BLEI'S INEQUALITY AND COORDINATEWISE MULTIPLE SUMMING OPERATORS

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Abstract. Two inequalities resembling the multilinear Hölder inequality for mixed-norm Lebesgue spaces are proved. When specialized to single-function inequalities they include a pair of inequalities due to Blei and a recent extension of Blei's inequality. The first of these inequalities is applied to give explicit indices in a known result for coordinatewise multiple summing operators. The second is used to prove a complementary result to the known one, again with explicit indices. As an application of the complementary result, a sufficient condition is given for a composition of operators to be multiple summing.

1. Introduction

A mixed-norm Lebesgue space is a space of complex-valued $\mu \times \nu$-measurable functions defined on the product of two measure spaces (X, μ) and (Y, ν) and satisfying

$$\left(\int \left(\int |f(x,y)|^p \, d\mu(x) \right)^{q/p} d\nu(y) \right)^{1/q} < \infty,$$

for given indices $p \geq 1$ and $q \geq 1$. These spaces and the closely related amalgam spaces have a prominent place in harmonic analysis. For example, the Littlewood 4/3 theorem is proved in [1] using two mixed-norm inequalities for matrices. (To get mixed norms on matrices simply take μ and ν above to be counting measures on finite sets.) The multilinear Hölder inequality for mixed-norm spaces follows easily by iterating the usual one, so

$$\int f_1 f_2, \dots f_n \, d(\mu \times \nu) \leq \prod_{j=1}^{n} \left(\int \left(\int |f_j|^{p_j} \, d\mu \right)^{q_j/p_j} d\nu \right)^{1/q_j}$$

provided $1/p_1 + \cdots + 1/p_n = 1/q_1 + \cdots + 1/q_n = 1$. It is important to note that the order of integration is the same in each factor. In [1], Lemma 2 on Page 430, two mixed-norm inequalities appear in which the order of integration differs in the factors. They are expressed as matrix inequalities:

$$\left(\sum_{i,j} |b_{i\,j}|^{4/3} \right)^{3/4} \leq \left(\sum_i \left(\sum_j |b_{i\,j}|^2 \right)^{1/2} \right)^{1/2} \left(\sum_j \left(\sum_i |b_{i\,j}|^2 \right)^{1/2} \right)^{1/2} \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 26D15, Secondary 47B10, 47H60.

Key words and phrases. Blei's inequality, mixed-norm space, multiple summing operator, Bohnenblust-Hille inequality.

Supported by the Natural Sciences and Engineering Research Council of Canada.

and

$$\begin{aligned}\left(\sum_{i,j,k}|a_{i\,j\,k}|^{6/5}\right)^{5/6} &\le \left(\sum_{i,j}\left(\sum_{k}|a_{i\,j\,k}|^{2}\right)^{1/2}\right)^{1/3} \\ &\quad\times\left(\sum_{i,k}\left(\sum_{j}|a_{i\,j\,k}|^{2}\right)^{1/2}\right)^{1/3}\left(\sum_{j,k}\left(\sum_{i}|a_{i\,j\,k}|^{2}\right)^{1/2}\right)^{1/3}.\end{aligned}\tag{1.2}$$

In Theorem 2.1 we present a pair of multilinear Hölder-type inequalities in which the order of integration differs in the mixed-norm factors. When specialized to single function inequalities, they include the two inequalities above, and a recent generalization from [8]. It appears that investigation of such inequalities in the past has been mostly restricted to the single-function case, see [2], [3], [11] and [14]. In [11], the author introduces permuted mixed norms and proves a Minkowski-type inequality for them. Although still a single-function result, this inequality is applicable to our situation and may be used to give alternative proofs of our Theorem 2.1. We prefer to present the concrete, elementary proof given in the next section.

The motivation for extending Blei's inequalities from [1] comes from the theory of multiple summing operators, which began with the comparison between unconditional and absolute convergence in Banach spaces and developed into an essential tool of functional analysis. Bohnenblust and Hille, in Theorem I of their ingenious 1931 paper [4], proved that for each natural number m there exists a constant BH_m such that for every N and every m-linear mapping $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$

$$\left(\sum_{i_1=1}^{N}\cdots\sum_{i_m=1}^{N}|U(e_{i_1},\ldots,e_{i_m})|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \le BH_m\|U\|,$$

and, moreover, proved that the exponent $\frac{2m}{m+1}$ is optimal. Here $e_1,\ldots,e_N$ denote the standard basis vectors in ℓ_∞^N. The case $m = 2$ is Littlewood's famous 4/3-inequality from [12] and is closely connected with (1.1). In modern terminology, see [15, Corollary 3.20], the Bohnenblust-Hille theorem may be stated as follows: For each natural number m there exists a constant BH_m such that if $X_1,\ldots,X_m$ are Banach spaces and $\varphi : X_1 \times \cdots \times X_m \to \mathbb{C}$ is bounded and multilinear, then φ is multiple $(\frac{2m}{m+1}, 1)$-summing, and

$$\pi^{\mathrm{mult}}_{\frac{2m}{m+1},1}(\varphi) \le BH_m\|\varphi\|.$$

For definitions and basic results, including the definition of $\pi^{\mathrm{mult}}_{r,1}$, see Section 3.

In [8], coordinatewise multiple summing operators were introduced and studied, then applied to give a multilinear extension of Kwapień's theorem, a multivariate polynomial version of the same result, and a theorem on products of vector-valued Dirichlet series. Their main result on coordinatewise multiple summing operators, Theorem 5.1 of [8], shows that if an operator is coordinatewise multiple summing in each subset of some partition of the coordinate set, then the operator is multiple summing. Unfortunately, the indices in this result are recursively defined, making them difficult to handle except in special cases. In Theorem 3.2, below, we prove a version of Theorem 5.1 from [8], giving explicit values for the indices, and simplifying its proof by applying Theorem 2.1. The simplification comes at the expense of the careful control of the constants established in [8]. Theorem 3.2 also includes a companion result, which involves operators that are coordinatewise multiple summing in the complement of each subset of the partition. As an application of the companion result, it is combined with the Bohnenblust-Hille theorem to give a sufficient condition for a composition of operators to be multiple summing, see Theorem 3.5.

2. Multilinear Blei's inequalities

Let (M_j, μ_j) be σ-finite measure spaces for $j = 1, 2, \ldots, n$, and introduce the product measure spaces (M^n, μ^n) and (M_j^n, μ_j^n) by

$$M^n = \prod_{k=1}^n M_k, \quad \mu^n = \prod_{k=1}^n \mu_k, \quad M_j^n = \prod_{\substack{k=1\\k\neq j}}^n M_k, \quad \mu_j^n = \prod_{\substack{k=1\\k\neq j}}^n \mu_k.$$

Note that $M_n^n = M^{n-1}$.

