# Sphere bundles and their invariants

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[BCFR23] Ulrik Buchholtz, C., Jarl Flaten, Egbert Rijke.
Central H-spaces and banded types. arXiv:2301.02636

[BCMR24] Ulrik Buchholtz, C., David Jaz Myers, Egbert Rijke.
Vector fields on spheres and Euler classes. WIP.

**Outline:**

- Bundles
- Central types
- Cup products, the Euler class and the Thom class
- The hairy ball theorem

# Introduction and conventions

Vector bundles play a major role in classical homotopy theory.

They arise as the tangent bundles to manifolds, as the canonical bundles over projective spaces, as normal bundles of embeddings, and in many other contexts.

Since vector spaces are contractible, we don't have a good theory of vector bundles in plain homotopy type theory.

However, we can study their associated sphere bundles as a homotopically interesting replacement.

Indeed, we'll see that some invariants of vector bundles can be defined just using their sphere bundles.

**Conventions.**

We work in book HoTT, with pushouts, truncations and enough univalent universes.

We often use univalence and function extensionality implicitly.

# Bundles

Fix $T : \mathcal{U}$. A **$T$-bundle** over $B$ is a map $P : B \to \mathcal{U}$ with $\|P(b) \simeq T\|$ for all $b : B$.

That is, it's a map $B \to \mathsf{BAut}(T)$, where $\mathsf{BAut}(T) :\equiv \sum_{X:\mathcal{U}} \|X \simeq T\|$.

By the usual correspondence between maps and type families, we have an equivalence

$$\Big(\sum_{E:\mathcal{U}} \sum_{f:E\to B} \prod_{b:B} \|\mathsf{fib}_f\, b \simeq T\|\Big) \;\simeq\; \big(B \to \mathsf{BAut}(T)\big),$$

so we say that $\mathsf{BAut}(T)$ classifies maps whose fibres are merely equivalent to $T$.

For $P : B \to \mathsf{BAut}(T)$, we get a pullback square:

$$\begin{tikzcd} \sum_{b:B} P(b) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{(X,p):\mathsf{BAut}(T)} X \arrow[d, "\mathsf{pr}_1"] \\ B \arrow[r, "P"] & \mathsf{BAut}(T) \end{tikzcd}$$

so the map on the right is called the **universal $T$-bundle**.

# Oriented bundles

**Recall**: ***T*-bundles** are maps $B \to \mathsf{BAut}(T) :\equiv \sum_{X:\mathcal{U}} \|X \simeq T\|$.

An **oriented *T*-bundle** is a map $B \to \mathsf{BAut}_1(T)$, where

$$\mathsf{BAut}_1(T) :\equiv \sum_{X:\mathcal{U}} \|X \simeq T\|_0.$$

Instead of merely having an equivalence to $T$, we have a homotopy class of equivalences.

**Example:** If $X$ is merely equivalent to the sphere $S^n$, then $\|X \simeq S^n\|$ is contractible, but $\|X \simeq S^n\|_0$ is equivalent to $\mathsf{Bool}$.

**Aside:** We have forgetful maps

$$\mathsf{BAut}_1(T) \longrightarrow \mathsf{BAut}(T) \longrightarrow \mathcal{U}.$$

$\mathsf{BAut}(T)$ is the component of the universe containing $T$, which is also called the 0-connected cover of the pointed type $(\mathcal{U}, T)$.

Since $\|X \simeq T\|_0 \simeq \big(|X|_1 = |T|_1\big)$, $\mathsf{BAut}_1(T)$ is the fibre of the map $\mathcal{U} \to \|\mathcal{U}\|_1$ at the point $|T|_1$. That is, it is the 1-connected cover of $(\mathcal{U}, T)$, which explains the notation.

# Central types 

For $x : X$, the **component** of $X$ containing $x$ is denoted $X_{(x)} :\equiv \sum_{y:X} \|y = x\|$.

