Mapping Spaces and Straightening-Unstraightening

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Notes by James Richardson on talks given by Aji Dhillon.
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Some Notation and Preamble

We will denote the category of simplicial sets by **sSet** and the category of simplicially enriched categories by **SCat**. Given objects $x, y \in \mathcal{C}$ in a simplicial category, we will denote their mapping space by $\underline{\mathcal{C}}(x, y)$, and the underlying set of 0-simplices by $\mathcal{C}(x, y)$. Simplicial sets with the Quillen model structure will be denoted $\mathbf{sSet_Q}$ and the weak equivalences will be called **Kan equivalences**; simplicial sets with the Joyal model structure will be denoted $\mathbf{sSet_J}$ and the weak equivalences called **Joyal equivalences**. Recall that these are maps $f : X \to Y$ for which the induced function

$$f^* : [Y, Z] \to [X, Z]$$

on categorical homotopy classes of maps is an isomorphism for all quasicategories Z. Note that if Z is a quasicategory we can calculate $[X, Z]$ as the coequaliser of the pair

$$\mathbf{sSet}\left(X \times E^1, Z\right) \rightrightarrows \mathbf{sSet}\left(X, Z\right)$$

where $E^1 = \mathrm{N}\left(\mathrm{I}\right)$ is the nerve of the walking isomorphism.

We will also occasionally mention the Bergner model structure on **SCat**. The fibrant objects in this model structure are the locally Kan simplicial categories and the weak equivalences are the **DK-equivalences**. These are simplicial functors $F : \mathcal{C} \to \mathcal{D}$ for which the induced maps $\underline{\mathcal{C}}(x, y) \to \underline{\mathcal{D}}(Fx, Fy)$ are Kan equivalences, and the induced map on homotopy categories is essentially surjective.

We have seen the **rigidification functor** $\mathfrak{C} : \mathbf{sSet} \to \mathbf{SCat}$ and its right adjoint the **coherent nerve** $\mathbf{N} : \mathbf{SCat} \to \mathbf{sSet}$. When we first motivated the definition of quasicategory, the idea was to think of vertices in a quasicategory as objects, 1-simplices as morphisms between them, and so on. With this in mind, given a quasicategory Z and objects $x, y \in Z$ we would hope to have a simplicial set $\mathrm{Hom}_Z\left(x, y\right)$ whose vertices are maps in $\mathcal{C}$ between x and y, and whose higher simplices capture ways to compare these maps. This is what rigidification achieves: vertices in Z become the objects in $\mathfrak{C}Z$, and the simplicial set $\underline{\mathfrak{C}Z}(x,y)$ is a strong candidate for what should be thought of as the mapping space from x to y. This approach even yields a composition map:

$$\underline{\mathfrak{C}Z}(y,z)\times\underline{\mathfrak{C}Z}(x,y)\to\underline{\mathfrak{C}Z}(x,z)$$

On the other hand, the Joyal model structure gives us a factory-made interpretation of what should be meant by the mapping space between two objects in a quasicategory. For any simplicial set X, a choice of vertices $x,y\in X_0$ is the same as a map $\partial\triangle^1\to X$. The slice category $\left(\partial\triangle^1\downarrow\mathbf{sSet_J}\right):=\mathbf{sSet}_{**}$ carries a model structure induced by the Joyal model structure, and we can consider the **mapping space**:

$$\mathrm{Map}_{\mathbf{sSet}_{**}}\left(\triangle^1,X\right)$$

Here the distinguished vertices in $\triangle^1$ are given by the inclusion $\partial\triangle^1\subseteq\triangle^1$.

There are a number of models of the homotopy function complex, which can be constructed in any model category; general theory (see [Hov99]) implies that they are all weakly equivalent in $\mathbf{sSet_Q}$. The models $\mathrm{Hom}^R_Z(x,y)$, $\mathrm{Hom}^L_Z(x,y)$ and $\mathrm{Hom}_Z(x,y)$ discussed in [Lur09] and [DS11a] all fit into this framework, so it follows that they are all weakly equivalent; see [DS11a] for proof.

Our main goal will be to show that these models also agree with the model coming from rigidification. Specifically, we are aiming towards this:

Theorem. *If X is a quasicategory, then, for any $x,y\in X$, $\mathrm{Hom}^R_X(x,y)$ and $\underline{\mathfrak{C}X}(x,y)$ are Kan equivalent.*

Along the way we will prove the following:

Theorem. *If $\mathcal{C}$ is a fibrant object in* **SCat** *then the counit of the adjunction $\mathfrak{C}\dashv\mathbf{N}$ induces Kan equivalences:*

$$u_{x,y}:\underline{\mathfrak{C}\mathbf{N}\mathcal{C}}(x,y)\to\underline{\mathcal{C}}(x,y)$$

for any $x,y\in\mathcal{C}$.