The following two theorems give complementary inequalities for functions defined on the product space M^n. Observe that, except for the names of the indices, each reduces to the same inequality in the case $n = 2$. This case is proved separately below. Note that for $p > 1$, p' is defined by $1/p + 1/p' = 1$.

Theorem 2.1. *If $n \geq 2$ and positive indices $q_1, \ldots, q_n$ satisfy $\sum_{j=1}^n \frac{1}{q_j} \leq 1$ then for any non-negative μ^n-measurable functions $f_1, f_2, \ldots, f_n$,*

$$\int_{M^n} f_1 f_2 \ldots f_n \, d\mu^n \leq \prod_{j=1}^n \left(\int_{M_j} \left(\int_{M_j^n} f_j^{q_j} \, d\mu_j^n \right)^{p_j/q_j} d\mu_j \right)^{1/p_j} \quad \text{and} \tag{2.1}$$

$$\int_{M^n} f_1 f_2 \ldots f_n \, d\mu^n \leq \prod_{j=1}^n \left(\int_{M_j^n} \left(\int_{M_j} f_j^{q_j} \, d\mu_j \right)^{s_j/q_j} d\mu_j^n \right)^{1/s_j}. \tag{2.2}$$

Here $\frac{1}{p_j} = \frac{1}{q_j} + 1 - \sum_{k=1}^n \frac{1}{q_k}$ and $\frac{1}{s_j} = \frac{1}{q_j} + \frac{1}{n-1}\left(1 - \sum_{k=1}^n \frac{1}{q_k}\right)$.

Proof. If $n = 2$ then $M_1^2 = M_2$, $M_2^2 = M_1$, $p_1 = s_1 = q_2'$ and $p_2 = s_2 = q_1'$. Two applications of Hölder's inequality give

$$\begin{aligned}
&\int_{M^2} f_1 f_2 \, d\mu^2 = \int_{M_1} \int_{M_2} f_1 f_2 \, d\mu_2 \, d\mu_1 \\
&\leq \int_{M_1} \left(\int_{M_2} f_1^{q_1} \, d\mu_2 \right)^{1/q_1} \left(\int_{M_2} f_2^{q_1'} \, d\mu_2 \right)^{1/q_1'} d\mu_1 \\
&\leq \left(\int_{M_1} \left(\int_{M_2} f_1^{q_1} \, d\mu_2 \right)^{q_2'/q_1} d\mu_1 \right)^{1/q_2'} \left(\int_{M_1} \left(\int_{M_2} f_2^{q_1'} \, d\mu_2 \right)^{q_2/q_1'} d\mu_1 \right)^{1/q_2}.
\end{aligned}$$

Since $q_2/q_1' \geq 1$, Minkowski's integral inequality shows that the second factor in the last expression is no greater than

$$\left(\int_{M_2} \left(\int_{M_1} f_2^{q_2} \, d\mu_1 \right)^{q_1'/q_2} d\mu_2 \right)^{1/q_1'}$$

and establishes the case $n = 2$ of both (2.1) and (2.2).

Next we prove the remaining cases of (2.1) by induction on n. First observe that $1 < p_j \leq q_j < \infty$ for each j. For the induction step we suppose $n \geq 3$ and deduce the result from the case $n-1$. Fix $q_1,\dots,q_n$ such that $\sum_{j=1}^n \frac{1}{q_j}\le 1$ and set $Q_j = q_j/q_n'$ for $j=1,2,\dots,n-1$. Observe that

$$\sum_{j=1}^{n-1}\frac{1}{Q_j} = q_n'\sum_{j=1}^{n-1}\frac{1}{q_j}\le q_n'\left(1-\frac{1}{q_n}\right)=1.$$

Thus, by the inductive hypothesis,

$$\int_{M^{n-1}} F_1F_2\dots F_{n-1}\,d\mu^{n-1}\le \prod_{j=1}^{n-1}\left(\int_{M_j}\left(\int_{M_j^{n-1}} F_j^{Q_j}\,d\mu_j^{n-1}\right)^{P_j/Q_j} d\mu_j\right)^{1/P_j},$$

for non-negative μ^{n-1}-measurable functions $F_1,\dots,F_{n-1}$, where $P_j = p_j/q_n'$ because,

$$1-\frac{1}{P_j}=\sum_{\substack{k=1\\k\neq j}}^{n-1}\frac{1}{Q_k}=q_n'\sum_{\substack{k=1\\k\neq j}}^{n-1}\frac{1}{q_k}=q_n'\left(1-\frac{1}{p_j}-\frac{1}{q_n}\right)=1-\frac{q_n'}{p_j}.$$

We apply this inequality to the functions $F_j = f_j^{q_n'}$, with the nth variable of $f_1,\dots,f_{n-1}$ fixed, to get,

$$\left(\int_{M^{n-1}}(f_1f_2\dots f_{n-1})^{q_n'}\,d\mu^{n-1}\right)^{1/q_n'}$$
$$\le\prod_{j=1}^{n-1}\left(\int_{M_j}\left(\int_{M_j^{n-1}} f_j^{q_j}\,d\mu_j^{n-1}\right)^{p_j/q_j} d\mu_j\right)^{1/p_j}.$$

For convenience, set

$$C=\left(\int_{M_n}\left(\int_{M^{n-1}} f_n^{q_n}\,d\mu_n^{n}\right)^{p_n/q_n} d\mu_n\right)^{1/p_n}.$$

Then Hölder's inequality, used twice, and the inequality above yield,

$$\begin{aligned}
\int_{M^n} f_1f_2\dots f_n\,d\mu^n &= \int_{M_n}\int_{M^{n-1}} f_1f_2\dots f_n\,d\mu^{n-1}\,d\mu_n\\
&\le \int_{M_n}\left(\int_{M^{n-1}} f_n^{q_n}\,d\mu^{n-1}\right)^{1/q_n}\left(\int_{M^{n-1}}(f_1f_2\dots f_{n-1})^{q_n'}\,d\mu^{n-1}\right)^{1/q_n'} d\mu_n\\
&\le C\left(\int_{M_n}\left(\int_{M^{n-1}}(f_1f_2\dots f_{n-1})^{q_n'}\,d\mu^{n-1}\right)^{p_n'/q_n'} d\mu_n\right)^{1/p_n'}\\
&\le C\left(\int_{M_n}\prod_{j=1}^{n-1}\left(\int_{M_j}\left(\int_{M_j^{n-1}} f_j^{q_j}\,d\mu_j^{n-1}\right)^{p_j/q_j} d\mu_j\right)^{p_n'/p_j} d\mu_n\right)^{1/p_n'}.
\end{aligned}$$