A pointed type $T$ is **central** if the map below is an equivalence:

$$(T \to T)_{(\mathsf{id})} \xrightarrow{\mathsf{ev}} T \qquad\qquad f \longmapsto f(\mathsf{pt})$$

**Note.** $T$ is a connected H-space.

**Proposition.** Let $T$ be a pointed type. Then the following are equivalent:

(1) $T$ is central.

(2) $T$ is a connected H-space and $T \to_* T$ is a set.

(3) $T$ is a connected H-space and $T \to_* \Omega T$ is contractible.

(4) $T$ is a connected H-space and $\Sigma T \to_* T$ is contractible.

**Example.** Every Eilenberg-Mac Lane space $K(G, n)$ with $G$ abelian and $n > 0$.

**Non-example.** $S^3$ is not central, since $\pi_4(S^3)$ is non-trivial.

# Delooping central types

**Recall.** A pointed type $T$ is **central** if the map $\mathsf{ev} : (T \to T)_{(\mathsf{id})} \to T$ is an equivalence.

**Proposition.** If $T$ is central, then $\mathsf{BAut}_1(T)$ is a delooping of $T$.

**Proof.**

$$\begin{aligned}
\Omega\,\mathsf{BAut}_1(T) &\equiv \big((T, |\mathsf{id}|_0) = (T, |\mathsf{id}|_0)\big) \\
&\simeq \sum_{e:T\simeq T} \big(|\mathsf{id}|_0 \circ |e|_0 = |\mathsf{id}|_0\big) \\
&\simeq (T \to T)_{(\mathsf{id})} \\
&\simeq T \quad \text{(by centrality)} \qquad \square
\end{aligned}$$

**Note.** In fact, one can show that $T$ has a unique delooping.

**Theorem.** Let $T$ be central. Then every pointed self-map of $T$ has a unique delooping. That is, the loop space functor gives an equivalence

$$\big(\mathsf{BAut}_1(T) \to_* \mathsf{BAut}_1(T)\big) \xrightarrow{\sim} (T \to_* T).$$

# $\mathsf{BAut}_1(T)$ is an H-space

Let $T$ be central.

**Proposition.** Let $(X,\omega),(Y,\gamma) : \mathsf{BAut}_1(T)$. Then $(X,\omega) = (Y,\gamma)$ has a $T$-orientation.

**Proof.** We need to construct an orientation $\|((X,\omega) = (Y,\gamma)) \simeq T\|_0$.

Since the goal is a set, we may induct on $\omega$ and $\gamma$, thus reducing the goal to $\|\Omega\,\mathsf{BAut}_1(T) \simeq T\|_0$, which follows from the previous proposition. □

**Definition.** Let $(X,\omega),(Y,\gamma) : \mathsf{BAut}_1(T)$. We define

$$(X,\omega) \otimes (Y,\gamma) :\equiv \big((X,\omega^*) = (Y,\gamma)\big) : \mathsf{BAut}_1(T),$$

with the orientation from above. Here $\omega^*$ is obtained from $\omega : \|X \simeq T\|_0$ by post-composing with the inversion map on $T$.

**Theorem.** The operation $\otimes$ makes $\mathsf{BAut}_1(T)$ into an H-space.

# Iterating

**Theorem.** Let $T$ be central. Then $\mathsf{BAut}_1(T)$ is central.
It follows that $T$ is an infinite loop space in a unique way.

**Proof.** $\mathsf{BAut}_1(T)$ is a (1-)connected H-space, so by an earlier result, it's enough to show that $\mathsf{BAut}_1(T) \to_* \mathsf{BAut}_1(T)$ a set.

But $\big(\mathsf{BAut}_1(T) \to_* \mathsf{BAut}_1(T)\big) \simeq (T \to_* T)$, and the latter is a set. □

**Corollary.** For $G$ abelian and $n > 0$, $\mathsf{BAut}_1(K(G,n))$ is a $K(G, n+1)$ and $\otimes$ gives a concrete description of its H-space structure.

