UNIVALENCE IN SIMPLICIAL SETS

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Abstract. We present an accessible account of Voevodsky's construction of a univalent universe of Kan fibrations.

Our goal in this note is to give a concise, self-contained account of the results of [Voe11, Section 5]: the construction of a homotopically universal small Kan fibration $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$; the proof that U_α is a Kan complex; and the proof that π is univalent.

We assume some background knowledge of the homotopy theory of simplicial sets, and category theory in general; [Hov99] and [ML98] are canonical and sufficient references. Other good sources include [May67], [GJ09], and [Joy09].

In Section 1, we construct $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$, and prove that it is a weakly universal α-small Kan fibration. In Section 2, we prove further that the base U_α is a Kan complex.

Section 3 is dedicated to constructing the fibration of weak equivalences between two fibrations over a common base. In Section 4 we define univalence for a general fibration, and prove our main theorem: that π is univalent. Finally, in Section 5, we derive from this a statement of "homotopical uniqueness" for the universal property of U_α.

Overall, we largely follow Voevodsky's original presentation, with some departures: some proofs in Sections 2 and 4 are simplified based on a result of André Joyal ([Joy11, Lemma 0.2], cf. our Lemmas 17, 18); and Section 3 also is somewhat modified and reorganised.

A recurring theme throughout is that when a map is defined by a "right-handed" universal property, showing that it is a fibration (resp. trivial fibration) corresponds to showing that the objects it represents extend along trivial (resp. all) cofibrations.

An alternative construction of $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ can be found in [Str11], and an alternative proof of univalence in [Moe11].

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1. Representability of fibrations

Definition 1. Let X be a simplicial set. A *well-ordered morphism* $f\colon Y \longrightarrow X$ is a pair consisting of a morphism into X (also denoted by f) and a function assigning to each simplex $x \in X_n$ a well-ordering on the fiber $Y_x := f^{-1}(x) \subseteq Y_n$.

If $f\colon Y \longrightarrow X$, $f'\colon Y' \longrightarrow X$ are well-ordered morphisms into X, an *isomorphism* of well-ordered morphisms from f to f' is an isomorphism $Y \cong Y'$ over X preserving the well-orderings on the fibers.

Remark 2. Since we require no compatibility conditions, there are infinitely many (specifically, 2^ω) well-orderings even on the map $1 \amalg 1 \longrightarrow 1$. The well-orderings are haphazard beasts, and not of intrinsic interest; they are essentially just a technical device to obtain Lemma 5.

Proposition 3. *Given two well-ordered sets, there is at most one isomorphism between them. Given two well-ordered morphisms over a common base, there is at most one isomorphism between them.*

Proof. The first statement is classical, and immediate by induction; the second follows from the first, applied in each fiber. □

Definition 4. Fix (once and for all) a regular cardinal α. Say a map $f\colon Y \longrightarrow X$ is *α-small* if each of its fibers Y_x has cardinality $< \alpha$.

Given a simplicial set X we define $\mathbf{W}_\alpha(X)$ to be the set of isomorphism classes of α-small well-ordered morphisms $f\colon Y \longrightarrow X$. Given a morphism $t\colon X' \longrightarrow X$ we define $\mathbf{W}_\alpha(t)\colon \mathbf{W}_\alpha(X) \longrightarrow \mathbf{W}_\alpha(X')$ by $\mathbf{W}_\alpha(t) = t^*$ (the pullback functor). This gives a contravariant functor $\mathbf{W}_\alpha\colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$.

Lemma 5. $\mathbf{W}_\alpha$ *preserves all limits.*

Proof. Suppose $F\colon \mathcal{I} \longrightarrow \mathbf{sSets}$ is some diagram, and $X = \operatorname{colim}_{\mathcal{I}} F$ is its colimit, with injections $\nu_i\colon F(i) \longrightarrow X$. We need to show that the canonical map $\mathbf{W}_\alpha(X) \longrightarrow \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$ is an isomorphism.