Since $\sum_{j=1}^{n-1}\frac{p_n'}{q_j}=1$, Hölder's inequality with indices $q_1/p_n',\dots,q_{n-1}/p_n'$ implies,

$$\int_{M^n} f_1f_2\dots f_n\,d\mu^n$$
$$\le C\prod_{j=1}^{n-1}\left(\int_{M_n}\left(\int_{M_j}\left(\int_{M_j^{n-1}} f_j^{q_j}\,d\mu_j^{n-1}\right)^{p_j/q_j} d\mu_j\right)^{q_j/p_j} d\mu_n\right)^{1/q_j},$$

and since $q_j/p_j \geq 1$, Minkowski's integral inequality gives,

$$\int_{M^n} f_1 f_2 \dots f_n \, d\mu^n \leq C \prod_{j=1}^{n-1} \left(\int_{M_j} \left(\int_{M_n} \int_{M_j^{n-1}} f_j^{q_j} \, d\mu_j^{n-1} \, d\mu_n \right)^{p_j/q_j} d\mu_j \right)^{1/p_j}$$
$$= \prod_{j=1}^{n} \left(\int_{M_j} \left(\int_{M_j^n} f_j^{q_j} \, d\mu_j^n \right)^{p_j/q_j} d\mu_j \right)^{1/p_j}$$

(Note that $M^{n-1} = M_n^n$.) This completes the proof of (2.1).

The induction step to prove (2.2) is similar but there are some notable differences so we give the details. Note that $1 < s_j \leq q_j < \infty$ for each j. Fix $q_1, \dots, q_n$ such that $\sum_{j=1}^n \frac{1}{q_j} \leq 1$ and set $\tilde{Q}_j = q_j/s'_n$ for $j = 1, 2, \dots, n-1$. Observe that,

$$\frac{1}{s_n} + \sum_{j=1}^{n-1} \frac{1}{q_j} = \sum_{j=1}^{n} \frac{1}{q_j} + \frac{1}{n-1}\left(1 - \sum_{j=1}^{n} \frac{1}{q_j}\right) = \frac{1}{n-1} + \frac{n-2}{n-1} \sum_{j=1}^{n} \frac{1}{q_j} \leq 1,$$

so

$$\sum_{j=1}^{n-1} \frac{1}{\tilde{Q}_j} = s'_n \sum_{j=1}^{n-1} \frac{1}{q_j} \leq s'_n \left(1 - \frac{1}{s_n}\right) = 1.$$

Thus, by the inductive hypothesis,

$$\int_{M^{n-1}} F_1 F_2 \dots F_{n-1} \, d\mu^{n-1} \leq \prod_{j=1}^{n-1} \left(\int_{M_j^{n-1}} \left(\int_{M_j} F_j^{\tilde{Q}_j} \, d\mu_j \right)^{S_j/\tilde{Q}_j} d\mu_j^{n-1} \right)^{1/S_j},$$

for non-negative μ^{n-1}-measurable functions $F_1, \dots, F_{n-1}$, where $S_j = s_j/s'_n$ because,

$$\begin{aligned}
\frac{1}{S_j} &= \frac{1}{\tilde{Q}_j} + \frac{1}{n-2}\left(1 - \sum_{k=1}^{n-1} \frac{1}{\tilde{Q}_k}\right) \\
&= s'_n \left(\frac{1}{q_j} + \frac{1}{n-2}\left(\frac{1}{s'_n} - \sum_{k=1}^{n-1} \frac{1}{q_k} \right) \right) \\
&= s'_n \left(\frac{1}{q_j} + \frac{1}{n-2}\left(1 - \frac{1}{q_n} - \frac{1}{n-1}\left(1 - \sum_{k=1}^{n} \frac{1}{q_k}\right) - \sum_{k=1}^{n-1} \frac{1}{q_k} \right) \right) \\
&= s'_n \left(\frac{1}{q_j} + \frac{1}{n-2}\left(1 - \frac{1}{n-1}\right)\left(1 - \sum_{k=1}^{n} \frac{1}{q_k}\right) \right) \\
&= s'_n \left(\frac{1}{q_j} + \frac{1}{n-1}\left(1 - \sum_{k=1}^{n} \frac{1}{q_k}\right) \right) = \frac{s'_n}{s_j}.
\end{aligned}$$

We apply this inequality with $F_j = \left(\int_{M_n} f_j^{s_j} \, d\mu_n \right)^{s'_n/s_j}$ for $j = 1, 2, \dots, n-1$. Note that the integration with respect to the nth variable produces non-negative

μ^{n-1}-measurable functions $F_1, F_2, \ldots, F_{n-1}$. We get,

$$\left(\int_{M^{n-1}} \prod_{j=1}^{n-1} \left(\int_{M_n} f_j^{s_j}\, d\mu_n\right)^{s_n'/s_j} d\mu^{n-1}\right)^{1/s_n'} \\ \leq \prod_{j=1}^{n-1} \left(\int_{M_j^{n-1}} \left(\int_{M_j} \left(\int_{M_n} f_j^{s_j}\, d\mu_n\right)^{q_j/s_j} d\mu_j\right)^{s_j/q_j} d\mu_j^{n-1}\right)^{1/s_j}.$$

For convenience, set

$$\tilde{C} = \left(\int_{M^{n-1}} \left(\int_{M_n} f_n^{q_n}\, d\mu_n\right)^{s_n/q_n} d\mu^{n-1}\right)^{1/s_n}.$$

Then Hölder's inequality, used three times, and the inequality above yield,

$$\begin{aligned}
\int_{M^n} f_1 f_2 \ldots f_n\, d\mu^n &= \int_{M^{n-1}} \int_{M_n} f_1 f_2 \ldots f_n\, d\mu_n\, d\mu^{n-1} \\
&\leq \int_{M^{n-1}} \left(\int_{M_n} f_n^{q_n}\, d\mu_n\right)^{1/q_n} \left(\int_{M_n} (f_1 f_2 \ldots f_{n-1})^{q_n'}\, d\mu_n\right)^{1/q_n'} d\mu^{n-1} \\
&\leq \tilde{C}\left(\int_{M^{n-1}} \left(\int_{M_n} (f_1 f_2 \ldots f_{n-1})^{q_n'}\, d\mu_n\right)^{s_n'/q_n'} d\mu^{n-1}\right)^{1/s_n'} \\
&\leq \tilde{C}\left(\int_{M^{n-1}} \prod_{j=1}^{n-1} \left(\int_{M_n} f_j^{s_j} d\mu_n\right)^{s_n'/s_j} d\mu_j^{n-1}\right)^{1/s_n'} \\
&\leq \tilde{C} \prod_{j=1}^{n-1} \left(\int_{M_j^{n-1}} \left(\int_{M_j} \left(\int_{M_n} f_j^{s_j}\, d\mu_n\right)^{q_j/s_j} d\mu_j\right)^{s_j/q_j} d\mu_j^{n-1}\right)^{1/s_j}.
\end{aligned}$$