So we can use this to represent cohomology:

$$H^{n+1}(X;G) \simeq \Big\| X \to \mathsf{BAut}_1(K(G,n)) \Big\|_0$$

with $\otimes$ giving the addition in cohomology.

**Aside.** If $X$ is a simply-connected pointed type and $\Omega X \to_* \Omega X$ is a set, then $X$ is central.

# Aside: open problems about central types

**Fact.** $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 2)$ is central, but $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ is not central.

**Question.** Is every central type a product of Eilenberg-Mac Lane spaces?

**Question.** If $T$ is central, is the base point component of $\Omega T$ central?

**Question.** If $T$ is central and $n : \mathbb{N}$, is $\|T\|_n$ central?

# Cup products



The cup product is a family of bilinear operations

$$H^n(X;\mathbb{Z}) \to_{\mathsf{Grp}} H^m(X;\mathbb{Z}) \to_{\mathsf{Grp}} H^{n+m}(X;\mathbb{Z}) \tag{1}$$

which make cohomology into a graded ring.

First defined in HoTT by Guillaume Brunerie, and extended to other rings by Lamiaux-Ljungström-Mörtberg, Brunerie-Ljungström-Mörtberg, and David Wärn. We'll give a new approach using oriented types.

**Additive natural transformations.** A map $f : K(\mathbb{Z}, n) \to K(\mathbb{Z}, m)$ induces a natural transformation $H^n(-) \to H^m(-)$.

If $f$ is $\Omega g$ for $g : K(\mathbb{Z}, n+1) \to_* K(\mathbb{Z}, m+1)$, then the transformation will be additive.

So, to get $H^m(X) \to_{\mathsf{Grp}} H^{n+m}(X)$, we want a map $K(\mathbb{Z}, m+1) \to_* K(\mathbb{Z}, n+m+1)$.

To get the left homomorphism in (1), we need a delooping of $K(\mathbb{Z}, m+1) \to_* K(\mathbb{Z}, n+m+1)$, which we take to be $K(\mathbb{Z}, m+1) \to_* K(\mathbb{Z}, n+m+2)$.

So our final goal is a map of type

$$K(\mathbb{Z}, n+1) \to_* K(\mathbb{Z}, m+1) \to_* K(\mathbb{Z}, n+m+2).$$

# Cup products via oriented types

**Goal:** $\star_{n,m} : K(\mathbb{Z}, n+1) \to_* K(\mathbb{Z}, m+1) \to_* K(\mathbb{Z}, n+m+2)$.

Let $K_1$ be any delooping of $\mathbb{Z}$, and define $K_{n+1} :\equiv \mathsf{BAut}_1(K_n)$ for $n \geq 1$.

For $n = 0$ or $m = 0$, we use an existing definition.

For $n \geq 1$ and $m \geq 1$, we define

$$\star_{n,m} : \mathsf{BAut}_1(K_n) \to_* \mathsf{BAut}_1(K_m) \to_* \mathsf{BAut}_1(K_{n+m+1})$$

by $(X, \omega) \star_{n,m} (Y, \gamma) :\equiv (\|X \star Y\|_{n+m+1}, \_)$, where the orientation requires a bit of work to describe.

$\star_{n,m}$ is bipointed: This follows from the fact that if $(X, \omega) : \mathsf{BAut}_1(T)$ and we have $x : X$, then $(X, \omega) = (T, |\mathsf{id}|_0)$. (For $T$ central.)

Since $K_n \star Y$ is pointed, $(K_n, |\mathsf{id}|_0) \star_{n,m} (Y, \gamma) = (K_{n+m+1}, |\mathsf{id}|_0)$.

**Theorem.** The family of maps $\star_{n,m}$ agrees with the cup product, up to sign.

# The Euler class

**Definition.** For $n > 0$, the **universal Euler class** is the map

$$\mathsf{e} : \mathsf{BAut}_1(S^n) \longrightarrow \mathsf{BAut}_1(K(\mathbb{Z}, n))$$

which sends an oriented $n$-sphere $(X, \omega)$ to $\|X\|_n$ with its natural $K(\mathbb{Z}, n)$-orientation coming from the equivalence $\|S^n\|_n \simeq K(\mathbb{Z}, n)$.

