CORRIGENDUM TO "ANGULAR EQUIVALENCE OF NORMED SPACES" [J. MATH. ANAL. APPL. 454(2) (2017) 942–960]

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Abstract. A correct proof is given for Theorem 2.1 of E. Kikianty and G. Sinnamon, 'Angular equivalence of normed spaces', *J. Math. Anal. Appl.*, 454(2):942–960, 2017. http://doi.org/10.1016/j.jmaa.2017.05.038.

The statement of Theorem 2.1 in the above paper is correct, but the proof contains an unsupported statement. The statement and a correct proof follow. Refer to the original article for notation and definitions, and for inequality (1.2).

Theorem 2.1. *Let X be a real vector space having two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are angularly equivalent and x is a non-zero vector in X. Then $x/\|x\|_1$ is an extreme point of the $\|\cdot\|_1$-unit ball if and only if $x/\|x\|_2$ is an extreme point of the $\|\cdot\|_2$-unit ball.*

Proof. We argue the contrapositive. Suppose $x/\|x\|_2$ is not an extreme point of the $\|\cdot\|_2$-unit ball. Then there are points y and z in X such that $(y+z)/2 = x/\|x\|_2$ and the closed line segment from y to z is contained in the $\|\cdot\|_2$-unit ball. If $s \in [0,1]$ then $(1-s)y+sz$ and $sy+(1-s)z$ are on the line segment and hence in the $\|\cdot\|_2$-unit ball. Thus,

$$2 = \|y+z\|_2 = \|(1-s)y+sz+sy+(1-s)z\|_2 \le \|(1-s)y+sz\|_2 + \|sy+(1-s)z\|_2 \le 2.$$

It follows that $\|(1-s)y+sz\|_2 = 1$. In particular, observe that $\|y\|_2 = \|z\|_2 = 1$. Now,

$$\begin{aligned} g_2^{\pm}(y,z) &= \lim_{t\to 0\pm} \tfrac{1}{t}(\|y+tz\|_2 - 1) \\ &= \lim_{s\to 0\pm} \tfrac{1-s}{s}(\|y + \tfrac{s}{1-s}z\|_2 - 1) \\ &= \lim_{s\to 0\pm} \tfrac{1}{s}(\|(1-s)y+sz\| - 1 + s) = 1. \end{aligned}$$

This shows that $g_2(y,z) = 1$, $\cos(\theta_2(y,z)) = 1$, and $\tan(\theta_2(y,z)/2) = 0$. By angular equivalence, $\tan(\theta_1(y,z)/2) = 0$ as well. This implies $\cos(\theta_1(y,z)) = 1$ and hence $g_1(y,z) = \|y\|_1\|z\|_1$. The last statement, which may be written as

$$g_1^-(y,z) + g_1^+(y,z) = 2\|y\|_1\|z\|_1,$$

combined with

$$g_1^-(y,z) \le g_1^+(y,z) \le \|y\|_1(\|y+z\|_1 - \|y\|_1) \le \|y\|_1\|z\|_1,$$

from (1.2), gives

$$\|y\|_1(\|y+z\|_1 - \|y\|_1) = \|y\|_1\|z\|_1.$$

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Since $\|y+z\|_1 = \|y\|_1 + \|z\|_1$ and $x/\|x\|_2 = (y+z)/2$, we have

$$\frac{x}{\|x\|_1} = \frac{y+z}{\|y+z\|_1} = \frac{\|y\|_1}{\|y\|_1 + \|z\|_1}\frac{y}{\|y\|_1} + \frac{\|z\|_1}{\|y\|_1 + \|z\|_1}\frac{z}{\|z\|_1},$$

which is a convex combination of the points $y/\|y\|_1$ and $z/\|z\|_1$. Thus, $x/\|x\|_1$ is an interior point of the line segment from $y/\|y\|_1$ to $z/\|z\|_1$. Since the endpoints of this segment lie in the $\|\cdot\|_1$-unit ball, convexity shows that the entire line segment does. Thus, $x/\|x\|_1$ is not an extreme point of the $\|\cdot\|_1$-unit ball.

Reversing the roles of the two norms gives the other implication.

□

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