Fourier inequalities and a new Lorentz space

Gord Sinnamon

Abstract. A scale of weighted Lorentz spaces is introduced. These spaces lie between the classical Λ-spaces and the more recent Γ-spaces. The norm of these new spaces is used to give a simple expression for a necessary and sufficient condition that characterizes the weights for which the Fourier transform is bounded as a map between weighted Lorentz Γ-spaces.

1 Introduction

The Lorentz spaces $\Lambda_p(w)$ were studied by G. G. Lorentz [5] in 1951. Here $1 \le p < \infty$ and w is a non-negative, decreasing function on $(0, \infty)$. Advances in the study of the Hardy averaging operator, notably in [1], lead to the consideration, in [6], of the Lorentz spaces $\Gamma_p(w)$ as a substitute for $\Lambda_p(w)$ in the case that w is not decreasing. Both of these spaces are essential for the study of monotone functions and arise naturally in a wide variety of situations. In particular, they are natural spaces on which to consider the boundedness of the Fourier transform and other operators in signal processing because amplitude-based thresholding of signals can be understood in terms of projections onto appropriately weighted Lorentz spaces.

Investigations into the boundedness of the Fourier transform between Lorentz Γ-spaces in [8] gave a weight condition that is not properly described by either a Λ-norm or a Γ-norm but by something in between the two. In this paper we introduce a new class of Lorentz spaces that lie between the Λ- and Γ-spaces. They provide a closer substitute for Λ when w fails to be decreasing and they can be used to express a weight condition that characterizes the boundedness of the Fourier transform between a Lorentz Γ-spaces for a range of indices.

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In the next section we define the Lorentz Λ- and Γ-spaces and introduce our new spaces, the Θ-spaces. The following section contains an extension of the Fourier inequalities of [8], reformulated in terms of the Lorentz Θ-norms.

2 The Lorentz Θ-spaces

Let (X, μ) be a σ-finite measure space and let L^1_μ and L^∞_μ be the usual spaces of integrable and bounded μ-measurable functions, respectively. For $f \in L^1_\mu + L^\infty_\mu$, the non-increasing rearrangement of f (see [3]) is a Lebesgue measurable function on $(0, \infty)$, and so is

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*.$$

We suppose $1 \leq p < \infty$ and w is a non-negative weight on $(0, \infty)$; but exclude the trivial case in which w is almost everywhere zero. Define the weighted Lebesgue spaces $L^p(w)$ by their norms,

$$\|h\|_{L^p(w)} = \left(\int_0^\infty |h|^p w \right)^{1/p},$$

and define

$$\|f\|_{\Lambda_p(w)} = \|f^*\|_{L^p(w)} \quad \text{and} \quad \|f\|_{\Gamma_p(w)} = \|f^{**}\|_{L^p(w)}.$$

Although it is not true, in general, that $(f+g)^* \leq f^* + g^*$, it is well known that if w is non-increasing, then

$$\|f+g\|_{\Lambda_p(w)} \leq \|f\|_{\Lambda_p(w)} + \|g\|_{\Lambda_p(w)}.$$

It follows that the Lorentz Λ-space,

$$\Lambda_p(w) \equiv \{ f \in L^1_\mu + L^\infty_\mu : \|f\|_{\Lambda_p(w)} < \infty \}$$

is a Banach function space with norm $\|\cdot\|_{\Lambda_p(w)}$, provided w is non-increasing. These expressions are still defined when w is not monotone, although $\|\cdot\|_{\Lambda_p(w)}$ will not be a norm and, under some conditions on w, $\Lambda_p(w)$ may not even be a vector space. (See [4] for details.)

On the other hand, $(f+g)^{**} \leq f^{**} + g^{**}$ does hold in general so, even if w is not monotone, the Lorentz Γ-space,

$$\Gamma_p(w) \equiv \{f \in L^1_\mu + L^\infty_\mu : \|f\|_{\Gamma_p(w)} < \infty\}$$

is a Banach function space with norm $\|\cdot\|_{\Gamma_p(w)}$.