To see that it is surjective, suppose we are given $[f_i\colon Y_i \longrightarrow F(i)] \in \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$. For each $x \in X_n$, choose some i and $\bar{x} \in F(i)$ with $\nu(\bar{x}) = x$, and set $Y_x := (Y_i)_{\bar{x}}$. By Proposition 3, this is well-defined up to *canonical* isomorphism, independent of the choices of representatives i, $\bar{x}$, Y_i, f_i. The total space of these fibers then defines a well-ordered morphism $f\colon Y \longrightarrow X$, with fibers smaller than α, and with pullbacks isomorphic to f_i as required.

For injectivity, suppose f, f' are well-ordered morphisms over X, and $\nu_i^* f \cong \nu_i^* f'$ for each i. By Proposition 3, these isomorphisms agree on each fiber, so together give an isomorphism $f \cong f'$. □

Define the simplicial set W_α by

$$\mathrm{W}_\alpha := \mathbf{W}_\alpha \circ \mathbf{y}^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Sets},$$

where $\mathbf{y}$ denotes the Yoneda embedding $\Delta \longrightarrow \mathbf{sSets}$.

Lemma 6. *The functor $\mathbf{W}_\alpha$ is representable, represented by W_α.*

Proof. Given $X \in \mathbf{sSets}$, we have isomorphisms, natural in X:

$$\begin{aligned}
\mathbf{W}_\alpha(X) &\cong \mathbf{W}_\alpha(\mathrm{colim}_{\int X} \Delta[n]) \\
&\cong \lim_{\int X} \mathbf{W}_\alpha(\Delta[n]) \\
&\cong \lim_{\int X} (\mathrm{W}_\alpha)_n \\
&\cong \lim_{\int X} \mathbf{sSets}(\Delta[n], \mathrm{W}_\alpha) \\
&\cong \mathbf{sSets}(\mathrm{colim}_{\int X} \Delta[n], \mathrm{W}_\alpha) \\
&\cong \mathbf{sSets}(X, \mathrm{W}_\alpha).
\end{aligned}$$

□

Notation 7. Given an α-small well-ordered map $f \colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$, the corresponding map $X \longrightarrow \mathrm{W}_\alpha$ will be denoted by $\ulcorner f \urcorner$.

Applying the natural isomorphism above to the identity map $\mathrm{W}_\alpha \longrightarrow \mathrm{W}_\alpha$ gives a universal α-small well-ordered simplicial set $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$. Explicitly, n-simplices of $\widetilde{\mathrm{W}}_\alpha$ are pairs

$$(f \colon Y \longrightarrow \Delta[n], s \in f^{-1}(1_{[n]}))$$

i.e. the fiber of $\widetilde{\mathrm{W}}_\alpha$ over an n-simplex $\ulcorner f \urcorner \in \mathrm{W}_\alpha$ is exactly (an isomorphic copy of) the main fiber of f. So, by construction:

Proposition 8. *The canonical projection $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ is universal for α-small well-ordered morphisms.*

Corollary 9. *The canonical projection $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ is weakly universal for α-small morphisms of simplicial sets; that is, any such morphism can be given (not necessarily uniquely) as a pullback of the projection.*

Proof. By the well-ordering principle and the axiom of choice, one can well-order the fibers, and then use the universal property of W_α. □

Definition 10. Let $\mathbf{U}_\alpha \subseteq \mathbf{W}_\alpha$ (respectively, $\mathrm{U}_\alpha \subseteq \mathrm{W}_\alpha$) be the subobject consisting of α-small well-ordered fibrations[1]; and define $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ as the pullback:

$$\begin{tikzcd}
\widetilde{\mathrm{U}}_\alpha \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{W}}_\alpha \arrow[d] \\
\mathrm{U}_\alpha \arrow[r, hook] & \mathrm{W}_\alpha
\end{tikzcd}$$

Lemma 11. *The map $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is a fibration.*

[1] Here and throughout, by "fibration" we always mean "Kan fibration".