Remark. Although we won't discuss them here, [DS11b] describes a model for mapping spaces in terms of generalised paths called **necklaces**. This description is technically useful, and I found it helpful in trying to understand $\mathfrak{C}X$; it lies between the more 'intuitive' or 'geometric' descriptions like $\mathrm{Hom}^R_X(x,y)$, which we will see below, and the rigidification $\underline{\mathfrak{C}X}(x,y)$.

1 Mapping Spaces and Joins

Definition 1. Let $C_1,C_2\in\mathbf{Cat}$. We can form their **join** $C_1*C_2\in\mathbf{Cat}$ as follows:

$$Ob(C_1*C_2)=Ob(C_1)\amalg Ob(C_2)$$

$$(C_1 * C_2)(x,y) = \begin{cases} C_i(x,y) & x,y \in C_i \\ * & x \in C_1, y \in C_2 \\ \emptyset & x \in C_2, y \in C_1 \end{cases}$$

For example, if C_2 is terminal then taking the join freely adjoins a terminal object to C_1. We are 'putting C_2 off to the right of C_1'. We may generalise this construction to simplicial sets:

Definition 2. Let $X, Y \in \mathbf{sSet}$ and consider the ordinal $[n] = \{0, \ldots, n\}$. We have the set of **partitions** of $[n]$:

$$\mathrm{Part}([n]) = \{(I,J) \mid I, J \subseteq [n], I \amalg J = [n], i \in I, j \in J \Rightarrow i < j\}$$

Note that picking a partition amounts to picking a division point. The **join** $X * Y \in \mathbf{sSet}$ is given by:

$$(X * Y)_n = \coprod_{(I,J) \in \mathrm{Part}([n])} X(I) \times Y(J)$$

Note that the sets in the partition can be empty; in this case we take $Z(\emptyset) := *$ when forming this coproduct. The face and degeneracy maps are induced by the maps on partitions given by taking preimages.

Example 3. The nerve preserves joins. That is, given categories $C, D \in \mathbf{Cat}$ we have:

$$\mathrm{N}(C * D) \cong \mathrm{N}(C) * \mathrm{N}(D)$$

In particular we have:

$$\triangle^{n+1} \cong \mathrm{N}([n+1]) \cong \mathrm{N}([n] * [0]) \cong \mathrm{N}([n]) * \mathrm{N}([0]) \cong \triangle^n * \triangle^0$$

Definition 4. For any simplicial set X we may form **cones** on X:

$$X^{\triangleleft} \;:=\; \triangle^0 * X \qquad X^{\triangleright} \;:=\; X * \triangle^0$$

We denote the quotients $C_L(X) := X^{\triangleleft}/X$ and $C_R(X) := X^{\triangleright}/X$. These are both canonically objects of $\mathbf{sSet}_{**}$.

Here our notation follows [DS11a]. Note that in [Lur09] $C_R(\triangle^n)$ is denoted by J^n, which is a convention we will follow in Section 3.

Definition 5. Given vertices $x, y \in X$ we may form the **left mapping space,** with n-simplices:

$$\mathrm{Hom}^L_X(x,y)_n = \mathbf{sSet}_{**}(C_L(\triangle^n), X)$$

and the **right mapping space,** with n-simplices:

$$\mathrm{Hom}^R_X(x,y)_n = \mathbf{sSet}_{**}(C_R(\triangle^n), X)$$

Lemma 6. *When X is a quasicategory,* $\mathrm{Hom}^L_X(x,y)$ *and* $\mathrm{Hom}^R_X(x,y)$ *are Kan complexes for any $x, y \in X$.*

Proof. We will prove this for $\mathrm{Hom}_X^L(x,y)$. Note that, using Example 3, we may describe the n-simplices of $\mathrm{Hom}_X^L(x,y)$ as:

$$\mathrm{Hom}_X^L(x,y)_n = \left\{\sigma : \triangle^{n+1} \to X \mid \sigma(0) = x, d_0(\sigma) = y\right\} \subseteq X_{n+1}$$

With this description the structure maps are inherited from X. Given a map from a horn $\Lambda_k^n \to \mathrm{Hom}_X^L(x,y)$ we can consider it as a map $\Lambda_k^{n+1} \to X$ by gluing in a degenerate n-cell. Since X is a quasicategory, we get a lift for $0 < k \leq n$, which gives us an extension of the original horn. Thus, $\mathrm{Hom}_X^L(x,y)$ is a quasicategory. To see that it is a Kan complex, note that 1-cells in $\mathrm{Hom}_X^L(x,y)$ are homotopies in X. The standard properties of homotopies imply that they are equivalences as maps in $\mathrm{Hom}_X^L(x,y)$. This means that every horn in $\mathrm{Hom}_X^L(x,y)$ is a special horn, so any horn has an extension. □

Remark 7. Although $\mathrm{Hom}_X^L(x,y)$ and $\mathrm{Hom}_X^R(x,y)$ both capture the intuition behind the notion of a mapping space in a quasicategory (their objects are maps and their higher cells may be thought of as higher homotopies), neither comes with an obvious composition map of the following form:

$$\mathrm{Hom}_X(y,z) \times \mathrm{Hom}_X(x,y) \to \mathrm{Hom}_X(x,z)$$

This is a disadvantage of these models compared to the mapping spaces $\underline{\mathfrak{C}X}(x,y)$ coming from rigidification.