Since $f^* \leq f^{**}$ we have $\Gamma_p(w) \subset \Lambda_p(w)$ for any p and w. It was shown in [6] that the two spaces are equal, with equivalent norms, whenever the expression $\|\cdot\|_{\Lambda_p(w)}$ is equivalent to a norm. For this reason it is natural to take $\Gamma_p(w)$ as a substitute for $\Lambda_p(w)$ in the case that w is not monotone.

Definition 1 *If $1 \leq p < \infty$ and w is a non-negative weight on $(0,\infty)$, set*

$$\Theta_p(w) = \{f \in L^1_\mu + L^\infty_\mu : \|f\|_{\Theta_p(w)} < \infty\},$$

where

$$\|f\|_{\Theta_p(w)} = \sup_{h^{**} \leq f^{**}} \|h^*\|_{L^p(w)}.$$

In the supremum, $h \in L^1 + L^\infty$ is a Lebesgue measurable function defined on $(0,\infty)$, not a μ-measurable function defined on X.

The main difficulty in showing that $\|\cdot\|_{\Theta_p(w)}$ is a norm is proving the triangle inequality. The next lemma provides the key to this. This result is essentially a special case of K-divisibility from the theory of interpolation spaces. The direct proof below avoids much of the technical detail of the general result.

Lemma 2 *If $h^{**} \leq f_1^{**} + f_2^{**}$ then there exist h_1 and h_2 defined on $(0,\infty)$ such that $h_1^{**} \leq f_1^{**}$, $h_2^{**} \leq f_2^{**}$ and $h_1^* + h_2^* = h^*$.*

Proof. Let S be the collection of non-negative, non-decreasing, concave functions on $(0,\infty)$. Then F_1, F_2 and H are all in S, where

$$F_1(t) = \int_0^t f_1^*, \quad F_2(t) = \int_0^t f_2^*, \quad \text{and} \quad H(t) = \int_0^t h^*.$$

Moreover, $H \leq F_1 + F_2$. We use Zorn's lemma to show that there exist $H_1, H_2 \in S$ such that $H_1 \leq F_1$, $H_2 \leq F_2$ and $H_1 + H_2 = H$. Define

$T = \{(A, B) \in S \times S : A \le F_1, B \le F_2, A + B \ge H\}$ and note that T is non-empty because $(F_1, F_2) \in T$. The partial order $(A, B) \le (A', B')$ on T is just $A \le A'$ and $B \le B'$. Suppose that $\{(A_j, B_j) : j \in J\}$ is a non-empty, totally ordered subset of T and define A and B by $A(t) = \inf_{j\in J} A_j(t)$ and $B(t) = \inf_{j\in J} B_j(t)$. It is evident that A and B are non-negative and non-decreasing and that (A, B) is a lower bound for the subset. To see that $(A, B) \in T$ we check that A and B are concave and that $A + B \ge H$. If $x < y < z$ with $y = (1 - \theta)x + \theta z$ then

$$\begin{aligned} A(y) = \inf_{j\in J} A_j(y) &\ge \inf_{j\in J}[(1-\theta)A_j(x) + \theta A_j(z)] \\ &\ge (1-\theta)\inf_{j\in J} A_j(t) + \theta \inf_{j\in J} A_j(z) \\ &= (1-\theta)A(x) + \theta A(z). \end{aligned}$$

Thus $A \in S$ and similarly $B \in S$. Also, if $j, k \in J$ then because the collection is totally ordered, either $(A_j, B_j) \le (A_k, B_k)$ or the reverse. Therefore, for each t,

$$A_j(t) + B_k(t) \ge \min(A_j(t) + B_j(t), A_k(t) + B_k(t)) \ge H(t)$$

and taking the infimum over all j and k we have $A + B \ge H$ so $(A, B) \in T$.

The hypotheses for Zorn's lemma are satisfied and it follows that T has a minimal element, call it (H_1, H_2).

