Constructive mathematics in univalent type theory

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Homotopy Type Theory Electronic Seminar Talk, 26th April 2018

Elementary-topos type theory (Lambek and Scott 1986)

Internal language of the free elementary topos with NNO.

They also call it "intuitionistic type theory".

1. Simply typed λ-calculus with finite product types.
2. Type $\mathbb{N}$ of natural numbers.
3. Type Ω of truth values (corresponding to subsingletons of 1).
4. Functions $(=_X) : X \times X \to \Omega$ for each type/object X.
5. Functions $(\wedge), (\vee), (\implies) : \Omega \times \Omega \to \Omega$.
6. Functions $\forall_X, \exists_X : (X \to \Omega) \to \Omega$ for each X.
7. Axioms (for equality, function and propositional extensionality, induction, ...).
8. Intuitionistic deductions rules.
9. Has the existence and disjunction properties.

   In the *empty context*, $\vee$ and $\exists$ behave like $+$ and Σ in MLTT.

- Although toposes have Π and Σ, this type theory doesn't include them.
- Can state property directly, but need to give structure indirectly.

Intensional Martin-Löf type theory (a spartan one here)

1. Dependently typed λ-calculus with Π, Σ, Id, $+$, 0, 1, $\mathbb{N}$, $\mathcal{U}$.
2. No axioms. Rules to derive types, contexts, terms.
3. Need to add definitional (or judgmental) equality.

   Doesn't occur in terms.

   Can only be written when it holds.

   Plays a role only in derivation rules.
4. All types, not just subsingletons are considered to be propositions.
5. And is $\times$, or is $+$, implies is $\to$, for all is Π, and exists is Σ.
6. Lacks function extensionality (and propositional extensionality doesn't make sense).

- Property and structure (or data) are conflated.
- Use setoids to collapse structure to property.
- The universe allows to define types of mathematical structures.
- Its identity type is underspecified (compatible with both K and UA).

Example: Dedekind reals in MLTT

A Dedekind real is a pair of functions $l, u : \mathbb{Q} \to \mathcal{U}$ together with some data, including

1. A function $\Pi(q : \mathbb{Q}), l(q) \to \Sigma(r : \mathbb{Q}), (q < r) \times l(r)$.

   This function, given any rational in the lower section, picks a bigger rational in the lower section.

   (Which rational number is picked?)

2. A function $\Pi(p, q), p < q \to l(p) + u(q)$.

   This function, given rational numbers $p < q$, decides which of $l(p)$ or $u(q)$ holds.

   What does it answer when both hold?

- E.g. the number π is represented by (infinitely) many different Dedekind sections.
- Need to work with an equivalence relation.
- No quotients, hence work with setoids (type $\&$ equivalence relation).

(Like Bishop proposed, although he worked with Cantor reals.)

Example: image

For $f : X \to Y$,

1. The candidate for the image is $\Sigma(y : Y), \Sigma(x : X), \mathrm{Id}(f(x), y)$.

   This is the type of $y : Y$ for which there is some x mapped to a point identified with y.
2. But this is in bijection with X.

   Again need an equivalence relation on this type, identifying (y, x, p) with (y, x', p')
3. Moreover, we don't actually work with $\mathrm{Id}(f(x), y)$.

   Instead we have that X and Y already are setoids, $f : X \to Y$ preserves the equivalence relation, and the image has underlying type $\Sigma(y : Y), \Sigma(x : X), f(x) \sim y$ with equivalence relation defined by
   $$((y, x, p) \sim (y, x', p')) = (y \sim y').$$

The identity type is hardly used when working with setoids.

Problems with setoids

1. Practical one (nicknamed "setoid hell").

   Incredible amount of bookkeeping is needed in proof assistants.

   (Bishop simply ignores the bookkeeping in his book.)

2. It is not known how to handle the universe as a setoid.

Univalent type theory (again a spartan one)

1. MLTT $+$ propositional truncation $+$ univalence.
2. Propositions redefined to be subsingletons.

   $\mathrm{isProp}(X) = \Pi(x, y : X), \mathrm{Id}(x, y)$.

   $\Omega = \Sigma(P : \mathcal{U}), \mathrm{isProp}(P)$.
3. Get functional and propositional extensionality.
4. The propositional truncation of X is the universal solution $|-| : X \to \|X\|$ to the problem of mapping X to a subsingleton:

   If $X \to P$ then $\|X\| \to P$.
5. $P \vee Q = \|P + Q\|$.