2 Straightening and Unstraightening

2.1 The Grothendieck Construction

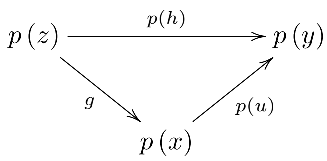
We start by reviewing the Grothendieck construction for categories. The primary tool we use in the proof of the main theorems will be a generalisation of the Grothendieck construction to quasicategories.

Definition 8. Let $p : C \to D$ be a functor. A morphism $u : x \to y$ in C is **p-cartesian** if for every $z \in X$ the natural map:

$$C(z,x) \to C(z,y) \underset{D(p(z),p(y))}{\times} D(p(z),p(x))$$

is an isomorphism.

More explicitly, this says that for any maps $h : z \to y$ and $g : p(z) \to p(x)$ making the diagram below commute:



there is a unique map $l : z \to x$ with $u \circ l = h$ and $p(l) = g$.

Example 9. Consider the codomain functor $\mathbf{cod} : C^{[1]} \to C$. A square in C is a pullback iff it is **cod**-cartesian when considered as an arrow in $C^{[1]}$.

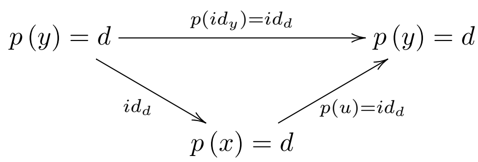
Definition 10. A functor $p : C \to D$ is **fibred in groupoids** if it satisfies the following:

1. Every arrow in C is p-cartesian.
2. For every object $x \in C$ and every map $f : d \to p(x)$ in D there is an arrow $u : y \to x$ in C with $p(u) = f$.

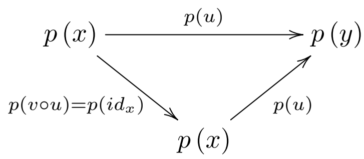
For any $d \in D$, the **fibre** over d is denoted $C_d := p^{-1}(d)$. This consist of all objects in C that are mapped to d, and all arrows that are mapped to id_d.

Lemma 11. *Let $p : C \to D$ be a functor fibred in groupoids. The fibre C_d is a groupoid.*

Proof. Let $u : x \to y$ be a map in C_d. We have the following diagram in D:



Since p is fibred in groupoids, u is p-cartesian, which means there is a unique map $v : y \to x$ with $u \circ v = id_y$ and $p(v) = id_d$. We claim that this map v gives an inverse for u in C_d. To see this, note that the diagram below commutes:



By uniqueness, the commutativity of this diagram implies that $v \circ u = id_x$. □

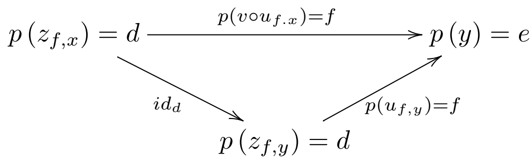
Lemma 12. *Let $p : C \to D$ be a functor fibred in groupoids. Given a map $f : d \to e$ in D, we can construct a functor $f^* : C_e \to C_d$. We have natural isomorphisms:*

$$(f \circ g)^* \cong g^* \circ f^* \qquad id^* \cong id$$

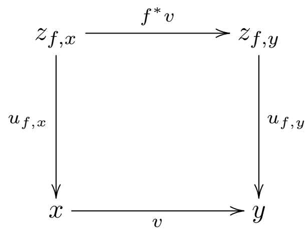
These organise into a ***pseudofunctor*** $D^{op} \to \mathbf{Gpd}$.

Proof. We will only describe the construction of f^*. For each map $h : d \to p(x)$ in D, fix a choice of arrow $u_{h,x} : z_{h,x} \to x$ with $p(u_{h,x}) = h$. This is possible by the second axiom of Definition 10. Given an object $x \in D_e$ we have $p(x) = e$, so the map $f : d \to e$ lifts to $u_{f,x} : z_{f,x} \to x$. Define $f^*(x) = z_{f,x} \in C_d$.

Given a map $v : x \to y$ in D_e consider the commutative diagram below in D:



Since $u_{f,y}$ is p-cartesian, there is a unique map $f^*v : z_{f,x} \to z_{f,y}$ with $p(f^*v) = id_d$ making the diagram below commute:



This defines the functor $f^* : C_e \to C_d$; its functoriality follows from the universal property of cartesian arrows. Note that the definition of f^* depends on the choice of maps in C lying above those in D, but a different choice gives a naturally isomorphic definition. □

Definition 13. Suppose we start with a pseudofunctor $\chi : D^{op} \to \mathbf{Gpd}$. The **Grothendieck construction** produces a functor fibred in groupoids $p : D_\chi \to D$ as follows:

$$Ob(D_\chi) = \{(d, \eta) \mid d \in D, \eta \in \chi(d)\}$$

$$D_\chi((d', \eta'), (d, \eta)) = \{(u : d' \to d, \alpha : (\chi u)\eta \to \eta')\}$$

Note that, since $\chi(d')$ is a groupoid, any map α as above must be an isomorphism. The functor $p : D_\chi \to D$ is the obvious forgetful functor.