Since $(H_1, H_2) \in T$ we have $H_1 + H_2 \ge H$. If $H_1 + H_2 \ne H$ then there exists an $x \in (0, \infty)$ such that $H_1(x) + H_2(x) > H(x)$. Let $\ell = \ell(t)$ be a tangent line to the concave function H at x. Then $H \le \ell$ and $H(x) = \ell(x)$. Let I be the interval $\{y > 0 : \ell(y) < H_1(y) + H_2(y)\}$. Define $K_1 \le H_1$ to be the unique function in S that agrees with H_1 off I and is a straight line on I. Define K_2 similarly and note that $K_1 + K_2 = \min(\ell, H_1 + H_2) \ge H$ so $(K_1, K_2) \in T$. Since $K_1(x) + K_2(x) \le \ell(x) = H(x) < H_1(x) + H_2(x)$ this contradicts the minimality of (H_1, H_2). We conclude that $H_1 + H_2 = H$.

Since functions in S are the integrals of their derivatives, which exist almost everywhere and are non-increasing, we set $h_1 = H_1'$ and $h_2 = H_2'$. Now $h_1^* = h_1$ and $h_2^* = h_2$ almost everywhere and hence $h_1^* + h_2^* = h^*$, the derivative of H, almost everywhere. It follows from the right continuity of the rearrangement that $h_1^* + h_2^* = h^*$ everywhere.

Theorem 3 *If $1 \leq p < \infty$ and w is non-negative then $\Theta_p(w)$ is a rearrangement-invariant normed function space whose norm satisfies*

$$\|f\|_{\Lambda_p(w)} \leq \|f\|_{\Theta_p(w)} \leq \|f\|_{\Gamma_p(w)} \tag{1}$$

for all $f \in L^1_\mu + L^\infty_\mu$.

Proof. It is clear that $\|\cdot\|_{\Theta_p(w)}$ is non-negative and is zero when f vanishes almost everywhere. Rearrangement-invariance and homogeneity for positive constants are also easily checked. The first inequality of (1) follows by taking $h = f^*$ in the definition of $\Theta_p(w)$, and the second,by observing that if $h^{**} \leq f^{**}$ then $h^* \leq h^{**} \leq f^{**}$.

If $\|f\|_{\Theta_p(w)} = 0$ and $0 < s < t$ it is easy to check that $(s/t)\chi^{**}_{(0,t)} \leq \chi^{**}_{(0,s)}$ and $\chi^*_{(0,s)} = \chi_{(0,s)} \leq f^*/f^*(s)$. It follows that $(sf^*(s)/t)\chi^{**}_{(0,t)} \leq f^{**}$ and hence $(sf^*(s)/t)\|\chi_{(0,t)}\|_{L^p(w)} = 0$. By assumption, w is not almost everywhere zero so for sufficiently large t, $\|\chi_{(0,t)}\|_{L^p(w)} > 0$. We conclude that $f^*(s) = 0$. Since s was arbitrary we see that f is μ-almost everywhere zero.

It remains to verify the triangle inequality. Fix $f_1, f_2 \in \Theta_p(w)$ and suppose that $h^{**} \leq (f_1 + f_2)^{**}$ for some $h \in L^1 + L^\infty$. Then $h^{**} \leq f_1^{**} + f_2^{**}$ so we may take the h_1 and h_2 guaranteed by Lemma 2 and observe that, since $h^* = h_1^* + h_2^*$,

$$\|h^*\|_{L^p(w)} \leq \|h_1^*\|_{L^p(w)} + \|h_2^*\|_{L^p(w)} \leq \|f_1\|_{\Theta_p(w)} + \|f_2\|_{\Theta_p(w)}.$$

Taking the supremum over all such h proves that

$$\|f_1 + f_2\|_{\Theta_p(w)} \leq \|f_1\|_{\Theta_p(w)} + \|f_2\|_{\Theta_p(w)}$$

and completes the proof.