Proof. Consider a horn to be filled

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \widetilde{\mathrm{U}}_\alpha \arrow[d, "\pi"] \\ \Delta[n] \arrow[r, "\ulcorner x \urcorner"'] & \mathrm{U}_\alpha \end{tikzcd}$$

for some $0 \leq k \leq n$. It factors through the pullback

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \bullet \arrow[r] \arrow[d, "x"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{U}}_\alpha \arrow[d, "\pi"] \\ \Delta[n] \arrow[r, equal] & \Delta[n] \arrow[r, "\ulcorner x \urcorner"'] & \mathrm{U}_\alpha \end{tikzcd}$$

where by the definition of U_α, x is a fibration. Thus the left square admits a diagonal filler, and hence so does the outer rectangle. □

Lemma 12. *An α-small well-ordered morphism $f \colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$ is a fibration if and only if $\ulcorner f \urcorner \colon X \longrightarrow \mathrm{W}_\alpha$ factors through U_α.*

Proof. For '⇒', assume that $f \colon Y \longrightarrow X$ is a fibration. Then the pullback of f to any representable is certainly a fibration:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "x^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \Delta[n] \arrow[r, "x"'] & X. \end{tikzcd}$$

so $\ulcorner f \urcorner(x) = x^* f \in \mathrm{U}_\alpha$, and hence $\ulcorner f \urcorner$ factors through U_α.

Conversely, suppose $\ulcorner f \urcorner$ factors through U_α. Then we obtain:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \widetilde{\mathrm{U}}_\alpha \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{W}}_\alpha \arrow[d] \\ X \arrow[r] & \mathrm{U}_\alpha \arrow[r, hook] & \mathrm{W}_\alpha, \end{tikzcd}$$

where the lower composite is $\ulcorner f \urcorner$, and the outer rectangle and the right square are pullbacks. Hence so is the left square, so by Lemma 11 f is a fibration. □

As an immediate consequence we obtain the following corollary.

Corollary 13. *The functor $\mathbf{U}_\alpha$ is representable, represented by U_α. The map $\pi \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is universal for α-small well-ordered fibrations, and weakly universal for α-small fibrations.*

2. Fibrancy of U_α

Our next goal is to prove the following theorem.

Theorem 14. *The simplicial set* U_α *is a Kan complex.*

Before proceeding with the proof we will gather four useful lemmas. The first two, on the theory of *minimal fibrations*, come originally from [Qui68] and [BGM59]. Since these two lemmas contain all that we need to know about minimal fibrations, we treat the notion as a black box, and refer the interested reader to [May67] for more.

Lemma 15 (Quillen's Lemma, [Qui68])**.** *Any fibration* $f\colon Y \longrightarrow X$ *may be factored as* $f = pg$*, where* p *is a minimal fibration and* g *is a trivial fibration.*

Lemma 16 ([BGM59, III.5.6]; see also [May67, Cor. 11.7])**.** *Suppose* X *is contractible, with* $x_0 \in X$*, and* $p\colon Y \longrightarrow X$ *is a minimal fibration with fiber* $F := Y_{x_0}$*. Then there is an isomorphism*

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "p"'] & & F \times X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

over X*.*

For the last outstanding lemma, the proof we give is due to André Joyal, somewhat simpler than Voevodsky's original proof. We include details here since the original [Joy11] is not currently publicly available. For this, and again for Theorem 28 below, we make crucial use of exponentiation along cofibrations; so we pause first to establish some facts about this.

Lemma 17 (Cf. [Joy11, Lemma 0.2])**.** *Suppose* $i\colon A \longrightarrow B$ *is a cofibration. Let* i_* *and* $i_!$ *denote respectively the right and the left adjoint to the pullback functor* $i^*\colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/A$*. Then:*

1. $i_*\colon \mathbf{sSets}/A \longrightarrow \mathbf{sSets}/B$ *preserves trivial fibrations;*
2. *the counit* $i^*i_* \longrightarrow 1_{\mathbf{sSets}/A}$ *is an isomorphism;*
3. *if* $p\colon E \longrightarrow A$ *is* α*-small, then so is* i_*p*.*

Proof.

1. By adjunction, since i^* preserves cofibrations.
2. Since i is mono, $i^*i_! \cong 1_{\mathbf{sSets}/A}$; so by adjointness, $i^*i_* \cong 1_{\mathbf{sSets}/A}$.
3. For any n-simplex $x\colon \Delta[n] \longrightarrow B$, we have $(i_*p)_x \cong \mathrm{Hom}_{\mathbf{sSets}/B}(i^*x, p)$. As a subobject of $\Delta[n]$, i^*x has only finitely many non-degenerate simplices, so $(i_*p)_x$ injects into a finite product of fibers of p and is thus of size $< \alpha$. □

Lemma 18 ([Joy11, Lemma 0.2])**.** *If* $t\colon Y \longrightarrow X$ *is a trivial fibration and* $j\colon X \longrightarrow X'$ *is a cofibration, then there exists a trivial fibration* $t'\colon Y' \longrightarrow X'$

and a pullback square of the form:

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, dotted, "t'"] \\ X \arrow[r, hook, "j"'] & X'. \end{tikzcd}$$

If t is α-small, then t' may be chosen to also be.