Given an oriented sphere bundle $P : B \to \mathsf{BAut}_1(S^n)$, its **Euler class** is

$$\mathsf{e}(P) :\equiv |\,\mathsf{e} \circ P|_0 : \Big\| B \to \mathsf{BAut}_1(K(\mathbb{Z}, n)) \Big\|_0 \equiv H^{n+1}(B; \mathbb{Z}).$$

**Moral:** The Euler class is just the conversion of an oriented $n$-sphere bundle to an oriented $K(\mathbb{Z}, n)$ bundle by truncating the fibres!

**Question.** How do we know we got it right?

# The Euler class and sections

Classically, the Euler class is the first obstruction to the existence of a nowhere zero section of a vector bundle.

**Theorem.** Let $n > 0$ and let $P : B \to \mathsf{BAut}_1(S^n)$. If $P$ merely has a section, then $\mathsf{e}(P) = 0$.

**Proof.** Since the conclusion is a proposition, we can suppose that $P$ has a section $s : \prod_{b:B} P(b)$.

Then $|s(b)|_n$ is a point in $(\mathsf{e} \circ P)(b) \equiv \|P(b)\|_n$.

Therefore, $(\mathsf{e} \circ P)(b)$ is a pointed type in $\mathsf{BAut}_1(K(\mathbb{Z}, n))$, so by the result mentioned earlier, it is equal to the basepoint. □

**Theorem.** Let $n > 0$ and let $P : S^{n+1} \to \mathsf{BAut}_1(S^n)$. If $\mathsf{e}(P) = 0$, then $P$ merely has a section.

# The Whitney sum formula

Classically, the unit sphere in the direct sum of two real vector spaces is the join of the unit spheres:

$$S(V \oplus W) \simeq S(V) \star S(W).$$

So we represent this direct sum operation on sphere bundles using the join:

$$\mathsf{BAut}(S^n) \longrightarrow \mathsf{BAut}(S^m) \longrightarrow \mathsf{BAut}(S^{n+m+1}).$$

This also works for oriented bundles:

$$\mathsf{BAut}_1(S^n) \longrightarrow \mathsf{BAut}_1(S^m) \longrightarrow \mathsf{BAut}_1(S^{n+m+1}).$$

**Whitney Sum Formula.** For $(X, \omega) : \mathsf{BAut}_1(S^n)$ and $(Y, \gamma) : \mathsf{BAut}_1(S^m)$,

$$\mathsf{e}((X, \omega) \star (Y, \gamma)) = \mathsf{e}(X, \omega) \star_{n,m} \mathsf{e}(Y, \gamma).$$

**Proof.** $\mathsf{e}((X, \omega) \star (Y, \gamma)) \equiv \|X \star Y\|_{n+m+1} \simeq \Big\| \|X\|_n \star \|Y\|_m \Big\|_{n+m+1}$

$$\equiv \|\mathsf{e}(X, \omega) \star \mathsf{e}(Y, \gamma)\|_{n+m+1} \equiv \mathsf{e}(X, \omega) \star_{n,m} \mathsf{e}(Y, \gamma).$$

□

# The Thom isomorphism

The **Thom space** $\mathsf{Thom}(P)$ of $P : B \to \mathsf{BAut}_1(S^n)$ is the pushout

$$\begin{tikzcd} \sum_{b:B} P(b) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & 1 \arrow[d, "\mathsf{pt}"] \\ B \arrow[r, "i"] & \mathsf{Thom}(P) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

**Theorem.** For $i \geq 0$, $\quad \big(B \to K(\mathbb{Z}, i)\big) \simeq \big(\mathsf{Thom}(P) \to_* K(\mathbb{Z}, i+n+1)\big)$

and so $$H^i(B) \simeq \tilde{H}^{i+n+1}(\mathsf{Thom}(P)).$$

**Proof.** We have

$$\begin{aligned} \big(\mathsf{Thom}(P) \to_* K(\mathbb{Z}, i+n+1)\big) &\simeq \Big(\prod_b \big(\Sigma P(b) \to_* K(\mathbb{Z}, i+n+1)\big)\Big) \\ &\simeq \big(B \to K(\mathbb{Z}, i)\big). \end{aligned}$$