The completeness of $\Theta_p(w)$ is not needed for what follows and we have not established it here. The interested reader may wish to extend the argument of Lemma 2 from two functions, f_1 and f_2, to a sequence, $f_1, f_2, \ldots$. With the extension in place, the proof of completeness follows along standard lines, showing that $\Theta_p(w)$ is a Banach function space.

3 Fourier Inequalities

The behaviour of the Fourier transform as a map between unweighted L^p spaces on $\mathbf{R}^n$ is well understood and substantial progress has been made in the case of weighted L^p in which the weight possesses some (radial) monotonicity property. For general weights, however, it remains a difficult question to find a simple weight condition that determines whether or not the Fourier transform is bounded as a map from one weighted L^p space to another.

In the scale of Lorentz spaces the situation is somewhat better. See [2] for Fourier inequalities between weighted Λ-spaces. In [8], necessary conditions and sufficient conditions are given for the boundedness of the Fourier transform from $\Gamma_p(u) \to \Gamma_q(w)$ when $p < q$. Moreover, when $0 < p \le q = 2$ the conditions coincide and reduce to a readily verifiable integral condition. In this section we show that the necessary conditions and sufficient conditions of [8] coincide for $0 < p \le 2 \le q < \infty$. The resulting condition does not simplify in the way that it did in the case $q = 2$, however. To express this new condition simply we use the Lorentz Θ-space norms just introduced.

We begin with a lemma stated in a somewhat more general setting than strictly necessary.

Lemma 4 *Let $0 < p \le 1 \le q < \infty$. Suppose (Y, μ), (X, ν), and (T, λ) are σ-finite measure spaces, $k(x,t)$ is a non-negative $\nu \times \lambda$-measurable function, and $a(y,x)$ is a non-negative $\mu \times \nu$-measurable function. Define K and A by,*

$$Kh(x) = \int_T k(x,t)h(t)\,d\lambda(t) \quad \textit{and} \quad Ag(y) = \int_X a(y,x)g(x)\,d\nu(x).$$

If k_t is defined by $k_t(x) = k(x,t)$, then

$$\sup_{h \ge 0} \frac{\|AKh\|_{L^q_\mu}}{\|Kh\|_{L^p_\nu}} \le \operatorname*{ess\,sup}_{t \in T} \frac{\|Ak_t\|_{L^q_\mu}}{\|k_t\|_{L^p_\nu}}.$$

Proof. Let

$$C = \operatorname*{ess\,sup}_{t \in T} \frac{\|Ak_t\|_{L^q_\mu}}{\|k_t\|_{L^p_\nu}}$$

so that

$$\left(\int_Y \left(\int_X a(y,x)k(x,t)\,d\nu(x)\right)^q d\mu(u)\right)^{1/q} \le C\left(\int_X k(x,t)^p\,d\nu(x)\right)^{1/p}$$

for λ-almost every $t \in T$.

Two applications of Minkowski's integral inequality finish the proof.

$$\begin{aligned}
\|AKh\|_{L^q_\mu} &= \left(\int_Y \left(\int_T\int_X a(y,x)k(x,t)\,d\nu(x)h(t)\,d\lambda(t)\right)^q d\mu(y)\right)^{1/q}\\
&\le \int_T \left(\int_Y \left(\int_X a(y,x)k(x,t)\,d\nu(x)\right)^q d\mu(y)\right)^{1/q} h(t)\,d\lambda(t)\\
&\le C\int_T \left(\int_X k(x,t)^p\,d\nu(x)\right)^{1/p} h(t)\,d\lambda(t)\\
&\le C\left(\int_X \left(\int_T k(x,t)h(t)\,d\lambda(t)\right)^p d\nu(x)\right)^{1/p}\\
&= C\|Kh\|_{L^p_\nu}.
\end{aligned}$$

Diving by $\|Kh\|_{L^p_\nu}$ and taking the supremum over all non-negative h yields the desired result.