Fact 14. *For a fixed category D, the Grothendieck construction induces an equivalence between pseudofunctors $\chi : D^{op} \to \mathbf{Gpd}$, and functors $p : C \to D$ fibred in groupoids.*

2.2 Straightening and Unstraightening for Quasicategories

We now introduce the analogue of the Grothendieck construction for quasicategories.

Definition 15. Let $K \in \mathbf{sSet}$ and $\mathcal{C} \in \mathbf{SCat}$, and suppose we have a simplicial functor $\phi : \mathfrak{C}K \to \mathcal{C}^{op}$. We will define a functor:

$$\mathbf{St}_\phi : (\mathbf{sSet} \downarrow K) \to \mathbf{SCat}(\mathcal{C}, \mathbf{sSet})$$

from the slice category $(\mathbf{sSet} \downarrow K)$ to the category of simplicial functors from $\mathcal{C}$ to $\mathbf{sSet}$.

Given $(f : X \to K) \in (\mathbf{sSet} \downarrow K)$, consider the simplicial category $\mathcal{M}_X$ defined by the following pushout in **SCat**:

$$\begin{array}{ccc} \mathfrak{C}X & \longrightarrow & \mathfrak{C}X^{\triangleright} \\ {\scriptstyle \phi \circ \mathfrak{C}f} \downarrow & & \downarrow {\scriptstyle j} \\ \mathcal{C}^{op} & \underset{i}{\longrightarrow} & \mathcal{M}_X \end{array}$$

Let $* \in X^{\triangleright}$ denote the cone point. Then the **straightening functor** associated to ϕ is defined on objects as follows:

$$\mathbf{St}_{\phi}(X) := \underline{\mathcal{M}_X}(i(-), *) : \mathcal{C} \to \mathbf{sSet}$$

Here we denote the object $f : X \to K$ merely by X.

Fact 16. *The straightening functor* $\mathbf{St}_{\phi} : (\mathbf{sSet} \downarrow K) \to \mathbf{SCat}(\mathcal{C}, \mathbf{sSet})$ *preserves colimits.*

There is an explicit description of $\underline{\mathfrak{C}X}(x, y)$ in [DS11b] which can be used to check this. The adjoint functor theorem then implies that $\mathbf{St}_{\phi}$ has a right adjoint:

$$\mathbf{Un}_{\phi} : \mathbf{SCat}(\mathcal{C}, \mathbf{sSet}) \to (\mathbf{sSet} \downarrow K)$$

which we will call the **unstraightening functor** associated to ϕ.

Exercise 17. Given a simplicial category $\mathcal{C}$, describe the n-simplices of the unstraightening $\mathbf{Un}_{\phi}(\mathcal{C})$ explicitly.

Definition 18. For any simplicial set K we may define the **contravariant model structure** on $(\mathbf{sSet} \downarrow K)$, with cofibrations given by monomorphisms and weak equivalences given by maps

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & K & \end{array}$$

such that the induced map

$$X^{\triangleright} \coprod_{X} K \to Y^{\triangleright} \coprod_{Y} K$$

is a Joyal equivalence.

For any simplicial category $\mathcal{C}$, we may equip the category of simplicial functors $\mathbf{SCat}(\mathcal{C}, \mathbf{sSet})$ with the **projective model structure**. The weak equivalences for this model structure are the pointwise Kan equivalences and the fibrations are the pointwise Kan fibrations. See [Lur09, Section A.3.3] for a discussion of model structures on enriched functor categories.

Fact 19. *The straightening functor* $\mathbf{St}_\phi : (\mathbf{sSet} \downarrow K) \to \mathbf{SCat}(\mathcal{C}, \mathbf{sSet})$ *preserves cofibrations, where we consider the contravariant model structure on the domain and the projective model structure on the codomain.*

The fibrant objects in the contravariant model structure are exactly the **right fibrations**. These are the maps that have the right lifting property with respect to all horn inclusions $\Lambda^n_k \subseteq \triangle^n$ for each $0 < k \leq n$; in particular these are inner fibrations. Note that if a functor $p : C \to D$ is fibred in groupoids then its nerve $\mathrm{N}(p) : \mathrm{N}(C) \to \mathrm{N}(D)$ is a right fibration. To see this, note that the nerve of any functor is an inner fibration, and that the nerve of a category is 2-coskeletal. Thus, to check if $\mathrm{N}(p)$ is a right fibration we need only check the horns Λ^1_1, Λ^2_2 and Λ^3_3. The axioms of Definition 10 are precisely what is required to give lifts for these three horns.