   $\exists(x : X), P(x) = \|\Sigma(x : X), P(x)\|$.

cubicaltt is an example of constructive univalent type theory:

1. Univalence is a theorem.
2. Has the canocity property.

   (Which gives the disjunction and existence properties.)

Examples revisited

1. The image of $f : X \to Y$ is $\Sigma(y : Y), \|\Sigma(x : X), \mathrm{Id}(f(x), y)\|$.
2. A Dedekind real is a pair of functions $l, u : \mathbb{Q} \to \Omega$ together with properties, including
   2.1 A function $\Pi(q : \mathbb{Q}), l(q) \to \|\Sigma(r : \mathbb{Q}), (q < r) \times l(r)\|$.
   2.2 A function $\Pi(p, q), p < q \to \|l(p) + u(q)\|$.

   - Such functions, as well as the ones we have omitted, are unique if they exist.
   - Moreover, the identity type gives the correct notion of equality.

   (Without quotienting.)

   - This relies on functional and propositional extensionality.

Univalent type theory $\approx$ MLTT + Topos TT + ∞

1. MLTT favours *data*. Σ
2. Topos TT favours *property*. $\exists$
3. UTT incorporates both. Σ and $\exists$

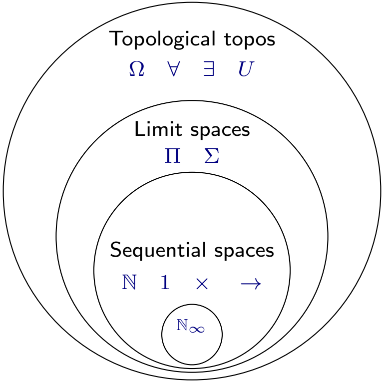
In his *"The formulae-as-types notion of contruction"*, Howard actually discusses *two* notions of existence.

So perhaps univalent logic is the true Curry-Howard logic.

Another example: all functions are continuous

I will approach this from a model, and then formulate it in type theory.

Johnstone's topological topos (1979)



1. The site is the category of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ with the canonical coverage.
2. Taking colimits of $\mathbb{N}_\infty$ in topological spaces gives sequential spaces.
3. The limit spaces arise as the subobjects of sequential spaces.

Examples of MLTT-definable objects of the topos

1. The interpretation of the type $\mathbb{N} \to 2$ gives the Cantor space $2^{\mathbb{N}}$.
2. The interpretation of the type $\mathbb{N} \to \mathbb{N}$ gives the Baire space $\mathbb{N}^{\mathbb{N}}$.
3. The interpretation of the simple types gives the Kleene–Kreisel continuous functionals. (Start from $\mathbb{N}$ and close under $\to$.)
4. The interpretation of the type

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \left(\sum_{\alpha : \mathbb{N} \to 2} \prod_{n:\mathbb{N}} \alpha_n = 0 \to \alpha_{n+1} = 0 \right)$$

   gives the one-point compactification of $\mathbb{N}$, with $\infty \stackrel{\text{def}}{=} (\lambda i.1, -)$.

   Here "$=$" is the identity type, interpreted as an equalizer.
5. The interpretation of the type

$$\sum_{x:\mathbb{N}_\infty} 2^{x=\infty}$$

   is a T_1, non-Hausdorff, but compact, space with two points at infinity,

$$\infty_0 \stackrel{\text{def}}{=} (\infty, \lambda p.0), \quad \infty_1 \stackrel{\text{def}}{=} (\infty, \lambda p.1).$$

The topological topos validates continuity axioms

Continuity axiom (Cont)

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\forall f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \forall \alpha : \mathbb{N}^{\mathbb{N}}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Uniform continuity axiom (UC)

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\forall f : 2^{\mathbb{N}} \to \mathbb{N}.\ \exists n : \mathbb{N}.\ \forall \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

- This assumes a classical meta-theory.
- There is another topological topos developed within a constructive meta-theory by Chuangjie Xu and myself.
- Also formalized in (cubical) Agda by Chuangjie. We can compute moduli of continuity using this Agda implementation of the model.
- Using stacks Coquand et al. have extended this to a model of univalent type theory

Can we replace $\exists$ by Σ?