Note that the third application of Hölder's inequality above uses the indices s_j/q_n' for $j = 1, \ldots, n-1$. This valid because

$$\sum_{j=1}^{n-1} \frac{q_n'}{s_j} = q_n'\left(\sum_{j=1}^{n-1} \frac{1}{q_j} + 1 - \sum_{j=1}^{n} \frac{1}{q_j}\right) = 1.$$

Since $q_j/s_j \geq 1$, Minkowski's integral inequality gives,

$$\begin{aligned}
\int_{M^n} f_1 f_2 \ldots f_n\, d\mu^n &\leq C \prod_{j=1}^{n-1} \left(\int_{M_j^{n-1}} \int_{M_n} \left(\int_{M_j} f_j^{q_j}\, d\mu_j\right)^{s_j/q_j} d\mu_n\, d\mu_j^{n-1}\right)^{1/s_j} \\
&= \prod_{j=1}^{n} \left(\int_{M_j^n} \left(\int_{M_j} f_j^{q_j}\, d\mu_j\right)^{s_j/q_j} d\mu_j^n\right)^{1/s_j}.
\end{aligned}$$

This completes the proof. □

The above theorem gives a useful corollary in the special case when the functions $f_1, f_2, \ldots, f_n$ are taken to be a powers of a single function.

Corollary 2.2. *Suppose $q > 0$, $n \geq 2$ and $r_1, \dots, r_n \in (0, q)$. If $h \geq 0$ is μ^n-measurable, then*

$$\left(\int_{M^n} h^Q \, d\mu^n\right)^{1/Q} \leq \prod_{j=1}^{n} \left(\int_{M_j} \left(\int_{M_j^n} h^q \, d\mu_j^n\right)^{r_j/q} d\mu_j\right)^{1/(R(q-r_j))} \quad \text{and} \tag{2.3}$$

$$\left(\int_{M^n} h^S \, d\mu^n\right)^{1/S} \leq \prod_{j=1}^{n} \left(\int_{M_j^n} \left(\int_{M_j} h^q \, d\mu_j\right)^{r_j/q} d\mu_j^n\right)^{1/(R(q-r_j))}. \tag{2.4}$$

Here $R = \sum_{j=1}^n \frac{r_j}{q-r_j}$, $Q = qR/(1+R)$ and $S = qR/(n-1+R)$.

Proof. For the first inequality, let $q_j = (1+R)(q-r_j)/r_j$ and $f_j = h^{q/q_j}$ for $j = 1, 2, \dots, n$. Then

$$\sum_{j=1}^{n} \frac{1}{q_j} = \frac{R}{1+R} \leq 1$$

and

$$1 - \frac{1}{p_j} = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{q_k} = \frac{R}{1+R} - \frac{1}{q_j} = 1 - \frac{q}{(1+R)(q-r_j)}$$

so $p_j = (1+R)(q-r_j)/q$ for $j = 1, 2, \dots, n$. With these substitutions, the inequality (2.1), raised to the power $(1+R)/(qR)$, gives (2.3).

For the second inequality, let $q_j = (n-1+R)(q-r_j)/r_j$ and $f_j = h^{q/q_j}$ for $j = 1, 2, \dots, n$. Then

$$\sum_{j=1}^{n} \frac{1}{q_j} = \frac{R}{n-1+R} \leq 1$$

and

$$\frac{1}{s_j} = \frac{1}{q_j} + \frac{1}{n-1}\left(1 - \frac{R}{n-1+R}\right)$$
$$= \frac{r_j}{q-r_j}\frac{1}{n-1+R} + \frac{1}{n-1+R} = \frac{q}{q-r_j}\frac{1}{n-1+R}$$

so $s_j = (n-1+R)(q-r_j)/q$ for $j = 1, 2, \dots, n$. With these substitutions, the inequality (2.2), raised to the power $(n-1+R)/(qR)$, gives (2.4). □

Inequality (2.3), with $n = 2$, μ_1 and μ_2 taken to be counting measure on the positive integers, $q = 2$, and $r_1 = r_2 = 1$ becomes (1.1).

Also with $n = 2$ and counting measures, but with general q, r_1 and r_2, (2.3) gives Lemma 3.1 from [8], providing explicit values for the recursively defined exponents in that result.

In the case $q = 2$, $r_1 = \cdots = r_n = 1$, $n \geq 2$, inequality (2.3) gives a variant of Blei's inequality which is used in [9] (Lemma 1) as an ingredient in the proof that the Bohnenblust-Hille inequality for polynomials is hypercontractive.

With counting measure, and $r_1 = \cdots = r_n$, (2.3) reduces to Lemma 5.1 of [7].

With counting measure, but with general q, $r_1, \dots, r_n$, (2.3) gives Lemma 2.3 from [17], providing explicit values for the recursively defined exponents.

Inequality (2.4) with $n = 3$, μ_1, μ_2, and μ_3 taken to be counting measure on the positive integers, $q = 2$, and $r_1 = r_2 = r_3 = 1$ becomes (1.2).