Example 20. If we take $K = \triangle^0$, the contravariant model structure of Definition 18 reduces to the Quillen model structure on **sSet**. To see this, we need only check that the cofibrations are the monomorphisms - which is true by definition - and that the fibrant objects are precisely the Kan complexes. As noted above, the fibrant objects are all simplicial sets X for which the unique map $X \to \triangle^0$ is a right fibration. By [Lur09, Prop 1.2.5.1], this map must in fact be a Kan fibration. Thus, the fibrant objects are exactly the Kan complexes.

Note that this gives a characterisation of Kan equivalences in **sSet**: a map $f : X \to Y$ is a Kan equivalence iff the induced map $C_R(f) : C_R(X) \to C_R(Y)$ is a Joyal equivalence.

3 A Special Case of Straightening-Unstraightening

For our purposes, we will focus on the special case of Definition 15 with $K = \triangle^0$ and $\phi = id_{\mathfrak{C}\triangle^0}$. Note that $\mathfrak{C}\triangle^0$ is the simplicial category with one object $0 \in \mathfrak{C}\triangle^0$ and mapping space $\underline{\mathfrak{C}\triangle^0}(0,0) = \triangle^0$. Thus, we have an isomorphism:

$$\mathbf{SCat}\left(\mathfrak{C}\triangle^0, \mathbf{sSet}\right) \cong \mathbf{sSet}$$

We also have an isomorphism:

$$\left(\mathbf{sSet} \downarrow \triangle^0\right) \cong \mathbf{sSet}$$

Thus, in this case, straightening and unstraightening reduces to an adjoint pair on **sSet**:

$$\mathbf{St} : \mathbf{sSet} \rightleftarrows \mathbf{sSet} : \mathbf{Un}$$

Now, straightening preserves colimits, so it is entirely determined by the cosimplicial object $Q^\bullet$ in **sSet** given by:

$$Q^n := \mathbf{St}\left(\triangle^n\right)$$

Using Definition 15, the fact that $\mathfrak{C}$ preserves colimits, and Example 3, we have

$$Q^n = \underline{\mathfrak{C}J^n}(x, y)$$

where we follow [Lur09] in denoting:

$$J^n := C_R(\triangle^n) = \triangle^{n+1} \coprod_{\triangle^n} \triangle^0$$

Here x is the final vertex of $\triangle^{n+1}$ and y is the unique vertex of $\triangle^0$. A complete description of **St** follows using the description of any simplicial set as a colimit over its simplices:

$$\mathbf{St}(X) = \underset{\triangle^n \to X}{\mathrm{Colim}}(Q^n)$$

Lemma 21. *There is a natural isomorphism:*

$$\mathrm{Hom}^R_{\mathbf{N}\mathcal{C}}(a,b) \cong \mathbf{Un}(\underline{\mathcal{C}}(a,b))$$

for any simplicial category $\mathcal{C}$*.*

Proof. For any $X \in \mathbf{sSet}$, we have, by definition:

$$\mathrm{Hom}^R_X(a,b)_n = \{\sigma : J^n \to X \mid \sigma(x) = a, \sigma(y) = b\}$$

where $x, y \in J^n$ are defined as above. When we take $X = \mathbf{N}(\mathcal{C})$, any simplex $\sigma \in \mathrm{Hom}^R_{\mathbf{N}\mathcal{C}}(a,b)_n$ corresponds under the adjunction $\mathfrak{C} \dashv \mathbf{N}$ to a map $\sigma : \mathfrak{C}J^n \to \mathcal{C}$, again taking x to a and y to b. Such a simplicial functor is entirely determined by the map $\underline{\mathfrak{C}J^n}(x,y) \to \underline{\mathcal{C}}(a,b)$. Thus, we have isomorphisms:

$$\begin{aligned}
\mathrm{Hom}^R_{\mathbf{N}\mathcal{C}}(a,b)_n &\cong \mathbf{sSet}(\underline{\mathfrak{C}J^n}(x,y), \underline{\mathcal{C}}(a,b)) \\
&= \mathbf{sSet}(Q^n, \underline{\mathcal{C}}(a,b)) \\
&= \mathbf{sSet}(\mathbf{St}(\triangle^n), \underline{\mathcal{C}}(a,b)) \\
&\cong \mathbf{sSet}(\triangle^n, \mathbf{Un}(\underline{\mathcal{C}}(a,b))) \\
&\cong \mathbf{Un}(\underline{\mathcal{C}}(a,b))_n
\end{aligned}$$

□

We will now give a more down-to-earth description of the simplicial set Q^n. Denote the power set of the ordinal $[n] = \{0, \ldots, n\}$ by $\mathcal{P}([n])$. As a poset (ordered by inclusion) this is isomorphic to a cube:

$$\begin{aligned}
\mathcal{P}([n]) &\xrightarrow{\cong} [1]^{n+1} \\
S &\mapsto (\varepsilon_0, \ldots, \varepsilon_n)
\end{aligned}$$

where $\varepsilon_i = 1$ if $i \in S$ and $\varepsilon_i = 0$ if $i \notin S$. Define $\mathcal{P}_{[n]} \subseteq \mathcal{P}([n])$ to be the fullsubcategory of the power set on all nonempty subsets. The nerve preserves products, so taking nerves gives:

$$K_{[n]} := \mathrm{N}\left(\mathcal{P}_{[n]}\right) \subseteq \mathrm{N}([1])^{n+1} = \left(\triangle^1\right)^{n+1}$$

Note that this simplicial set $K_{[n]}$ is the barycentric subdivision of $\triangle^n$.