Proof. Take $(Y', t') := j_*(Y, t)$. By part 1 of Lemma 17, this is a trivial fibration; by part 2, $j^*Y' \cong Y$; and by part 3, it is small. □

We are now ready to prove that U_α is a Kan complex.

Proof of Theorem 14. We need to show that we can extend any horn in U_α to a simplex:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & \mathrm{U}_\alpha \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

By Corollary 13, such a horn corresponds to an α-small well-ordered fibration $q\colon Y \longrightarrow \Lambda^k[n]$. To extend $\ulcorner q \urcorner$ to a simplex, we just need to construct an α-small fibration Y' over $\Delta[n]$ which restricts on the horn to Y:

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, dotted, "q'"] \\ \Lambda^k[n] \arrow[r, hook] & \Delta[n]. \end{tikzcd}$$

By the axiom of choice one can then extend the well-ordering of q to q', so the map $\ulcorner q' \urcorner\colon \Delta[n] \longrightarrow \mathrm{U}_\alpha$ gives the desired simplex.

By Quillen's Lemma, we can factor q as

$$Y \xrightarrow{q_t} Y_0 \xrightarrow{q_m} \Lambda^k[n],$$

where q_t is a trivial fibration and q_m is a minimal fibration. Both are still α-small: each fiber of q_t is a subset of a fiber of q, and since a trivial fibration is onto, each fiber of q_m is a quotient of a fiber of q.

By Lemma 16, we have an isomorphism $Y_0 \cong F \times \Lambda^k[n]$, and hence a pullback diagram:

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \times \Delta[n] \arrow[d] \\ \Lambda^k[n] \arrow[r, hook] & \Delta[n] \end{tikzcd}$$

By Lemma 18, we can then complete the upper square in the following diagram, with both right-hand vertical maps α-small fibrations:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "q_t"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\
Y_0 \arrow[r, hook] \arrow[d, "q_m"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \times \Delta[n] \arrow[d] \\
\Lambda^k[n] \arrow[r, hook] & \Delta[n]
\end{tikzcd} .$$

Since α is regular, the composite of the right-hand side is again α-small; so we are done. □

3. Representability of weak equivalences

To define univalence, we first need to construct the *object of weak equivalences* between fibrations $p_1 \colon E_1 \longrightarrow B$ and $p_2 \colon E_2 \longrightarrow B$ over a common base. In other words, we want an object representing the functor sending $(X, f) \in \mathbf{sSets}/B$ to the set $\mathrm{Eq}_X(f^*E_1, f^*E_2)$. As we did for $\mathbf{U}_\alpha$, we proceed in two steps, first exhibiting it as a subfunctor of a functor more easily seen (or already known) to be representable.

For the remainder of the section, fix fibrations E_1, E_2 as above over a base B. Since **sSets** is locally Cartesian closed, we can construct the exponential object between them:

Definition 19. Let $\mathrm{Hom}_B(E_1, E_2) \longrightarrow B$ denote the internal hom from E_1 to E_2 in $\mathbf{sSets}/B$.

Then for any X, a map $X \longrightarrow \mathrm{Hom}_B(E_1, E_2)$ corresponds to a map $f \colon X \longrightarrow B$, together with a map $u \colon f^*E_1 \longrightarrow f^*E_2$ over X.

Together with the Yoneda lemma, this implies the explicit description: an n-simplex of $\mathrm{Hom}_B(E_1, E_2)$ is a pair

$$(b \colon \Delta[n] \longrightarrow B, u \colon b^*E_1 \longrightarrow b^*E_2).$$

Lemma 20. *$\mathrm{Hom}_B(E_1, E_2) \longrightarrow B$ is a Kan fibration.*

Proof. The functor $(-) \times_B E_1 \colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/B$ preserves trivial cofibrations (since **sSets** is right proper); so its right adjoint $\mathrm{Hom}_B(E_1, -)$ preserves fibrant objects. □

Within $\mathrm{Hom}_B(E_1, E_2)$, we now want to construct the subobject of weak equivalences.