The first equivalence is easy, and the second uses that $\Sigma P(b) \to_* K(\mathbb{Z}, i+n+1)$ is a pointed type with a $K(\mathbb{Z}, i)$-orientation and that $K(\mathbb{Z}, i) \simeq_* K(\mathbb{Z}, i)$ is a set. $\square$

# The Thom class

**Recall.** For $P : B \to \mathsf{BAut}_1(S^n)$ an oriented sphere bundle and $i \geq 0$,

$$\big(B \to K(\mathbb{Z}, i)\big) \simeq \big(\,\mathsf{Thom}(P) \to_* K(\mathbb{Z}, i+n+1)\big).$$

Taking $i = 0$ gives

$$\big(B \to \mathbb{Z}\big) \simeq \big(\,\mathsf{Thom}(P) \to_* K(\mathbb{Z}, n+1)\big).$$

The **Thom class** $\mathsf{th}(P) : \mathsf{Thom}(B) \to_* K(\mathbb{Z}, n+1)$ is the image of $\lambda b.1$.

**Proposition.** Up to sign, the Thom isomorphism above is given by taking a pointwise cup product with Thom class.

**Theorem.** The restriction of the Thom class $\mathsf{th}(P)$ along the "zero section" $i : B \to \mathsf{Thom}(P)$ gives the Euler class $\mathsf{e}(P)$.

To prove the Theorem, we give a different (less natural) definition of the Thom class, which restricts by definition to the Euler class, and then we prove that the two Thom classes are equal.

# The tangent bundles of spheres

**Theorem.** For each $n \geq 0$, there is an oriented sphere bundle

$$\tau^{n+1} : S^{n+1} \to \mathsf{BAut}_1(S^n)$$

along with equivalences

$$\theta^{n+1} : \prod_{x:S^{n+1}} (S^0 \star \tau^{n+1}(x) \simeq S^{n+1}).$$

In fact, we prove a more general result, which lets us define tangent bundles to join powers $E^{\star n}$ for certain $E$.

As a special case, we get tangent bundles for real and complex projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$.

See David Jaz Myers' HoTTEST talk for a beautiful description of this construction.

# The hairy ball theorem

**Theorem.** The tangent bundle $\tau^{n+1} : S^{n+1} \to \mathsf{BAut}_1(S^n)$ has a section if and only if $n+1$ is odd.

**Proof 1.** For $n+1$ odd, we construct explicit sections.

For general $n$, we show that if you have a section, then we get a homotopy

$$\prod_{x:S^{n+1}} (x = -x),$$

where $-x$ denotes the action of the antipodal map.

The antipodal map has degree $(-1)^{n+2}$, so this gives a contradiction when $n+1$ is even. $\square$

**Proof 2.** We prove that $\mathsf{e}(\tau^{n+1}) = 1 + (-1)^{n+1}$, up to sign.

By earlier results on Euler classes, $\tau^{n+1}$ has a section if and only if the Euler class vanishes. $\square$

# The power of $\mathsf{BAut}_1$

Using $\mathsf{BAut}_1$, we can:

**Thanks!**

- Define oriented bundles.
- Define addition in cohomology.
- Define multiplication in cohomology.
- Define the Euler class and prove the Whitney sum formula.
- Prove the Thom isomorphism.
- Define the Thom class and prove that it restricts to the Euler class.
- Use these to prove the hairy ball theorem.

**References.**


Ulrik Buchholtz, C., Jarl Flaten, Egbert Rijke. Central H-spaces and banded types. arXiv:2301.02636

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**Formalization:** `https://github.com/HoTT/Coq-HoTT` and `https://github.com/jarlg/central-types`, plus parts not yet public.