For each $i \in [n]$ we may consider the following face of the cube:

$$\left(\triangle^1\right)^{\{0,\ldots,i-1\}} \times \{1\} \times \left(\triangle^1\right)^{\{i+1,\ldots,n\}} \subseteq K_{[n]}$$

The fact that there is a 1 in the i^{th} entry guarantees that this face lies in $K_{[n]}$. We obtain Q^n by collapsing each of these faces $\left(\triangle^1\right)^{\{0,\ldots,i-1\}} \times \{1\} \times \left(\triangle^1\right)^{\{i+1,\ldots,n\}}$ onto $\left(\triangle^1\right)^{\{i+1,\ldots,n\}}$ by taking the following pushout:

$$\begin{array}{ccc}
\coprod_{i\in[n]} \left(\left(\triangle^1\right)^{\{0,\ldots,i-1\}} \times \{1\} \times \left(\triangle^1\right)^{\{i+1,\ldots,n\}}\right) & \longrightarrow & K_{[n]} \\
\downarrow & & \downarrow \\
\coprod_{i\in[n]} \left(\triangle^1\right)^{\{i+1,\ldots,n\}} & \longrightarrow & Q^n
\end{array}$$

Given any poset map $f : [n] \to [m]$ we obtain a map as below:

$$\begin{array}{rcl}
\mathcal{P}_f : \mathcal{P}_{[n]} & \longrightarrow & \mathcal{P}_{[m]} \\
S & \mapsto & f(S)
\end{array}$$

This gives a map $K_f : K_{[n]} \to K_{[m]}$ on nerves, which induces a map on the pushouts $Q^n \to Q^m$. This gives the cosimplicial structure on $Q^\bullet$.

Definition 22. For any $[n]$, taking the supremum gives a poset map:

$$\begin{array}{rcl}
\mathcal{P}_{[n]} & \longrightarrow & [n] \\
S & \mapsto & \sup(S)
\end{array}$$

Taking nerves gives a map $K_{[n]} \to \triangle^n$, which induces a map on the pushout $\pi^n : Q^n \to \triangle^n$. These organise into a map of cosimplicial objects:

$$\pi : Q^\bullet \to \triangle^\bullet$$

Given any simplicial set X, this map π induces a map on the colimits below:

$$\pi_X : \mathbf{St}(X) = \underset{\triangle^n \to X}{\mathrm{Colim}}\,(Q^n) \longrightarrow X = \underset{\triangle^n \to X}{\mathrm{Colim}}\,(\triangle^n)$$

This gives a natural transformation $\pi : \mathbf{St} \Rightarrow id$.

Theorem 23. *The map $\pi_X : \mathbf{St}(X) \to X$ of Definition 22 is a Kan equivalence.*

Proof. Let $\mathcal{A}$ be the collection of all simplicial sets for which the result is true. We claim that $\mathcal{A}$ is closed under filtered colimits. To see this, suppose we have a filtered diagram $F : J \to \mathbf{sSet}$, with $F_i \in \mathcal{A}$ for every object $i \in J$. The maps π give a pointwise Kan equivalence $\mathbf{St} \circ F \to F$. Since J is filtered, the colimit functor $\mathrm{Colim} : \mathbf{sSet}^J \to \mathbf{sSet}$ takes pointwise Kan equivalences to Kan equivalences. (This is a fact of the homotopy theory of $\mathbf{sSet_Q}$. It follows in part because we have a Kan fibrant replacement functor $\mathbf{Ex}^\infty$ that preserves filtered colimits.)

This reduces us to the case of simplicial sets with finitely many nondegenerate simplices, since any simplicial set X may be written as a filtered colimit of its finite subobjects. We will prove the result by induction on dimension, and on the number of nondegenerate simplices of that dimension.