Although the lemma applies quite generally, we will need it only in the case that the operator K is given by

$$Kh(z) = \int_0^\infty \omega_z(t)h(t)\,dt.$$

Here and throughout, the function ω_z is defined by

$$\omega_z(t) = \min(z^{-2}, t^{-2}).$$

Notice that for each z, $\omega_z(t)$ is non-increasing, and $t^2\omega_z(t)$ is non-decreasing. A non-negative function on $(0,\infty)$ that satisfies these two monotonicity conditions is said to be in $\Omega_{2,0}$. The next lemma shows that every function in $\Omega_{2,0}$ can be approximated from below by images of non-negative functions under the operator K.

Lemma 5 *If $f \in \Omega_{2,0}$ then there exists a non-negative function $\tilde f$ such that $\frac{1}{2}\tilde f \le f \le \tilde f$ and $\tilde f$ is the pointwise limit of an increasing sequence of functions of the form Kh for $h \ge 0$.*

Proof. Define g by $g(z) = zf(z^{1/2})$ and observe that g is quasiconcave. That is, $g(z)$ is non-decreasing and $g(z)/z$ is non-increasing. According to Lemma 2.3 of [7], $\tilde{g}$, the least concave majorant of g, satisfies $\frac{1}{2}\tilde{g} \leq g \leq \tilde{g}$ and is the pointwise limit of an increasing sequence of functions of the form

$$\int_0^\infty \min(1, z/t)\bar{h}(t)\,dt$$

for some non-negative functions $\bar{h}$. Define $\tilde{f}$ by $\tilde{f}(z) = z^{-2}\tilde{g}(z^2)$ to obtain $\frac{1}{2}\tilde{f} \leq f \leq \tilde{f}$. Also, this $\tilde{f}$ is the pointwise limit of an increasing sequence of functions of the form

$$z^{-2}\int_0^\infty \min(1, z^2/t)\bar{h}(t)\,dt = Kh(z),$$

where $h(t) = 2t\bar{h}(t^2)$. This completes the proof.

Combining these two lemmas with results from [8] yields a weight condition that is sufficient to ensure that every operator of type $(1,\infty)$ and $(2,2)$ is bounded between certain Lorentz spaces. A sublinear operator T is said to be of type (p,q) provided T is a bounded map from L^p_μ to L^q_μ. From now on we take the measure μ to be Lebesgue measure on $\mathbb{R}^n$ for some $n \geq 1$ and write L^p_μ as $L^p(\mathbb{R}^n)$.

In the next three theorems, the weights v and w will be related by $v(t) = t^{p-2}w(1/t)$. Consequently, (3), below, may be viewed as a condition on the weights u and w, rather than on u and v. It should be understood in this sense in the statements and proofs of Theorems 6, 7, and 8.

Theorem 6 *Let $0 < p \leq 2 \leq q < \infty$ and u, v and w be non-negative functions on $(0,\infty)$ such that $v(t) = t^{p-2}w(1/t)$. If T is a sublinear operator of type $(1,\infty)$ and $(2,2)$ then there exists a $C > 0$ such that*

$$\|Tf\|_{\Lambda_q(u)} \leq C\|f\|_{\Gamma_p(w)} \tag{2}$$

for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ whenever

$$\sup_{z>0} \frac{\|\omega_z\|_{\Theta_{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}} < \infty. \tag{3}$$

Proof. Theorem 3.1 of [8] shows that (2) holds whenever

$$\sup_{A\in\mathcal{A}} \sup_{f\in\Omega_{2,0}} \frac{\|Af\|_{L^{q/2}(u)}}{\|f\|_{L^{p/2}(v)}} < \infty.$$

The set of operators $\mathcal{A}$ is defined in [8]. The only property we need is that each $A \in \mathcal{A}$ is an integral operator with non-negative kernel. For each $A \in \mathcal{A}$ we may apply Lemma 5 and then Lemma 4 (with p and q replaced by $p/2$ and $q/2$) to get

$$\sup_{f\in\Omega_{2,0}} \frac{\|Af\|_{L^{q/2}(u)}}{\|f\|_{L^{p/2}(v)}} \approx \sup_{h\geq 0} \frac{\|AKh\|_{L^{q/2}(u)}}{\|Kh\|_{L^{p/2}(v)}} = \sup_{z>0} \frac{\|A\omega_z\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}}.$$