Continuity axiom (Cont):

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

$$\Pi f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}.\ \Pi\alpha : \mathbb{N}^{\mathbb{N}}.\ \Sigma n : \mathbb{N}.\ \Pi\beta : \mathbb{N}^{\mathbb{N}}.\ \alpha =_n \beta \to f\alpha = f\beta.$$

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Continuity axiom (Cont): $\times$

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Uniform continuity axiom (UC):

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

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Can we replace $\exists$ by Σ?

Continuity axiom (Cont): × Moreover, no topos can validate this.

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Uniform continuity axiom (UC):

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

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Continuity axiom (Cont): $\times$ Moreover, no topos can validate this.

All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

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Uniform continuity axiom (UC): ✓

All functions $2^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous.

$$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi\alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Theorem of intensional Martin-Löf type theory

If all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous then $0 = 1$.

$$\left(\prod_{f\,:\,\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \prod_{\alpha : \mathbb{N}^{\mathbb{N}}} \sum_{n : \mathbb{N}} \prod_{\beta : \mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta \right) \to 0 = 1.$$

Theorem of intensional Martin-Löf type theory

$$\left(\prod_{f\,:\,\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\ \prod_{\alpha:\mathbb{N}^{\mathbb{N}}}\ \sum_{n:\mathbb{N}}\ \prod_{\beta:\mathbb{N}^{\mathbb{N}}} \alpha =_n \beta \to f\alpha = f\beta\right) \to 0 = 1.$$

I could instead say "not all functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous", but:

1. This would give the false impression that there might exist a non-continuous function to be found by looking hard enough.
   (In the topological topos all functions are continuous, and yet this holds.)

2. It is $0 = 1$ that our proof actually does give from the assumption.
   (A technicality that leads to the next item.)

3. We would need a universe to map the type $0 = 1$ to the type $\emptyset$,
   and our proof doesn't require universes.
   (So we are more general.)

Exiting truncations I

The elimination rule is $(X \to P) \to (\|X\| \to P)$

for subsingleton P.

We can disclose a secret $\|X\|$ to P provided we have a map $X \to P$.

Example. If $A(n)$ is decidable then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Proof sketch. If we have any n with $A(n)$, we can find the minimal n, using the decidability of $A(n)$, but "having a minimal n such that $A(n)$" is a subsingleton.

Exiting truncations II

Assume that $A(n)$ is a subsingleton for every $n : \mathbb{N}$.

If for any given n we have that $A(n)$ implies that $A(m)$ is decidable for all $m < n$, then

$$\|\Sigma n : \mathbb{N}.\ A(n)\| \to \Sigma n : \mathbb{N}.\ A(n).$$

Theorem

$$\begin{aligned}\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ &\left\|\Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\right\| \\ &\to \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.\end{aligned}$$

Proof. Set $A(n) = \left(\Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta\right)$ in the lemma.

Corollary. The topological topos also validates the uniform-continuity principle

$$\Pi f : 2^{\mathbb{N}} \to \mathbb{N}.\ \Sigma n : \mathbb{N}.\ \Pi \alpha, \beta : 2^{\mathbb{N}}.\ \alpha =_n \beta \implies f\alpha = f\beta.$$

Because the premise of the theorem is validated.

(In the topological topos, the theorem can be seen as getting global existence from local existence by compactness.)

LPO

$$\Pi(p : \mathbb{N} \to 2), (\exists(n : \mathbb{N}), p(n) = 0) + (\Pi(n : \mathbb{N}), p(n) = 1)$$

1. A meta-theorem is that we can't inhabit LPO or $\neg$LPO.

2. Each of them is consistent with MLTT.

   Classical models validate LPO.

   Effective and continuous models validate $\neg$LPO.

3. LPO is undecided, and we'll keep it that way.

4. But we'll say it is a constructive taboo.

We now make $\mathbb{N}$ larger by adding a point at infinity

Let $\mathbb{N}_\infty$ be the type of decreasing binary sequences.

$$\mathbb{N}_\infty \stackrel{\mathsf{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}), \Pi(n : \mathbb{N}), \alpha(n) = 0 \to \alpha(n+1) = 0.$$

Side-remark:

1. $\mathbb{N}$ is the *initial algebra* of the functor $1 + (-)$.

