Simplicially enriched categories and relative categories

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Classical homotopy theory usually deals with topological spaces and simplicial sets, so it is natural to ask if there is a way to construct quasicategories that model these homotopy theories. The goal of this presentation is to describe methods that let us construct quasicategories out of other objects that also present homotopy theories. In particular we want to be able to turn simplicially enriched categories and model categories into quasicategories.

References are:

- [Lur09] for an introduction and a many of applications of simplicially enriched categories.
- [Ber07] for details about the model structure on the category of simplicially enriched categories.
- [GZ12] for the general theory of localization of relative categories.
- [DK80] for the simplicial localization of relative categories.

1 Simplicially enriched categories

Notation 1. We denote the category of (small) simplicially enriched categories by SCat.

We want to define a functor that turns simplicially enriched categories into simplicial sets. As usual it is easier to start defining its right adjoint.

Definition 2. Start by defining the functor $\mathfrak{C} : \Delta \hookrightarrow \mathsf{SCat}$ on objects. For every natural number n construct the category $\mathfrak{C}(n)$ that has as objects the set $\{0 \dots n\}$ and as morphisms:

$$\mathrm{Hom}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ \mathbf{N}(P_{i,j}) & \text{if } i \leq j \end{cases}$$

where $P_{i,j}$ is the poset of subintervals of $[i \dots j]$ that contain both i and j.

Composition $\mathrm{Hom}(i_0,i_1) \times \mathrm{Hom}(i_1,i_2) \to \mathrm{Hom}(i_0,i_2)$ is the nerve of the (poset) morphism:

$$\begin{aligned} P_{i_0,i_1} \times P_{i_1,i_2} &\to P_{i_0,i_2} \\ (I_1, I_2) &\mapsto I_1 \cup I_2 \end{aligned}$$

Moreover, for every monotone map $\sigma : [n] \to [m]$ construct the (simplicially enriched) functor $\mathfrak{C}(\sigma) : \mathfrak{C}(n) \to \mathfrak{C}(m)$ that sends an object $i \in [n]$ to $\sigma(i) \in [m]$ and its action on homs is the nerve of the (poset) morphism:

$$\begin{aligned} P_{i,j} &\to P_{\sigma(i),\sigma(j)} \\ I &\mapsto \sigma(I) \end{aligned}$$

The simplicial category $\mathfrak{C}([n])$ can be regarded as a "thickened" version of the category $[n]$: the objects are be the same, but the mapping spaces, although contractible, have a richer structure. Finally define $\mathfrak{C}$ to be $\mathrm{Lan}_{\mathcal{Y}}\,\mathfrak{C}$, the left Kan extension of $\mathfrak{C}$ along the Yoneda embedding. This is called the *rigidification funtor*.

Example 3. The following picture illustrates the poset $P_{i,j}$ in the case $i = 0, j = 3$.

$$\begin{tikzcd}
 & \{0,1,3\} \arrow[dr] & \\
\{0,3\} \arrow[ur] \arrow[rr] \arrow[dr] & & \{0,1,2,3\} \\
 & \{0,2,3\} \arrow[ur] &
\end{tikzcd}$$

So the nerve $\mathbf{N}P_{0,3}$ is:

$$\begin{tikzcd}
 & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[rr] \arrow[dr] \arrow[r, phantom, shift left=3, "///"] \arrow[r, phantom, shift right=3, "///"] & & \bullet \\
 & \bullet \arrow[ur] &
\end{tikzcd}$$

This construction can be in fact regarded as a cofibrant replacement of the usual nerve functor.

Remark 4. There is a model structure on SCat^{Δ} such that $\mathfrak{C}$ is a cofibrant replacement of the usual (non-thickened) inclusion $\Delta \hookrightarrow \mathsf{SCat}$. ■

Definition 5. By construction we know that $\mathfrak{C}$ has a right adjoint $\mathbf{N}_c : \mathsf{SCat} \to \mathsf{sSet}$, that we call *coherent nerve*. Explicitly we have $\hom_{\mathsf{sSet}}(\Delta^n, \mathbf{N}_c C) = \hom_{\mathsf{SCat}}(\mathfrak{C}\Delta^n, C)$.

A concrete relation between the coherent nerve and the usual nerve is the following.

