a pair of functors $F_0\colon \mathsf{C} \to \mathsf{C}'$ and $F_1\colon \mathsf{T} \to \mathsf{T}'$ strictly compatible with $p$ and $\chi$ such that $F_0$ preserves a terminal object, $\Sigma$-types and $\mathsf{Id}$-types.

(2) A *weak equivalence of categorical models* is a morphism such that $F_0$ induces an equivalence of the homotopy categories, i.e., the localizations with respect to homotopy equivalences.

The homotopical category of categorical models is denoted by $\mathsf{CompCat}_{\mathsf{Id},\mathbf{\Sigma}}$. We will prove that it is DK-equivalent to the category of tribes.

**Proposition 9.7.** *A categorical model with its subcategory of fibrations is a tribe. Moreover, a morphism of categorical models is a homomorphism of tribes. This defines a homotopical functor* $T\colon \mathsf{CompCat}_{\mathsf{Id},\mathbf{\Sigma}} \to \mathsf{Trb}$.

*Proof.* Axiom (T1) is satisfied by the assumption while (T2) follows from the fact that $\chi$ carries cartesian morphisms to pullback squares. A factorization of a morphism $f\colon \Gamma \to \Delta$ is given by

$$\Gamma \longrightarrow \Delta.\Gamma.(\mathrm{id}\,.f)^*\mathsf{Id}_\Delta \longrightarrow \Delta$$

as constructed in [GG08, Lem. 11] which proves (T3). Finally, (T4) follows by [GG08, Prop. 14].

A morphism of categorical models preserves fibrations by definition. On the other hand, the anodyne morphisms can be characterized as those admitting deformation retractions by [GG08, Lem. 13(i)] so the conclusion follows by preservation of $\mathsf{Id}$-types. □

Given a tribe $\mathcal{T}$ we define a category $\mathcal{T}_{\mathrm{f}}^{[1]}$ as the full subcategory of $\mathcal{T}^{[1]}$ spanned by fibrations.

**Proposition 9.8.** *If $\mathcal{T}$ is a tribe, then let $\chi$ denote the inclusion $\mathcal{T}_{\mathrm{f}}^{[1]} \hookrightarrow \mathcal{T}^{[1]}$. Moreover, let $p\colon \mathcal{T}_{\mathrm{f}}^{[1]} \to \mathcal{T}$ be the composite* $\operatorname{cod}\chi$. *Then $(\mathcal{T}, \mathcal{T}_{\mathrm{f}}^{[1]}, \chi)$ is a categorical model. Moreover, a homomorphism of tribes induces a morphism of the associated categorical models, yielding a homotopical functor $C\colon \mathsf{Trb} \to \mathsf{CompCat}$.*

*Proof.* The category $\mathcal{T}$ has a terminal object by assumption. The functor $p$ is a Grothendieck fibration and $\chi$ preserves cartesian morphisms since $p$-cartesian morphisms are exactly pullbacks along fibrations.

For every fibration $q\colon x \twoheadrightarrow y$, the pullback functor $q^*\colon \mathcal{T}_{\mathrm{f}}^{[1]}(y) \to \mathcal{T}_{\mathrm{f}}^{[1]}(x)$ has a left adjoint given by composition and hence by Remark 9.5 strong $\mathbf{\Sigma}$-types exist.

Moreover, for every fibration $q\colon x \twoheadrightarrow y$, we choose a factorization

$$\begin{tikzcd} & \mathsf{Id}_x \arrow[d, two heads] \\ x \arrow[ur, tail, "\mathsf{r}_x", "\sim"'] \arrow[r] & x \times_y x. \end{tikzcd}$$

To see that $(\mathsf{Id}_x, \mathsf{r}_x)$ is an $\mathsf{Id}$-type, we need to verify that for every morphism $f\colon y' \to y$, the morphism $f^*\mathsf{r}_x\colon f^*x \to f^*\mathsf{Id}_x$ is anodyne. Indeed, this follows from Lemma 2.15.

The verification that a homomorphism of tribes induces a morphism of categorical model is straightforward. □

**Theorem 9.9.** *The functor $T\colon \mathsf{CompCat}_{\mathsf{Id},\mathbf{\Sigma}} \to \mathsf{Trb}$ of Proposition 9.7 is a DK-equivalence.*

*Proof.* We will show that the functor $C$ of Proposition 9.8 is a homotopy inverse of $T$. Clearly, $TC = \mathrm{id}_{\mathsf{Trb}}$. Given a categorical model $(\mathsf{C}, \mathsf{T}, \chi)$, we construct a natural morphism $(F_0, F_1)\colon \mathsf{C} \to CT\mathsf{C}$. First, we set $F_0 = \mathrm{id}_{\mathsf{C}}$. Moreover, $\chi\colon \mathsf{T} \to \mathsf{C}^{[1]}$ factors as $F_1\colon \mathsf{T} \to \mathsf{C}_{\mathrm{f}}^{[1]}$ followed by $\mathsf{C}_{\mathrm{f}}^{[1]} \hookrightarrow \mathsf{C}^{[1]}$. Since homotopy equivalences in $\mathsf{C}$ and $T\mathsf{C}$ agree, this morphism is a weak equivalence. □

Finally, we prove our main theorem.

**Theorem 9.10.** *The homotopical category of categorical models of Martin-Löf Type Theory with dependent sums and intensional identity types is DK-equivalent to the homotopical category of finitely complete $(\infty, 1)$-categories.*

*Proof.* We consider the composite

$$\mathsf{CompCat}_{\mathsf{Id},\Sigma} \longrightarrow \mathsf{Trb} \longrightarrow \mathsf{FibCat} \longrightarrow \mathsf{Lex}_\infty$$

where the first functor is an equivalence by Theorem 9.9, the second one by Theorem 8.1 and the last one by [Szu17b, Thm. 4.9]. □

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