2. $\mathbb{N}_\infty$ is the *final coalgebra* of this functor.

(This requires function extensionality.)

We now make $\mathbb{N}$ larger by adding a point at infinity

Let $\mathbb{N}_\infty$ be the type of decreasing binary sequences.

$$\mathbb{N}_\infty \stackrel{\text{def}}{=} \Sigma(\alpha : 2^{\mathbb{N}}), \Pi(n : \mathbb{N}), \alpha(n) = 0 \to \alpha(n+1) = 0.$$

1. The type $\mathbb{N}$ embeds into $\mathbb{N}_\infty$ by mapping the number $n : \mathbb{N}$ to the sequence $\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$.
2. A point not in the image of this is $\infty \stackrel{\text{def}}{=} 1^\omega$.
3. The assertion that every point of $\mathbb{N}_\infty$ is of one of these two forms is equivalent to LPO.
4. What is true is that no point of $\mathbb{N}_\infty$ is different from all points of these two forms.
5. The embedding $\mathbb{N} + 1 \to \mathbb{N}_\infty$ is an isomorphism iff LPO holds.
6. But the complement of its image is empty. We say it is dense.

Theorem

$$\Pi(p : \mathbb{N}_\infty \to 2), (\Sigma(x : \mathbb{N}_\infty), p(x) = 0) + \neg\Sigma(x : \mathbb{N}_\infty), p(n) = 0$$

1. This is LPO with $\mathbb{N}$ replaced by $\mathbb{N}_\infty$.

2. We don't use continuity axioms, which anyway are not available in MLTT.

3. However, this is motivated by topological (not homotopical) considerations.

   In Johnstone's *topological topos*, $\mathbb{N}_\infty$ gets interpreted as the one-point compactification of discrete $\mathbb{N}$.

   Here we are seeing a *logical manifestation* of *topological compactness*.

The talk ended here

I also said many things that are not in the slides.

1. The talk is here:
   `http://uwo.ca/math/faculty/kapulkin/seminars/hottest.html`
2. The cubical Agda implementation of the modified topological topos is here:
   `http://www.cs.bham.ac.uk/~mhe/chuangjie-xu-thesis-cubical/html`

WLPO is also undecided

$$\Pi(p : \mathbb{N} \to 2), (\Pi(n : \mathbb{N}), p(n) = 1) + \neg\Pi(x : \mathbb{N}), p(n) = 1$$

But we have:

Theorem $\Pi(p : \mathbb{N}_\infty \to 2), (\Pi(n : \mathbb{N}), p(\underline{n}) = 1) + \neg\Pi(n : \mathbb{N}), p(\underline{n}) = 1$

The point is that now we quantify over $\mathbb{N}$, although the function p is defined on $\mathbb{N}_\infty$.

Some consequences

1. Every function $f : \mathbb{N}_\infty \to \mathbb{N}$ is constant or not.
2. Any two functions $f, g : \mathbb{N}_\infty \to \mathbb{N}$ are equal or not.
3. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ has a minimum value, and it is possible to find a point at which the minimum value is attained.
4. For any function $f : \mathbb{N}_\infty \to \mathbb{N}$ we can find a point $x : \mathbb{N}_\infty$ such that if f has a maximum value, the maximum value is x.
5. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is not continuous, or not-not continuous.
6. There is a non-continuous function $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.

Two notions

Definition (Omniscient type)
A type X is omniscient if for every $p : X \to 2$, the assertion that we can find $x : X$ with $p(x) = 0$ is decidable.

In symbols:

$$\Pi(p : X \to 2), (\Sigma(x : X), p(x) = 0) + (\neg\Sigma(x : X), p(x) = 0).$$

Definition (Searchable type)
A type X is searchable if for every $p : X \to 2$ we can find $x_0 : X$, called a *universal witness* for p, such that if $p(x_0) = 1$, then $p(x) = 1$ for all $x : X$.

In symbols,

$$\Pi(p : X \to 2), \Sigma(x_0 : X), p(x_0) = 1 \to \Pi(x : X), p(x) = 1.$$

Their relationship

$$\text{omniscient}(X) = \Pi(p : X \to 2), (\Sigma(x : X), p(x) = 0) + (\neg\Sigma(x : X), p(x) = 0).$$

$$\text{searchable}(X) = \Pi(p : X \to 2), \Sigma(x_0 : X), p(x_0) = 1 \to \Pi(x : X), p(x) = 1.$$

NB. These types are not subsingletons in general.