Taking the supremum over all $A \in \mathcal{A}$ and using Corollary 2.4 of [8] gives us

$$\sup_{A\in\mathcal{A}} \sup_{f\in\Omega_{2,0}} \frac{\|Af\|_{L^{q/2}(u)}}{\|f\|_{L^{p/2}(v)}} = \sup_{z>0} \sup_{h^{**}\leq\omega_z^{**}} \frac{\|h^*\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}} = \sup_{z>0} \frac{\|\omega_z\|_{\Theta_{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}}$$

This completes the proof.

In the case of the Fourier transform on $\mathbb{R}^n$, denoted $\mathcal{F}$, the necessary condition given in [8] agrees with the sufficient condition just established.

Theorem 7 *Let* $0 < p \leq 2 \leq q < \infty$ *and* u, v *and* w *be non-negative functions on* $(0,\infty)$ *such that* $v(t) = t^{p-2}w(1/t)$. *If there exists a constant* C *such that*

$$\|\mathcal{F}f\|_{\Lambda_q(u)} \leq C\|f\|_{\Gamma_p(w)} \tag{4}$$

for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, *then* (3) *holds.*

Proof. Corollary 4.8 of [8] shows that if (4) holds, then

$$\sup_{z>0} \sup_{A\in\mathcal{A}} \frac{\|A\omega_z\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}}$$

is finite. But the definition of the Lorentz Θ-norm and Corollary 2.4 of [8] show that this supremum is just

$$\sup_{z>0} \sup_{h^{**}\leq\omega_z^{**}} \frac{\|h^*\|_{L^{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}} = \sup_{z>0} \frac{\|h^*\|_{\Theta_{q/2}(u)}}{\|\omega_z\|_{L^{p/2}(v)}}.$$

This completes the proof.

Theorem 8 *Let $0 < p \leq 2 \leq q < \infty$ and u, v and w be non-negative functions on $(0,\infty)$ such that $v(t) = t^{p-2}w(1/t)$. Let $\mathcal{T}$ denote the collection of all sublinear operators of type $(1,\infty)$ and $(2,2)$. Each of the following is equivalent to* (3).

$$\mathcal{F} : \Gamma_p(w) \to \Lambda_q(u) \text{ is bounded.} \tag{5}$$

$$T : \Gamma_p(w) \to \Lambda_q(u) \text{ is bounded for all } T \in \mathcal{T}. \tag{6}$$

$$\mathcal{F} : \Gamma_p(w) \to \Gamma_q(u) \text{ is bounded.} \tag{7}$$

$$T : \Gamma_p(w) \to \Gamma_q(u) \text{ is bounded for all } T \in \mathcal{T}. \tag{8}$$

$$\mathcal{F} : \Gamma_p(w) \to \Theta_q(u) \text{ is bounded.} \tag{9}$$

$$T : \Gamma_p(w) \to \Theta_q(u) \text{ is bounded for all } T \in \mathcal{T}. \tag{10}$$

Proof. Since the Fourier transform is of type $(1,\infty)$ and $(2,2)$, (6) implies (5) and (8) implies (7). Theorem 7 shows that (5) implies (3) and Theorem 6 shows that (3) implies (6).

Inequality (1) shows that (7) implies (9) implies (5) and (8) implies (10) implies (6). To complete the proof we show that (6) implies (8). The sublinear operator $f \mapsto f^{**}$ is of type (∞,∞) and $(2,2)$ (by Hardy's inequality) so for any $T \in \mathcal{T}$ the composition $f \mapsto (Tf)^{**}$ is also in $\mathcal{T}$. Applying (6) to this operator yields (8) for the operator T. This completes the proof.

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Gord Sinnamon
Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, Canada
Email: **sinnamon@uwo.ca**