Lemma 21. *Let $f \colon E_1 \longrightarrow E_2$ be a weak equivalence over B, and suppose $g \colon B' \longrightarrow B$. Then the induced map between pullbacks $g^*E_1 \longrightarrow g^*E_2$ is a weak equivalence.*

Proof. The pullback functor $g^* \colon \mathbf{sSets}/B \longrightarrow \mathbf{sSets}/B'$ preserves trivial fibrations; so by Ken Brown's Lemma [Hov99, Lemma 1.1.12], it preserves all weak equivalences between fibrant objects. □

Thus, weak equivalences from E_1 to E_2 form a subfunctor of the functor of maps from E_1 to E_2. To show that this is representable, we need just to show:

Lemma 22. *Let $f\colon E_1 \longrightarrow E_2$ be a morphism over B. If for each simplex $b\colon \Delta[n] \longrightarrow B$ the induced map $f_b\colon b^*E_1 \longrightarrow b^*E_2$ is a weak equivalence, then f is a weak equivalence.*

Proof. Without loss of generality, B is connected; otherwise, apply the result over each connected component separately. Take some vertex $b\colon \Delta[0] \longrightarrow B$, and set $F_i := b^*E_i$.

Now $\pi_0(f)$ factors as $\pi_0(E_1) \cong \pi_0(F_1) \xrightarrow{\pi_0(f_b)} \pi_0(F_2) \cong \pi_0(E_2)$, so is an isomorphism, since by hypothesis $\pi_0(f_b)$ is. Similarly, for any vertex $e\colon \Delta[0] \longrightarrow F_1$, we have by the long exact sequence for a fibration:

$$\begin{tikzcd}
\pi_{n+1}(B,b) \arrow[r] \arrow[d, "1"'] & \pi_n(F_1,e) \arrow[r] \arrow[d, "\pi_n(f_b)"'] & \pi_n(E_1,e) \arrow[r] \arrow[d, "\pi_n(f)"'] & \pi_n(B,b) \arrow[r] \arrow[d, "1"] & \pi_{n-1}(F_1,e) \arrow[d, "\pi_{n-1}(f_b)"] \\
\pi_{n+1}(B,b) \arrow[r] & \pi_n(F_2,f(e)) \arrow[r] & \pi_n(E_2,f(e)) \arrow[r] & \pi_n(B,b) \arrow[r] & \pi_{n-1}(F_2,f(e))
\end{tikzcd}$$

Each $\pi_n(f_b)$ is an isomorphism, so by the Five Lemma, so is each $\pi_n(f)$. Thus f is a weak equivalence. $\square$

Definition 23. Let $\mathrm{Eq}_B(E_1,E_2)$ be the simplicial subset of $\mathrm{Hom}_B(E_1,E_2)$ consisting of the n-simplices of the form:

$$(b\colon \Delta[n] \longrightarrow B, w\colon b^*E_1 \longrightarrow b^*E_2)$$

such that w is a weak equivalence. (By Lemma 21, this indeed defines a simplicial subset.)

From Lemma 22, we immediately have:

Corollary 24. *Let $(f,u)\colon X \longrightarrow \mathrm{Hom}_B(E_1,E_2)$. Then u is a weak equivalence if and only if (f,u) factors through $\mathrm{Eq}_B(E_1,E_2)$.*

Thus, maps $X \longrightarrow \mathrm{Eq}_B(E_1,E_2)$ correspond to pairs of maps

$$(f\colon X \longrightarrow B, w\colon f^*E_1 \longrightarrow f^*E_2),$$

where w is a weak equivalence. $\square$

While Lemma 22 was stated just as required by representability, its proof actually gives a slightly stronger statement:

Lemma 25. *Let $f\colon E_1 \longrightarrow E_2$ be a morphism over B. If for some vertex $b\colon \Delta[0] \longrightarrow B$ in each connected component the map of fibers $f_b\colon b^*E_1 \longrightarrow b^*E_2$ is a weak equivalence, then f is a weak equivalence.* $\square$