Clearly, the result holds for $X = \emptyset$. For the inductive step, suppose X is obtained from Y by attaching an n-cell. That is, X is defined as the following pushout:

$$\begin{array}{ccc} \partial\triangle^n & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \triangle^n & \longrightarrow & X \end{array}$$

Note that the boundary inclusion is a cofibration. The straightening functor preserves this pushout, and by Fact 19 it preserves cofibrations, so we obtain $\mathbf{St}\,(X)$ as the following pushout:

$$\begin{array}{ccc} \mathbf{St}\,(\partial\triangle^n) & \longrightarrow & \mathbf{St}\,(Y) \\ \big\downarrow & & \big\downarrow \\ \mathbf{St}\,(\triangle^n) & \longrightarrow & \mathbf{St}\,(X) \end{array}$$

In both of the pushouts above, one leg of the span is a cofibration; thus, these are in fact are homotopy pushouts. Therefore, it suffices to prove the result for Y, $\partial\triangle^n$ and $\triangle^n$, since this will give a pointwise Kan equivalence of the two cospans above. For Y and $\partial\triangle^n$, the result holds by the inductive hypothesis. Thus, it only remains to show that the map

$$\pi^n : Q^n \to \triangle^n$$

is a Kan equivalence. But we may contract Q^n onto the final vertex of $K_{[n]}$, so both Q^n and $\triangle^n$ are Kan equivalent to a point. Thus, π^n must be a Kan equivalence. □

Theorem 24. *The adjunction*

$$\mathbf{St} : \mathbf{sSet_Q} \rightleftarrows \mathbf{sSet_Q} : \mathbf{Un}$$

is a Quillen equivalence for the Quillen model structure on **sSet**.

Proof. By Fact 19, **St** preserves cofibrations. Moreover, Theorem 23 implies that **St** preserves Kan equivalences, by applying the 2-out-of-3 property to the naturality square below:

$$\begin{array}{ccc} \mathbf{St}(X) & \longrightarrow & \mathbf{St}(Y) \\ \wr\downarrow & & \downarrow\wr \\ X & \longrightarrow & Y \end{array}$$

Thus, the adjunction $\mathbf{St} \dashv \mathbf{Un}$ is a Quillen pair. Theorem 23 implies that the left derived functor $\mathrm{L}\mathbf{St} : \mathrm{Ho}(\mathbf{sSet_Q}) \to \mathrm{Ho}(\mathbf{sSet_Q})$ is naturally isomorphic to the identity, so in fact this is a Quillen equivalence. $\square$

The following generalisation of Theorem 24 is proved in [Lur09, Section 2.2]:

Fact 25. *For any simplicial functor $\phi : \mathfrak{C}K \to \mathcal{C}^{op}$ the adjunction*

$$\mathbf{St}_\phi : (\mathbf{sSet} \downarrow K) \rightleftarrows \mathbf{SCat}(\mathcal{C}, \mathbf{sSet}) : \mathbf{Un}_\phi$$

is a Quillen adjunction for the contravariant model structure on $(\mathbf{sSet} \downarrow K)$ and the projective model structure on $\mathbf{SCat}(\mathcal{C}, \mathbf{sSet})$. Moreover, if $\phi : \mathfrak{C}K \to \mathcal{C}^{op}$ is a DK-equivalence then it is a Quillen equivalence.

In particular, if we take $\phi = id_{\mathfrak{C}K}$ then we will denote the corresponding Quillen equivalence as follows:

$$\mathbf{St_K} : (\mathbf{sSet} \downarrow K) \rightleftarrows \mathbf{SCat}(\mathfrak{C}K^{op}, \mathbf{sSet}) : \mathbf{Un_K}$$

4 The Comparison Theorem

Let $X \in \mathbf{sSet}$. By Lemma 21, we have an isomorphism

$$\mathrm{Hom}^R_{\mathbf{N}\mathfrak{C}X}(x, y) \cong \mathbf{Un}\left(\underline{\mathfrak{C}X}(x, y)\right)$$

for any $x, y \in X$. The unit of the adjunction $\mathfrak{C} \dashv \mathbf{N}$ induces the map below:

$$\mathrm{Hom}^R_X(x, y) \longrightarrow \mathrm{Hom}^R_{\mathbf{N}\mathfrak{C}X}(x, y) \xrightarrow{\cong} \mathbf{Un}\left(\underline{\mathfrak{C}X}(x, y)\right)$$

Let

$$f : \mathbf{St}\left(\mathrm{Hom}^R_X(x, y)\right) \to \underline{\mathfrak{C}X}(x, y)$$

denote the adjunct of this map under $\mathbf{St} \dashv \mathbf{Un}$.

Theorem 26. *If X is a quasicategory the map*

$$f : \mathbf{St}\left(\mathrm{Hom}^R_X(x, y)\right) \to \underline{\mathfrak{C}X}(x, y)$$

is a Kan equivalence.