Proposition 6. *The usual nerve factors through the coherent nerve by first including* Cat *in* SCat *(by taking the hom spaces to be discrete simplicial sets):*

$$\begin{tikzcd}
\mathsf{Cat} \arrow[rr, "\mathbf{N}"] \arrow[dr, hook] & & \mathsf{sSet} \\
 & \mathsf{SCat} \arrow[ur, "\mathbf{N}_c"] &
\end{tikzcd}$$

■

Notice that there is a functor $T : \mathsf{SCat} \to \mathsf{Cat}$ that is the identity on objects and truncates the hom spaces (applying π_0). Then the following diagram does **not** commute:

$$\begin{tikzcd} \mathsf{Cat} \arrow[rr, "\mathbf{N}"] & & \mathsf{sSet} \\ & \neq & \\ & \mathsf{SCat} \arrow[uul, "T"] \arrow[uur, "\mathbf{N}_c"'] & \end{tikzcd}$$

Having a way to construct simplicial sets out of simplicially enriched categories it is natural to ask when the resulting simplicial set is a quasicategory.

Proposition 7. *If a simplicially enriched category C is locally Kan (every hom space is a Kan complex) then $\mathbf{N}_c C$ is a quasicategory.*

Proof. We give a sketch of the proof, a complete proof can be found in [Lur09, 1.1.5.1].

Given a lifting problem:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathbf{N}_c(C) \\ \Delta^n & \end{tikzcd}$$

with $n \geq 2, 0 < k < n$ we consider the adjoint problem:

$$\begin{tikzcd} \mathfrak{C}(\Lambda^n_k) \arrow[r, "\alpha"] \arrow[d] & C \\ \mathfrak{C}(\Delta^n) & \end{tikzcd}$$

Recall that horns are constructed as the following coequalizer:

$$\coprod_{\substack{0 \leq i < j < n \\ i,j \neq k}} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta^{n-1} \to \Lambda^n_k$$

Since $\mathfrak{C}$ preserves coproducts we deduce the following two facts:

1. There is a natural identification between the objects of the category $\mathfrak{C}(\Lambda^n_k)$ and the objects of the category $\mathfrak{C}(\Delta^n)$. Moreover, this identification is given by the map $\mathfrak{C}(\Lambda^n_k) \to \mathfrak{C}(\Delta^n)$ induced by the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

2. Under the above identification, if the conditions:

   - $0 \leq i \leq j \leq n$

- $k \neq 0, n$
- $i \neq 0$ or $j \neq n$

are satisfied, we have a natural isomorphism:

$$\hom_{\mathfrak{C}(\Lambda^n_k)}(i,j) \simeq \hom_{\mathfrak{C}(\Delta^n)}(i,j).$$

Again the isomorphism is induced by the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

This reduces the problem to finding a lifting only for the diagrams:

$$\begin{tikzcd} \hom_{\mathfrak{C}(\Lambda^n_k)}(0,n) \arrow[r, "\alpha"] \arrow[d] & \hom_C(\alpha(0),\alpha(n)) \\ \hom_{\mathfrak{C}(\Delta^n)}(0,n) & \end{tikzcd}$$

Notice that the vertical map is the inclusion of the $(n-1)$-dimensional cube without the interior and a face into $(\Delta^1)^{n-1}$, the $(n-1)$-dimensional cube, hence a trivial cofibration. Since C is locally fibrant, $\hom_C(\alpha(0),\alpha(n))$ is a Kan complex and thus the lift exists. ■

Using this last proposition we can construct some quasicategories of interest.

Example 8. Consider the category of topological spaces and enrich it over simplicial sets by applying the singular complex functor to each mapping space. Since the singular complex of a topological space is a Kan complex we obtain a locally Kan simplicially enriched category. The coherent nerve of this category is then the quasicategory of topological spaces.

Example 9. Similarly, the full subcategory of sSet spanned by Kan complexes is locally Kan. Its coherent nerve is called the *quasicategory of spaces*.

2 Relative categories

Definition 10. A *relative category* is a category C together with wide subcategory of weak equivalences $\mathcal{W}$. A *relative functor* between relative categories is a functor that maps weak equivalences to weak equivalences. We denote the category of small relative categories as RelCat.

Notice that any model category is in particular a relative category, so the following constructions apply in the case of model categories.

Definition 11. Given a relative category we define its *homotopy category* as a category $\mathsf{Ho}(C)$ that comes equipped with a functor $\gamma : C \to \mathsf{Ho}(C)$ that maps weak equivalences to isomorphisms and that is universal in the sense that for any other functor $C \to D$ that maps weak equivalences in C to isomorphisms in D there is a unique factorization:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \\ \mathsf{Ho}(C) \arrow[ur, dashed] & \end{tikzcd}$$

The idea now is to to define a simplicially enriched category that models the homotopy theory presented by a relative category.