The inequalities (2.1) and (2.2) of Theorem 2.1 can be used to prove the boundedness of a certain multilinear functional. In the next theorem we establish the norm of this functional. For $1 \le p < \infty$, the space L^p_μ is the collection of all complex-valued μ-measurable functions f for which

$$\|f\|_{L^p_\mu} \equiv \left(\int |f|^p \, d\mu \right)^{1/p} < \infty.$$

Theorem 2.3. *Suppose $n \ge 2$ and positive real numbers $q_1, q_2, \dots, q_n$ satisfy $\sum_{k \ne j} \frac{1}{q_k} < 1$ for $j = 1, 2, \dots, n$. Fix complex-valued functions $\varphi_j \in L^{q_j}_{\mu_j}$ and set*

$$T(f_1, f_2, \dots, f_n) = \int_{M^n} f_1 \varphi_1 f_2 \varphi_2 \dots f_n \varphi_n \, d\mu^n.$$

Then T is a well-defined, bounded multilinear functional on $\prod_{j=1}^n L^{s_j}_{\mu_j^n}$ and its norm is $\|\varphi_1\|_{L^{q_1}_{\mu_1}} \|\varphi_2\|_{L^{q_2}_{\mu_2}} \dots \|\varphi_n\|_{L^{q_n}_{\mu_n}}$. Here s_j is defined by,

$$\frac{1}{s_j} = \frac{1}{q_j} + \frac{1}{n-1} \left(1 - \sum_{k=1}^n \frac{1}{q_k} \right). \tag{2.5}$$

Proof. For each j, $1 < q_j < \infty$. But $\frac{1}{q_j} < \sum_{k=1}^n \frac{1}{q_k} < 1 + \frac{1}{q_j}$, so

$$0 \le \frac{n-2}{n-1} \frac{1}{q_j} < \frac{1}{s_j} < \frac{1}{n-1} + \frac{n-2}{n-1} \frac{1}{q_j} \le 1.$$

Therefore, $1 < s_j < \infty$ for each j. Also, we may sum (2.5) to get

$$\sum_{k=1}^n \frac{1}{s_k} = \frac{n}{n-1} - \frac{1}{n-1} \sum_{k=1}^n \frac{1}{q_k} \tag{2.6}$$

and conclude that

$$\frac{1}{q_j} = \frac{1}{s_j} + \left(1 - \left(\frac{n}{n-1} - \frac{1}{n-1} \sum_{k=1}^n \frac{1}{q_k} \right) \right) = \frac{1}{s_j} + \left(1 - \sum_{k=1}^n \frac{1}{s_k} \right). \tag{2.7}$$

Now suppose $f_j \in L^{s_j}_{\mu_j^n}$ for $j = 1, \dots, n$. If $\sum_{k=1}^n \frac{1}{q_k} \le 1$ then (2.2) implies,

$$\int_{M^n} |f_1 \varphi_1 f_2 \varphi_2 \dots f_n \varphi_n| \, d\mu^n \le \prod_{j=1}^n \left(\int_{M_j^n} \left(\int_{M_j} |f_j|^{q_j} |\varphi_j|^{q_j} \, d\mu_j \right)^{s_j/q_j} d\mu_j^n \right)^{1/s_j}.$$

If $\sum_{k=1}^n \frac{1}{q_k} \ge 1$ then (2.6) implies $\sum_{k=1}^n \frac{1}{s_k} \le 1$ and (2.7) shows that (2.1) holds with q_j replaced by s_j and p_j replaced by q_j. That is,

$$\int_{M^n} |f_1 \varphi_1 f_2 \varphi_2 \dots f_n \varphi_n| \, d\mu^n \le \prod_{j=1}^n \left(\int_{M_j} \left(\int_{M_j^n} |f_j|^{s_j} |\varphi_j|^{s_j} \, d\mu_j^n \right)^{q_j/s_j} d\mu_j \right)^{1/q_j}.$$

But φ_j is constant on M_j^n and f_j is constant on M_j so the inequality given in the case $\sum_{k=1}^n \frac{1}{q_k} \le 1$ and the inequality given in the case $\sum_{k=1}^n \frac{1}{q_k} \ge 1$ both reduce to

$$\int_{M^n} |f_1 \varphi_1 f_2 \varphi_2 \dots f_n \varphi_n| \, d\mu^n \le \left(\prod_{j=1}^n \|\varphi_j\|_{L^{q_j}_{\mu_j}} \right) \left(\prod_{j=1}^n \|f_j\|_{L^{s_j}_{\mu_j^n}} \right).$$

Since the right-hand side above is finite, the integral defining T converges absolutely so T is well defined. It is clear that T is multilinear. Moreover, if $\|f_j\|_{L^{s_j}(M_j^n)} \le 1$

for each j, the above calculation shows that

$$|T(f_1, f_2, \dots, f_n)| \leq \int_{M^n} |f_1\varphi_1 f_2\varphi_2 \dots f_n\varphi_n| \, d\mu^n \leq \prod_{j=1}^{n} \|\varphi_j\|_{L^{q_j}_{\mu_j}}.$$

Thus T is bounded and the norm is at most $\|\varphi_1\|_{L^{q_1}_{\mu_1}} \|\varphi_2\|_{L^{q_2}_{\mu_2}} \dots \|\varphi_n\|_{L^{q_n}_{\mu_n}}$. To show that the norm is attained, first observe that if $\varphi_j = 0$ μ_j-a.e. for some j then $T = 0$. Otherwise, set

$$f_j = \varepsilon_j \prod_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{|\varphi_k|}{\|\varphi_k\|_{L^{q_k}_{\mu_k}}} \right)^{q_k/s_j}$$

for $j = 1, 2, \dots, n$, where $\varepsilon_j = \overline{\mathrm{sgn}(\varphi_{j+1})}$ for $j = 1, 2, \dots, n-1$ and $\varepsilon_n = \overline{\mathrm{sgn}(\varphi_1)}$. Then $\|f_j\|_{L^{s_j}_{\mu^n_j}} \leq 1$ for each j, and a calculation shows that

$$f_1 f_2 \dots f_n = \varepsilon_1 \varepsilon_2 \dots \varepsilon_n \prod_{k=1}^{n} \left(\frac{|\varphi_k|}{\|\varphi_k\|_{L^{q_k}_{\mu_k}}} \right)^{q_k - 1}.$$

But $\varepsilon_j \varphi_{j+1} = |\varphi_{j+1}|$ for $j = 1, \dots, n-1$ and $\varepsilon_n \varphi_1 = |\varphi_1|$ so,

$$f_1\varphi_1 f_2\varphi_2 \dots f_n\varphi_n = \prod_{k=1}^{n} \frac{|\varphi_k|^{q_k}}{\|\varphi_k\|^{q_k-1}_{L^{q_k}_{\mu_k}}}$$

and we have

$$T(f_1, f_2, \dots, f_n) = \|\varphi_1\|_{L^{q_1}_{\mu_1}} \|\varphi_2\|_{L^{q_2}_{\mu_2}} \dots \|\varphi_n\|_{L^{q_n}_{\mu_n}}.$$

□

3. Coordinatewise multiple summing operators

To begin, we recall some known definitions and results for easy reference. For details see [6], [10] and [16]. If $1 \leq r < \infty$, Z and Y are Banach spaces and $T : Z \to Y$ is linear, we say T is *r-summing* provided there exists a constant $C \geq 0$ such that for any finite sequence $z_1, \dots, z_N$ in Z,

$$\Big(\sum_{i=1}^{N} \|T(z_i)\|^r_Y \Big)^{1/r} \leq C \sup_{\|z^*\|_{Z^*} \leq 1} \Big(\sum_{i=1}^{N} |z^*(z_i)|^r \Big)^{1/r}.$$

The least constant C is denoted $\pi_r(T)$.