Corollary 26. *The map $\mathrm{Eq}_B(E_1,E_2) \longrightarrow B$ is a fibration.*

Proof. Suppose we wish to fill a square:

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook, "i"] & \mathrm{Eq}_B(E_1, E_2) \arrow[d] \\
\Delta[n] \arrow[r, "b"] \arrow[ur, dotted] & B
\end{tikzcd}$$

By the universal property of $\mathrm{Eq}_B(E_1, E_2)$ this corresponds to showing that we can extend a weak equivalence $w\colon i^*b^*E_1 \longrightarrow i^*b^*E_2$ over $\Lambda^k[n]$ to a weak equivalence $\overline{w}\colon b^*E_1 \longrightarrow b^*E_2$ over $\Delta[n]$.

By Lemma 20, we can certainly find some map $\overline{w}$ extending w. But then since $\Delta[n]$ is connected, Lemma 25 implies that $\overline{w}$ is a weak equivalence. $\square$

4. Univalence

Let $p\colon E \longrightarrow B$ be a fibration. We then have two fibrations over $B \times B$, given by pulling back E along the projections. Call the object of weak equivalences between these $\mathrm{Eq}(E) := \mathrm{Eq}_{B\times B}(\pi_1^*E, \pi_2^*E)$. Concretely, simplices of $\mathrm{Eq}(E)$ are triples

$$(b_1, b_2 \in B_n,\ w\colon b_1^*E \longrightarrow b_2^*E).$$

By Corollary 24, a map $f\colon X \longrightarrow \mathrm{Eq}(E)$ corresponds to a pair of maps $f_1, f_2\colon X \longrightarrow B$ together with a weak equivalence $f_1^*E \longrightarrow f_2^*E$ over X. In particular, there is a diagonal map $\delta\colon B \longrightarrow \mathrm{Eq}(E)$, corresponding to the triple $(1_B, 1_B, 1_E)$, and sending a simplex $b \in B_n$ to the triple $(b, b, 1_{E_b})$.

There are also source and target maps $s, t\colon \mathrm{Eq}(E) \longrightarrow B$, given by the composites $\mathrm{Eq}(E) \longrightarrow B \times B \xrightarrow{\pi_i} B$, sending (b_1, b_2, w) to b_1 and b_2 respectively. These are both retractions of δ; and by Corollary 26, if B is fibrant then they are moreover fibrations.

Definition 27. A fibration $p\colon E \longrightarrow B$ is called *univalent* if $\delta\colon B \longrightarrow \mathrm{Eq}(E)$ is a weak equivalence.

Since δ is always a monomorphism (thanks to its retractions), this is equivalent to saying that $B \longrightarrow \mathrm{Eq}(E) \longrightarrow B \times B$ is a (trivial cofibration, fibration) factorisation of the diagonal $\Delta\colon B \longrightarrow B \times B$, i.e. that $\mathrm{Eq}(E)$ is a *path object* for B.

Theorem 28. *The fibration $\pi\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is univalent.*

Proof. We will show that t is a trivial fibration. Since it is a retraction of δ, this implies by 2-out-of-3 that δ is a weak equivalence.

So, we need to fill a square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "i"'] & \mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha) \arrow[d, "t"] \\
B \arrow[r] \arrow[ur, dotted] & \mathrm{U}_\alpha
\end{tikzcd}$$

where $i \colon A \hookrightarrow B$ is a cofibration.

By the universal properties of U_α and $\mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha)$, these data correspond to a weak equivalence $w \colon E_1 \longrightarrow E_2$ between small well-ordered fibrations over A, and an extension $\overline{E}_2$ of E_2 to a small, well-ordered fibration over B; and a filler corresponds to an extension $\overline{E}_1$ of E_1, together with a weak equivalence $\overline{w}$ extending w:

$$\begin{tikzcd}
E_1 \arrow[rr, dashed] \arrow[dr, "w"] \arrow[ddr] & & \overline{E}_1 \arrow[dr, dashed, "\overline{w}"] \arrow[ddr, dashed] & \\
& E_2 \arrow[rr] \arrow[d] & & \overline{E}_2 \arrow[d] \\
& A \arrow[rr] & & B
\end{tikzcd}$$

As usual, it is sufficient to construct this first without well-orderings on $\overline{E}_2$; these can then always be chosen so as to extend those of E_2.