For the proof of Theorem 26 we need the following facts:

Fact 27 ([Lur09, Prop 2.1.4.9])**.** *Let $K \in \mathbf{sSet}$. Every right anodyne map is a trivial cofibration in the contravariant model structure on $(\mathbf{sSet} \downarrow K)$.*

Definition 28. Suppose we have a map of simplicial sets $X \to K$. Given a vertex $k \in K$, we define the **fibre** over k as the simplicial set X_k given by the following pullback:

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & \triangle^0 \arrow[d, "k"] \\ X \arrow[r] & K \end{tikzcd}$$

Fact 29 ([Lur09, Prop 2.2.3.15])**.** *By comparing the pushouts that define* $\mathbf{St_K}X : \mathfrak{C}K^{op} \to \mathbf{sSet}$ *and* $\mathbf{St_{\{k\}}}X_k \in \mathbf{sSet}$ *we get a canonical map:*

$$\mathbf{St_{\{k\}}}X_k \to (\mathbf{St_K}X)(k)$$

If the original map $X \to K$ *is a right fibration, then this map is a Kan equivalence.*

We will now prove Theorem 26:

Proof. We have the following factorisation of f:

$$\begin{tikzcd} \mathbf{St}\left(\mathrm{Hom}^R_X(x,y)\right) \arrow[rr, "f"] \arrow[dr, "f'"'] & & \underline{\mathfrak{C}X}(x,y) \\ & \mathbf{St_X}(X \downarrow y)(x) \arrow[ur, "f''"'] & \end{tikzcd}$$

Here $(X \downarrow y)$ is the **slice** quasicategory with n-simplices given by:

$$(X \downarrow y)_n = \left\{\sigma : \triangle^{n+1} \to X \mid \sigma(n+1) = y\right\}$$

The map f'' is induced by the canonical map

$$(X \downarrow y)^{\triangleright} \coprod_{X \downarrow y} X \longrightarrow X$$

after applying the functor $\mathfrak{C}$. The map f' is given by the following composite:

$$\mathbf{St}\left(\mathrm{Hom}^R_X(x,y)\right) \xrightarrow{\cong} \mathbf{St_{\{x\}}}\left((X \downarrow y)_x\right) \longrightarrow \mathbf{St_X}(X \downarrow y)(x)$$

where the first isomorphism can be obtained by hand, and the second map is the one that appears in Fact 29.

For any quasicategory X, the map $(X \downarrow y) \to X$ is a right fibration. This was Theorem 6 in Dinesh's talk and [Lur09, Corollary 2.1.2.2]. Thus, Fact 29 implies that f' is a Kan equivalence.

The map f'' makes the diagram below commute:

$$\begin{tikzcd} \mathbf{St_X}\{y\}(x) \arrow[rr, "(\mathbf{St_X}i)(x)"] \arrow[dr, "\cong"'] & & \mathbf{St_X}(X \downarrow y)(x) \arrow[dl, "f''"] \\ & \underline{\mathfrak{C}X}(x,y) & \end{tikzcd}$$

The isomorphism on the left is apparent after unwinding the definition of $\mathbf{St_X}\{y\}(x)$. The inclusion $i : \{y\} \to (X \downarrow y)$ is right anodyne since it is the inclusion of a terminal object (see [Lur09, Prop 4.1.1.3]). Thus, by Fact 27, i is a trivial cofibration in the contravariant model structure on $(\mathbf{sSet} \downarrow X)$, and so the left Quillen functor $\mathbf{St_X}$ takes i to a pointwise Kan equivalence. Thus, by 2-out-of-3, f'' is a Kan equivalence. So $f = f'' \circ f'$ is also a Kan equivalence. □

Corollary 30. *If X is a quasicategory, then, for any $x, y \in X$, $\mathrm{Hom}^R_X(x, y)$ and $\underline{\mathfrak{C}X}(x, y)$ are Kan equivalent.*

Proof. This follows by Theorems 23 and 26. □

Corollary 31. *If $\mathcal{C}$ is a locally Kan simplicial category then the counit of the adjunction $\mathfrak{C} \dashv \mathbf{N}$ induces Kan equivalences:*

$$u_{x,y} : \underline{\mathfrak{C}\mathbf{N}\mathcal{C}}(x, y) \to \underline{\mathcal{C}}(x, y)$$

for any $x, y \in \mathcal{C}$.

Proof. If $\mathcal{C}$ is locally Kan then $\mathbf{N}\mathcal{C}$ is a quasicategory. Consider the commutative square below:

$$\begin{array}{ccc}
\mathbf{St}\left(\mathrm{Hom}^R_{\mathbf{N}\mathcal{C}}(x, y)\right) & \xrightarrow{\cong} & \mathbf{St}\left(\mathbf{Un}\left(\underline{\mathcal{C}}(x, y)\right)\right) \\
{\scriptstyle f}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle \varepsilon} \\
\underline{\mathfrak{C}\mathbf{N}\mathcal{C}}(x, y) & \xrightarrow[u]{} & \underline{\mathcal{C}}(x, y)
\end{array}$$

The top map is the image of the isomorphism from Lemma 21, the vertical map on the left is a Kan equivalence by Theorem 26, and the vertical map on the right is the counit of the adjunction $\mathbf{St} \dashv \mathbf{Un}$. By Theorem 24, this adjunction is a Quillen equivalence, so, since every simplicial set is cofibrant, the counit is a Kan equivalence. Thus, by 2-out-of-3, the final map u is also a Kan equivalence. □

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