Let N be a positive integer. The *weak ℓ^1-norm* of $x \in X^N$ is

$$w_1(x) = \sup_{\|x^*\|_{X^*} \leq 1} \sum_{i=1}^{N} |x^*(x_i)| = \sup \Big\{ \Big\| \sum_{i=1}^{N} a_i x_i \Big\|_X : |a_i| \leq 1, i = 1, \dots, N \Big\}.$$

Let $X_1, X_2, \dots, X_m$ and Y be Banach spaces and $U : X_1 \times \dots \times X_m \to Y$ be multilinear. For $1 \leq r < \infty$ we say U is *multiple $(r,1)$-summing* provided there exists a constant $C \geq 0$ such that for every choice of positive integers $N_1, \dots, N_m$ and $x_k = (x_k(1), \dots, x_k(N_k)) \in X_k^{N_k}$ for $k = 1, \dots, m$,

$$\Big(\sum_{i_1=1}^{N_1} \cdots \sum_{i_m=1}^{N_m} \|U(x_1(i_1), \dots, x_m(i_m))\|^r_Y \Big)^{1/r} \leq C w_1(x_1) \dots w_1(x_m).$$

The least constant C for which the inequality holds is denoted $\pi_{r,1}^{\text{mult}}(U)$. It is easy to verify that $\pi_{r,1}^{\text{mult}}$ gives a norm on the space $\Pi_{r,1}^{\text{mult}}(X_1 \times \cdots \times X_m; Y)$, of all multiple $(r,1)$-summing operators from $X_1 \times \cdots \times X_m$ to Y.

The concept of multiple summing operators was introduced independently in [5] and [13], although as we have mentioned it has its beginning in the classical paper of Bohnenblust and Hille from 1931. (When we wish to emphasize that U is linear rather than multilinear, we drop the "multiple" before $(r,1)$-summing, and write $\pi_{r,1}(U)$ for the best constant.)

Let $2 \le q < \infty$. A Banach space Y has *cotype* q provided there exists a constant $C \ge 0$ such that for each positive integer N and each $y \in Y^N$,

$$\left(\sum_{i=1}^{N} \|y_i\|_Y^q \right)^{1/q} \le C \left(\int_0^1 \Big\| \sum_{i=1}^{N} r_i(t) y_i \Big\|_Y^2 \, dt \right)^{1/2}.$$

The least constant C for which the inequality holds is denoted $C_q(Y)$, see [6] and [10]. Here $r_1, r_2, \ldots$ denote the Rademacher functions on $[0,1]$.

We will need the following special case of Kahane's inequality, see [10]: For each positive r there is a positive constant $K_{r,2}$ such that for any Banach space X, any positive integer N, and all $x \in X^N$,

$$\left(\int_0^1 \Big\| \sum_{i=1}^{N} r_i(t) x_i \Big\|_X^2 \, dt \right)^{1/2} \le K_{r,2} \left(\int_0^1 \Big\| \sum_{i=1}^{N} r_i(t) x_i \Big\|_X^r \, dt \right)^{1/r}.$$

Coordinatewise multiple summing operators were first defined in [8]. Our definition agrees, but with some minor changes in notation to simplify our presentation. For Banach spaces $X_1, X_2, \ldots, X_m$, $m \ge 2$, and a proper subset C of $\{1, \ldots, m\}$, that is $C \ne \emptyset$ and $C \ne \{1, \ldots, m\}$, we write $X^C = \prod_{k \in C} X_k$ and identify, in the obvious way, the space $X_1 \times \cdots \times X_m$ with $X^C \times X^{\overline{C}}$, where $\overline{C}$ denotes the complement of C in $\{1, \ldots, m\}$. With this identification if $x \in X^C$ and $z \in X^{\overline{C}}$, then $(x,z) \in X_1 \times \cdots \times X_m$. We take the norm on finite products of Banach spaces to be the maximum of the component norms so the identification is isometric.

If $U : X_1 \times \cdots \times X_m \to Y$ is multilinear, then U^C is defined by $U^C(z)(x) = U(x,z)$ for all $x \in X^C$ and $z \in X^{\overline{C}}$. Clearly, if $z \in X^{\overline{C}}$ is fixed, $U^C(z) : X^C \to Y$ is a multilinear map. Let $1 \le r < \infty$. If $U^C(z) \in \Pi_{r,1}^{\text{mult}}(X^C; Y)$ for each $z \in X^{\overline{C}}$ we say that U is *multiple $(r,1)$-summing in the coordinates of C*. In this case we view U^C as a map from $X^{\overline{C}}$ to $\Pi_{r,1}^{\text{mult}}(X^C; Y)$ and denote its (coordinatewise) norm by,

$$\|U^C\|_{CW(r,1)} \equiv \|U^C : X^{\overline{C}} \to \Pi_{r,1}^{\text{mult}}(X^C; Y)\| = \sup\{\pi_{r,1}^{\text{mult}}(U^C(z)) : \|z\|_{X^{\overline{C}}} \le 1\}.$$

To introduce multi-indices for summation, fix positive integers $N_1, \ldots, N_m$ and write $N^C = \prod_{k \in C} \{1, \ldots, N_k\}$. For $x_k = (x_k(1), \ldots, x_k(N_k)) \in X_k^{N_k}$, $k = 1, \ldots, m$, and $i \in N^C$ we set $x(i) = (x_k(i_k))_{k \in C}$ and obtain $x(i) \in X^C$. The identification made above gives $(x(i), x(j)) \in X_1 \times \cdots \times X_m$ whenever $i \in N^C$ and $j \in N^{\overline{C}}$.

The first statement of Theorem 3.2 below is based on Theorem 5.1 of [8] but is considerably simpler because explicit formulas for the indices are provided. The proof is based on Corollary 2.2. The key lemma, Lemma 3.1 below, is essentially given in the proof of Theorem 4.1 from [8] but is isolated here for easy reference. This lemma is used again in proof of the second statement of Theorem 3.2, which is complementary to the first.