Recalling Lemmas 17–18, we define $\overline{E}_1$ and $\overline{w}$ as the pullback

$$\begin{tikzcd}
\overline{E}_1 \arrow[r] \arrow[d, "\overline{w}"'] \arrow[dr, phantom, "\lrcorner", very near start] & i_* E_1 \arrow[d, "i_* w"] \\
\overline{E}_2 \arrow[r, "\eta"'] & i_* E_2
\end{tikzcd}$$

in $\mathbf{sSets}/B$, where η is the unit of $i^* \dashv i_*$ at $\overline{E}_2$. To see that this construction works, it remains to show:

(a) $i^* \overline{E}_1 \cong E_1$ in $\mathbf{sSets}/A$, and under this, $i^* \overline{w}$ corrsponds to w;
(b) $\overline{E}_1$ is small over B;
(c) $\overline{E}_1$ is a fibration over B, and $\overline{w}$ is a weak equivalence.

For (a), pull the defining diagram of $\overline{E}_1$ back to $\mathbf{sSets}/A$; by Lemma 17 part 2, we get a pullback square

$$\begin{tikzcd}
i^* \overline{E}_1 \arrow[r] \arrow[d, "i^* \overline{w}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \arrow[d, "w"] \\
E_2 \arrow[r, "1_{E_2}"] & E_2
\end{tikzcd}$$

in $\mathbf{sSets}/A$, giving the desired isomorphism.

For (b), Lemma 17 part 3 gives that $i_* E_1$ is α-small over B, so $\overline{E}_1$ is a subobject of a pullback of α-small maps.

For (c), note first that by factoring w, we may reduce to the cases where it is either a trivial fibration or a trivial cofibration.

In the former case, by Lemma 17 part 1 $i_* w$ is also a trivial fibration, and hence so is $\overline{w}$; so $\overline{E}_1$ is fibrant over $\overline{E}_2$, hence over B.

In the latter case, E_1 is then a deformation retract of E_2 over A; we will show that $\overline{E}_1$ is also a deformation retract of $\overline{E}_2$ over B. Let $H \colon E_2 \times \Delta[1] \longrightarrow E_2$ be a deformation retraction of E_2 onto E_1. We want some homotopy $\overline{H}\colon \overline{E}_2 \times \Delta[1] \longrightarrow \overline{E}_2$ extending H on $E_2 \times \Delta[1]$, $1_{\overline{E}_1} \times \Delta[1]$ on $\overline{E}_1 \times \Delta[1]$, and $1_{\overline{E}_2}$ on $\overline{E}_2 \times \{0\}$. Since these three maps agree on the intersections of their domains, this is exactly an instance of the homotopy lifting extension property, i.e. a square-filler

$$\begin{tikzcd}
(E_2 \times \Delta[1]) \cup (\overline{E}_1 \times \Delta[1]) \cup (\overline{E}_2 \times \{0\}) \arrow[r, "H \cup 1 \cup 1"] \arrow[d, hook] & \overline{E}_2 \arrow[d] \\
\overline{E}_2 \times \Delta[1] \arrow[r] \arrow[ur, dotted, "\overline{H}"] & B
\end{tikzcd}$$

which exists since the left-hand map is a trivial cofibration.

For $\overline{H}$ to be a deformation retraction, we need to see that $\overline{H}_{\{1\}}\colon \overline{E}_2 \longrightarrow \overline{E}_2$ factors through $\overline{E}_1$. By the definition of $\overline{E}_1$, a map $f\colon X \longrightarrow \overline{E}_2$ over $b\colon X \longrightarrow B$ factors through $\overline{E}_1$ just if the pullback $i^*f\colon i^*X \longrightarrow E_2$ factors through E_1. In the case of $\overline{H}_{\{1\}}$, the pullback is by construction $i^*(\overline{H}_{\{1\}}) = (i^*\overline{H})_{\{1\}} = H_{\{1\}}\colon E_2 \longrightarrow E_2$, which factors through E_1 since H was a deformation retraction onto E_1.