Proposition A type X is searchable iff it has a point and is omniscient:

$$\text{searchable}(X) \iff X \times \text{omniscient}(X).$$

A few theorems rely on pointedness, using the notion of searchability.

Closure under Σ

If X is omniscient/searchable and Y is an X-indexed family of omniscient/searchable types, then so is its disjoint sum $\Sigma(x : X), Y(x)$.

Closure under Π

Not to be expected in general.

E.g. $\mathbb{N}_\infty$ and 2 are omniscient, but in continuous and effective models of type theory, the function space $\mathbb{N}_\infty \to 2$ is not.

In the topological topos, $\mathbb{N}_\infty \to 2$ is isomorphic to $\mathbb{N}$.

Closure under finite products

Theorem A product of searchable types indexed by a finite type is searchable.

We will need this form of closure under Π

Theorem (micro Tychonoff)
A product of searchable types indexed by a subsingleton type is itself searchable.

That is, if X is a subsingleton, and Y is an X-indexed family of searchable types, then the type $\Pi(x : X), Y(x)$ is searchable.

This cannot be proved if searchability is replaced by omniscience (that is, if we don't assume that every $Y(x)$ is pointed).

This is easy with excluded middle, but we are not assuming it.

Theorem A subsingleton-indexed product of searchable types is searchable.

1. Let X subsingleton, $Y(x)$ searchable for every $x : X$.
2. $Z \stackrel{\text{def}}{=} \Pi(x : X), Y(x)$.

   We have $\Pi(x : X), (Z \simeq Y(x))$ and $(X \to 0) \to (Z \simeq 1)$.
3. Let $p : Z \to 2$.
4. Construct $z_0(x) \stackrel{\text{def}}{=} \ldots$ in Z using the first equivalence.
5. $X \to p(z_0) = 1 \to \Pi(z : Z), p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z), X \to p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z), p(z) = 0 \to (X \to 0)$.
6. $(X \to 0) \to p(z_0) = 1 \to \Pi(z : Z), p(z) = 1$.

   $p(z_0) = 1 \to \Pi(z : Z), (X \to 0) \to p(z) = 1$.
7. By transitivity of $\to$, we get

   $p(z_0) = 1 \to \Pi(z : Z), p(z) = 0 \to p(z) = 1$, so

   $p(z_0) = 1 \to \Pi(z : Z), p(z) = 1$. Q.E.D.

Amusing consequence, tangential to our development

Corollary. *The type* $\mathbb{N}^{LPO}$ *is searchable.*

- The reason is that LPO implies that $\mathbb{N}$ is searchable, and so this is a product of searchable types indexed by a subsingleton.

  Even though the searchability of $\mathbb{N}$ is undecided!

- If LPO holds, the type of the corollary is $\mathbb{N}$.
- If LPO fails, it is a singleton.
- As LPO is undecided, we don't know what the type $\mathbb{N}^{\mathsf{LPO}}$ "really is".
- Whatever it is, however, it is always searchable.

Disjoint sum with a point at infinity

Theorem

The disjoint sum of a countable family of searchable sets with a point at infinity is searchable.

We need to say how we add a point at infinity constructively.

The type $1 + \Sigma(n : \mathbb{N}), X(n)$ won't do, of course.

We will do this in a couple of steps.

Injectivity of the universe of types

Theorem
For any embedding $e : A \to B$, every $X : A \to U$ extends to some $Y : B \to U$ along e, up to equivalence,

$$\Pi(a : A), (Y(e(a)) \simeq X(a)).$$

A map $e : A \to B$ is called an embedding iff its fibers $e^{-1}(b)$,

$$\Sigma(a : A), f(a) = b,$$

are all subsingletons.

Injectivity of the universe of types

Theorem

For any embedding $e : A \to B$, every $X : A \to U$ extends to some $Y : B \to U$ along e, up to equivalence.