Lemma 3.1. *Let Y be a Banach space of cotype $q \geq 2$ and $1 \leq r < q$. If $m \geq 2$, C is a proper subset of $\{1, \dots, m\}$, $U : X_1 \times \cdots \times X_m \to Y$ is multiple $(r,1)$-summing in the coordinates of C, and $x_k = (x_k(1), \dots, x_k(N_k)) \in X_k^N$ satisfy $w_1(x_k) \leq 1$ for $k = 1, \dots, m$, then*

$$\left(\sum_{i \in N^C} \left(\sum_{j \in N^{\overline{C}}} \|U(x(i), x(j))\|_Y^q \right)^{r/q} \right)^{1/r} \leq (C_q(Y) K_{r,2})^{|\overline{C}|} \|U^C\|_{CW(r,1)}.$$

Proof. Fix $x_k = (x_k(1), \dots, x_k(N_k)) \in X_k^{N_k}$ satisfying $w_1(x_k) \leq 1$ for $k = 1, \dots, m$. Fix $i \in N^C$. By Lemma 2.2 of [8] and the multilinearity of U,

$$\begin{aligned}
&\left(\sum_{j \in N^{\overline{C}}} \|U(x(i), x(j))\|_Y^q \right)^{1/q} \\
&\leq (C_q(Y) K_{r,2})^{|\overline{C}|} \left(\int_{[0,1]^{\overline{C}}} \Big\| \sum_{j \in N^{\overline{C}}} \prod_{k \in \overline{C}} r_{j_k}(t_k) U(x(i), x(j)) \Big\|_Y^r dt \right)^{1/r} \\
&= (C_q(Y) K_{r,2})^{|\overline{C}|} \left(\int_{[0,1]^{\overline{C}}} \|U(x(i), R^{\overline{C}}(t))\|_Y^r \, dt \right)^{1/r},
\end{aligned}$$

where

$$R^{\overline{C}}(t) = \left(\sum_{j_k=1}^{N_k} r_{j_k}(t_k) x_k(j_k) \right)_{k \in \overline{C}}.$$

Since each $|r_{j_k}(t_k)| \leq 1$,

$$\Big\| \sum_{j_k=1}^{N_k} r_{j_k}(t_k) x_k(j_k) \Big\|_{X_k} \leq w_1(x_k) \leq 1$$

for each $k \in \overline{C}$ and hence $\|R^{\overline{C}}(t)\|_{X^{\overline{C}}} \leq 1$. But U is multiple $(r,1)$-summing in the coordinates of C so, summing over all $i \in N^C$, we have

$$\begin{aligned}
&\left(\sum_{i \in N^C} \left(\sum_{j \in N^{\overline{C}}} \|U(x(i), x(j))\|_Y^q \right)^{r/q} \right)^{1/r} \\
&\qquad \leq (C_q(Y) K_{r,2})^{|\overline{C}|} \left(\int_{[0,1]^{\overline{C}}} \sum_{i \in N^C} \|U(x(i), R^{\overline{C}}(t)\|_Y^r \, dt \right)^{1/r} \\
&\qquad \leq (C_q(Y) K_{r,2})^{|\overline{C}|} \|U^C\|_{CW(r,1)}.
\end{aligned}$$

□

Theorem 3.2. *Let $m \geq 2$, let $\{1, \dots, m\}$ be the disjoint union of $n \geq 2$ non-empty subsets $C_1, \dots, C_n$, let Y be a Banach space with cotype $q \geq 2$, and let $r_1, \dots, r_n \in [1, q)$. Set*

$$R = \sum_{j=1}^n r_j/(q - r_j), \quad Q = qR/(1+R) \quad \textit{and} \quad S = qR/(n-1+R).$$

If $U : X_1 \times \cdots \times X_m \to Y$ is multiple $(r_k, 1)$-summing in the coordinates of C_k for each $k = 1, \dots, n$, then U is multiple $(Q, 1)$-summing, and

$$\pi^{\mathrm{mult}}_{Q,1}(U) \le \prod_{k=1}^{n} \Big((C_q(Y) K_{r_k,2})^{|\overline{C_k}|} \|U^{C_k}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k))}.$$

If $V : X_1 \times \cdots \times X_m \to Y$ is multiple $(r_k, 1)$-summing in the coordinates of $\overline{C}_k$ for each $k = 1, \dots, n$, then V is multiple $(S, 1)$-summing, and

$$\pi^{\mathrm{mult}}_{S,1}(V) \le \prod_{k=1}^{n} \Big((C_q(Y) K_{r_k,2})^{|C_k|} \|V^{\overline{C_k}}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k))}.$$

Proof. Suppose $x_k = (x_k(1), \dots, x_k(N_k)) \in X_k^{N_k}$ satisfy $w_1(x_k) \le 1$ for $k = 1, \dots m$. Inequality 2.3 and Lemma 3.1 give,

$$\begin{aligned}
&\Big(\sum_{i \in N^{C_1} \times \cdots \times N^{C_n}} \|U(x(i))\|_Y^Q \Big)^{1/Q} \\
&\le \prod_{k=1}^{n} \Big(\sum_{i \in N^{C_k}} \Big(\sum_{j \in N^{\overline{C_k}}} \|U(x(i), x(j))\|_Y^q \Big)^{r_k/q} \Big)^{1/(R(q-r_k))} \\
&\le \prod_{k=1}^{n} \Big((C_q(Y) K_{r_k,2})^{|\overline{C_k}|} \|U^{C_k}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k)}.
\end{aligned}$$

Inequality (2.4) and Lemma 3.1 give,

$$\begin{aligned}
&\Big(\sum_{i \in N^{C_1} \times \cdots \times N^{C_n}} \|V(x(i))\|_Y^S \Big)^{1/S} \\
&\le \prod_{k=1}^{n} \Big(\sum_{i \in N^{\overline{C_k}}} \Big(\sum_{j \in N^{C_k}} \|V(x(i), x(j))\|_Y^q \Big)^{r_k/q} \Big)^{1/(R(q-r_k))} \\
&\le \prod_{k=1}^{n} \Big((C_q(Y) K_{r_k,2})^{|C_k|} \|V^{\overline{C_k}}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k)}.
\end{aligned}$$

The conclusion follows. □

These results are of particular interest in the special case when $C_k = \{k\}$ for each $k = 1, \dots, m$.