So $\overline{w}$ embeds $\overline{E}_1$ as a deformation retract of $\overline{E}_2$ over B; thus $\overline{E}_1$ is a fibration over B and $\overline{w}$ a weak equivalence, as desired. □

5. Uniqueness in the universal property of $\mathbf{U}_\alpha$

Finally, as promised, we will give a uniqueness statement for the representation of a small fibration as a pullback of $\pi\colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$: we show that the space of such representations is contractible.

Let $p\colon E \longrightarrow B$ be any fibration. We define a functor

$$\mathbf{P}_p\colon \mathbf{sSets}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

taking $\mathbf{P}_p(X)$ to be the set of pairs of a map $f\colon X \times B \longrightarrow \mathrm{U}_\alpha$, and a weak equivalence $w\colon X \times E \longrightarrow f^*\widetilde{\mathrm{U}}_\alpha$ over $X \times B$; equivalently, the set of squares

$$\begin{tikzcd}
X \times E \arrow[r, "f'"] \arrow[d, "X \times p"'] & \widetilde{\mathrm{U}}_\alpha \arrow[d, "\pi"] \\
X \times B \arrow[r, "f"'] & \mathrm{U}_\alpha
\end{tikzcd}$$

such that the induced map $X \times E \longrightarrow f^*\widetilde{\mathrm{U}}_\alpha$ is a weak equivalence. Lemma 21 ensures that this is functorial in X, by pullback.

Lemma 29. *The functor $\mathbf{P}_p$ is representable, represented by the simplicial set $(\mathrm{P}_p)_n := \mathbf{P}_p(\Delta[n])$.*

Proof. Let $\mathbf{Q}_p(X)$ be the set of all commutative squares (f, f') from p to $\widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$; we know that $\mathbf{Q}_p$ is represented by $\mathrm{Q}_p := E^{\widetilde{\mathrm{U}}_\alpha} \times_{E^{\mathrm{U}_\alpha}} B^{\mathrm{U}_\alpha}$.

Now, $\mathbf{P}_p$ is a subfunctor of $\mathbf{Q}_p$. By Lemma 22, an element $(f, f') \in \mathbf{Q}_p(X)$ lies in $\mathbf{P}_p(X)$ if and only if for each $x\colon \Delta[n] \longrightarrow X$, the pullback $x^*(f, f')$ lies in $\mathbf{P}_p(X)$; that is, if its representing map $X \longrightarrow \mathrm{Q}_p$ factors through P_p. □

Proposition 30. *Let p be an α-small fibration. Then P_p is contractible.*

Proof. By Corollary 13, take some map $\ulcorner p \urcorner \colon B \longrightarrow \mathrm{U}_\alpha$ such that $E \cong \ulcorner p \urcorner^* \widetilde{\mathrm{U}}_\alpha$.

Now, for any X, maps $X \longrightarrow \mathrm{P}_p$ correspond by definition to pairs of maps $f \colon X \times B \longrightarrow \mathrm{U}_\alpha$, $w \colon X \times E \longrightarrow f^* \widetilde{\mathrm{U}}_\alpha$. But $X \times E \cong (\ulcorner p \urcorner \cdot \pi_2)^* \widetilde{\mathrm{U}}_\alpha$ over X; so such pairs also correspond to maps $\bar{f} \colon X \times B \longrightarrow \mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha)$ such that $s \cdot \bar{f} = \ulcorner p \urcorner \cdot \pi_2 \colon X \times B \longrightarrow \mathrm{U}_\alpha$.

From this, we conclude that $\mathrm{P}_p \longrightarrow 1$ is a trivial fibration: filling a square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathrm{P}_p \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & 1 \end{tikzcd}$$

corresponds to filling the square

$$\begin{tikzcd} Y \times B \arrow[r] \arrow[d] & \mathrm{Eq}(\widetilde{\mathrm{U}}_\alpha) \arrow[d, "s"] \\ X \times B \arrow[r, "\ulcorner p \urcorner \cdot \pi_2"] \arrow[ur, dotted] & \mathrm{U}_\alpha \end{tikzcd}$$

but if $Y \longrightarrow X$ is a cofibration, then so is $Y \times B \longrightarrow X \times B$; and by univalence, s is a trivial fibration; so a filler exists. □

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