Definition 12. We construct a functor L^H : RelCat $\to$ SCat that maps a relative category to a simplicially enriched category whose objects are the same and whose hom spaces $L^H(C)(x, y)$ are the simplicial set such that the k-simplices are the reduced hammocks of width k.

A reduced k-hammock between x and y, objects of C, is a commutative diagram of the form:

$$
\begin{tikzcd}
& C_{0,0} \arrow[r, dash] \arrow[d] & C_{0,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & C_{0,n-1} \arrow[d] \arrow[dr, dash] & \\
x \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] \arrow[ddr, dash] & C_{1,0} \arrow[r, dash] \arrow[d] & C_{1,1} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] \arrow[d] & C_{1,n-1} \arrow[r, dash] \arrow[d] & y \arrow[dl, dash] \arrow[ddl, dash] \\
& \vdots \arrow[r, dash] \arrow[d] & \vdots \arrow[r, dash] \arrow[d] & \vdots \arrow[r, dash] \arrow[d] & \vdots \arrow[d] & \\
& C_{k,0} \arrow[r, dash] & C_{k,1} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k,n-1} &
\end{tikzcd}
$$

where headless arrows are arrows that can be oriented in any of the two directions and such that:

(i) $n \geq -1$.

(ii) The vertical maps are weak equivalences.

(iii) In each column (of arrows) all the arrows point to the same direction. And if an arrow points to the left then it is a weak equivalence.

(iv) Maps in adjacent columns (of arrows) point in different directions.

(v) No column (of arrows) contains only identities.

The simplicial set structure comes by equipping hammocks with:

- i-th face map: given by omitting the i-th row and reducing.
- i-th degeneracy: given by repeating the i-th row and reducing.

Where reducing means:

(iv′) Composing adjacent (arrow) columns if they point in the same direction.

(v′) Omitting (arrow) columns that consist only on identities.

The previous construction is pretty involved, but if the relative category has more structure we can consider only some very simple hammocks, as the following proposition indicates.

Proposition 13. *If C is a right proper model category then $L^H(C)(x,y) \simeq \mathbf{N}(H(x,y))$ where $H(x,t)$ is the category that has as objects diagrams of the form $x \xleftarrow{\sim} z \to y$, and as morphisms usual morphisms $z \to z'$ that make the natural diagram commute.* ∎

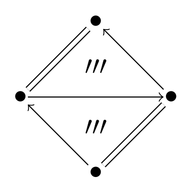
Although given a relative category we can get a quasicategory using the functor L^H : $\mathsf{RelCat} \to \mathsf{SCat}$ followed by the right derived functor of $\mathbf{N}_c : \mathsf{SCat} \to \mathsf{sSet}$, this construction involves many steps, so it is useful to have a more direct approach.

Definition 14. We define a functor $L : \mathsf{RelCat} \to \mathsf{sSet}$.

First recall the definition of the simplicial set K, which is constructed as the pushout:

$$\begin{tikzcd}
\Delta^1 \coprod \Delta^1 \arrow[r, "{[02,13]}"] \arrow[d] & \Delta^3 \arrow[d] \\
\Delta^0 \coprod \Delta^0 \arrow[r] & K
\end{tikzcd}$$

Thus K looks like:



We also have the inclusion $\Delta^1 \xrightarrow{[12]} \Delta^3 \to K$.

Then we define $L(C)$ by first constructing the pushout:

$$\begin{tikzcd}
\coprod_{u \in \mathcal{W}} \Delta^1 \arrow[r] \arrow[d] & \mathbf{N}(C) \arrow[d] \\
\coprod_{w \in \mathcal{W}} K \arrow[r] & \bullet
\end{tikzcd}$$

and then taking a fibrant replacement (in sSet_J).

Now one can ask about the relation between $L(-)$ and $\mathbf{N}_c(L^H(-))$, for this we need the Bergner model structure.

Theorem 15. *There is a model structure in* SCat *called the Bergner model structure, such that the fibrant objects are locally Kan simplicially enriched categories (i.e. the hom spaces are Kan complexes).* ∎

Theorem 16. *For any relative category C the simplicial sets $L(C)$ and $\mathbf{N}_c(\widetilde{L^H(C)})$ are weakly equivalent in sSet_J, where $\widetilde{L^H(C)}$ is a fibrant replacement of $L^H(C)$ in the Bergner model structure.* ∎

References

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