Two constructions:

1. We have the "maximal" extension $Y = X/e$.

$$\begin{aligned}(X/e)(b) &= \Pi\left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s) \\ &\simeq \Pi(a : A),\ e(a) = b \ \to\ X(a).\end{aligned}$$

2. And also the "minimal" extension $Y = X \setminus e$.

$$\begin{aligned}(X \setminus e)(b) &= \Sigma\left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s) \\ &\simeq \Sigma(a : A),\ e(a) = b \ \times\ X(a).\end{aligned}$$

The first one works our purposes.

Injectivity of the universe of types

Let $e : A \to B$ be an embedding and $X : A \to U$.

Consider the extended type family $X/e : B \to U$ defined above:

$$(X \setminus e)(b) = \Pi \left(s : e^{-1}(b)\right), X(\mathrm{pr}_1\, s)$$

We have

1. For all $b : B$ *not* in the image of the embedding,

$$(X/e)(b) \simeq 1.$$

2. If for all $a : A$, the type $X(a)$ is searchable too, then for all $b : B$ the type $(X/e)(b)$ is searchable, by micro-Tychonoff.

3. Hence if additionally B is searchable, the type $\Sigma(b : B), (X/e)(b)$ is searchable too.

4. We are interested in $A = \mathbb{N}$ and $B = \mathbb{N}_\infty$, which gives the disjoint sum of $X(a)$ with a point at infinity.

A map $L : (\mathbb{N} \to U) \to U$

Let $e : \mathbb{N} \to \mathbb{N}_\infty$ be the natural embedding.

Given $X : \mathbb{N} \to U$, first take $X/e : \mathbb{N}_\infty \to U$

This step adds a point at infinity to the sequence.

We then sum over $\mathbb{N}_\infty$, to get $L(X)$:

$$L(X) = \Sigma(u : \mathbb{N}_\infty), (X/e)(u).$$

Then L maps any sequence of searchable types to a searchable type.

Iterating this map $L : (\mathbb{N} \to U) \to U$

We get (very large!) searchable ordinals, with the property that any inhabited *decidable* subset has a least element.

They are all countable.

Or rather they each have a countable subset with empty complement.

An ordinal is a type X with a transitive and extensional relation $(-) < (-) : X \to X \to U$ which is well-founded in the above sense.

1. Extensional means that any two elements with the same predecessors are equal.
2. The accessibility of points of X is inductively defined.

   We say that $x : X$ is accessible whenever every $y < x$ is accessible.

For the sake of completeness, we characterize the injectives in UF

We have seen that universes are injective, and applied this to construct searchable types and ordinals.

Independently of this, it is natural to try to understand what the injective types are.

1. In topos theory, the injectives are the retracts powers of the subobject classifier.

2. We show that, in UF, they are the retracts of powers of universes.

3. Before concluding, we prove this and offer a finer analysis.

The Yoneda embedding

1. For any type X, point $x : X$ and family $A : X \to U$,

$$(\Pi(y : X).\, \mathrm{Id}\; x\, y \to A(y)) \simeq A(x).$$

   This is the Yoneda Lemma.

2. Say that A is representable if we have $x : X$ with $A(y) \simeq \mathrm{Id}\; x\, y$.

3. A having a universal element amounts to $\Sigma(x : X).A(x)$ being a singleton, or contractible.

4. The representability of A is equivalent to the contractibility of $\Sigma(x : X).A(x)$, and hence representability is a proposition.

Therefore, assuming univalence,

Theorem. *The map* $\mathrm{Id} : X \to U^X$ *is an embedding.*

Standard reasoning with injectives

1. Any power I^X of an injective type I is again injective.

2. A retract of an injective type is again injective.

3. An injective type is a retract of every type in which it is embedded.

Characterization of the injective types

Combining this with the Yoneda Embedding:

Theorem. *The injective types are precisely the retracts of powers of the universes.*

We also have:

Theorem. *The injective sets are precisely the retracts of powers of the universe of subsingletons.*

Theorem. *The injective* $n + 1$*-types are precisely the retracts of powers of the universe of* n*-types.*

More logical manifestations of topological concepts and theorems

The above claims/constructions have been formalized in Agda with univalent assumptions.

Much more can be found at

- `http://www.cs.bham.ac.uk/~mhe/agda-new/`
- `http://www.cs.bham.ac.uk/~mhe/agda-new/graph-tred.pdf`

Including

1. Compact types.
2. Totally separated types, total separated reflection, interaction with compactness and discretness.
3. Topological properties of simple types familiar from Kleene–Kreisel spaces.