Corollary 3.3. *Let $m \ge 2$, let Y be a Banach space with cotype $q > 2$, and let $r_1, \dots, r_m \in [1, q)$. Define R, Q, and S as in Theorem 3.2. If $U : X_1 \times \cdots \times X_m \to Y$ is $(r_k, 1)$-summing in the k coordinate for $k = 1, \dots, m$, then U is multiple $(Q, 1)$-summing, and*

(3.1)
$$\pi^{\mathrm{mult}}_{Q,1}(U) \le \Big(C_q(Y) \prod_{k=1}^{m} K_{r_k,2}^{r_k/(R(q-r_k))} \Big)^{m-1} \prod_{k=1}^{m} \Big(\|U^{\{k\}}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k))}.$$

If $V : X_1 \times \cdots \times X_m \to Y$ is multiple $(r_k, 1)$-summing in all coordinates except k, for $k = 1, \dots, m$, then V is multiple $(S, 1)$-summing, and

$$\pi^{\mathrm{mult}}_{S,1}(V) \le \Big(C_q(Y) \prod_{k=1}^{m} K_{r_k,2}^{r_k/(R(q-r_k))} \Big) \prod_{k=1}^{m} \Big(\|V^{\overline{\{k\}}}\|_{CW(r_k,1)} \Big)^{r_k/(R(q-r_k))}. \tag{3.2}$$

If $m > 2$, the two parts of Corollary 3.3 can be used one after the other to give an estimate of $\pi^{\mathrm{mult}}_{Q,1}(U)$ with a somewhat different constant. The idea is to apply inequality (3.1) with U replaced by $U^{\overline{\{j\}}}(x_j)$ to show that the hypotheses of inequality (3.2) are satisfied. We state and prove it in a form that is easily compared with the first statement of Corollary 3.3. Observe that only the factors arising from Kahane's inequality differ. It can be shown that the constant is improved by this process. We leave it to the interested reader to compare the constants arising in Corollaries 3.3, 3.4 and, in the case $r_1 = \cdots = r_n$, Corollary 5.2 of [8].

Corollary 3.4. *Let $m > 2$, let Y be a Banach space with cotype $q \geq 2$, and let $r_1, \dots, r_m \in [1, q)$. If $U : X_1 \times \cdots \times X_m \to Y$ is $(r_k, 1)$-summing in the k coordinate for $k = 1, \dots, m$, then U is multiple $(Q, 1)$-summing, where $Q = qR/(1 + R)$ with $R = \sum_{j=1}^{m} r_j/(q - r_j)$. Moreover,*

$$\pi^{\mathrm{mult}}_{Q,1}(U) \leq A\left(C_q(Y) \prod_{k=1}^{m} K_{r_k,2}^{r_k/(R(q-r_k))}\right)^{m-1} \prod_{k=1}^{m} \left(\|U^{\{k\}}\|_{CW(r_k,1)}\right)^{r_k/(R(q-r_k))}.$$

where

$$A = \left(\prod_{k=1}^{m} K_{r_k,2}^{r_k/(R(q-r_k))}\right)^{-1} \left(\prod_{k=1}^{m} K_{Q_k,2}^{1-(r_k/(R(q-r_k)))}\right)^{1/(m-1)}$$

and $Q_k = q\left(R - \frac{r_k}{q-r_k}\right)\left(1 + R - \frac{r_k}{q-r_k}\right)^{-1}$.

Proof. For $j = 1, \dots, m$ and $\|x_j\|_{X_j} \leq 1$, let $U_j = U^{\overline{\{j\}}}(x_j)$. Since U is $(r_k, 1)$-summing in the k coordinate for each k it is easily verified that U_j is $(r_k, 1)$-summing in the k coordinate for each $k \neq j$. Moreover,

$$\begin{aligned}\|U_j{}^{\{k\}}\|_{CW(r_k,1)} &= \sup\left\{\pi_{r_k,1}(U_j^{\{k\}}(z)) : \|z\|_{X^{\overline{\{j,k\}}}} \leq 1\right\} \\ &\leq \sup\left\{\pi_{r_k,1}(U^{\{k\}}(z)) : \|z\|_{X^{\overline{\{k\}}}} \leq 1\right\} = \|U^{\{k\}}\|_{CW(r_k,1)}.\end{aligned}$$

We apply inequality (3.1) to see that each U_j is multiple $(Q_j, 1)$-summing, where

$$R_j = \sum_{k\neq j} \frac{r_k}{q - r_k} = R - \frac{r_j}{q - r_j} \quad \text{and} \quad Q_j = \frac{qR_j}{1 + R_j}.$$

It also shows that

$$\begin{aligned}\pi^{\mathrm{mult}}_{Q_j,1}(U_j) &\leq \left(C_q(Y) \prod_{\substack{k=1\\k\neq j}}^{n} K_{r_k,2}^{r_k/(R_j(q-r_k))}\right)^{m-2} \prod_{\substack{k=1\\k\neq j}}^{n} \left(\|U_j{}^{\{k\}}\|_{CW(r_k,1)}\right)^{r_k/(R_j(q-r_k))} \\ &\leq \left(C_q(Y) \prod_{\substack{k=1\\k\neq j}}^{n} K_{r_k,2}^{r_k/(R_j(q-r_k))}\right)^{m-2} \prod_{\substack{k=1\\k\neq j}}^{n} \left(\|U^{\{k\}}\|_{CW(r_k,1)}\right)^{r_k/(R_j(q-r_k))}.\end{aligned}$$

But $U_j = U^{\overline{\{j\}}}(x_j)$ so, taking the supremum over all $x_j \in X_j$ such that $\|x_j\|_{X_j} \leq 1$, we have

$$\begin{aligned}&\|U^{\overline{\{j\}}}\|_{CW(Q_j,1)} \\ &\quad\leq \left(C_q(Y) \prod_{\substack{k=1\\k\neq j}}^{n} K_{r_k,2}^{r_k/(R_j(q-r_k))}\right)^{m-2} \prod_{\substack{k=1\\k\neq j}}^{n} \left(\|U^{\{k\}}\|_{CW(r_k,1)}\right)^{r_k/(R_j(q-r_k))}.\end{aligned}$$

Since U_j is multiple $(Q_j, 1)$-summing for each x_j with $\|x_j\|_{X_j} \le 1$ it follows that U is multiple $(Q_j, 1)$-summing in the coordinates of $\overline{\{j\}}$. Thus we may apply inequality (3.2) (with V replaced by U) to conclude that U is multiple $(S, 1)$-summing, where

$$\mathcal{R} = \sum_{j=1}^{m} \frac{Q_j}{q - Q_j} = \sum_{j=1}^{m} \frac{\frac{qR_j}{1+R_j}}{q - \frac{qR_j}{1+R_j}} = \sum_{j=1}^{m} R_j = (m-1)R$$

and

$$S = \frac{q\mathcal{R}}{m - 1 + \mathcal{R}} = \frac{qR}{1+R} = Q.$$

Thus U is multiple $(Q, 1)$-summing as stated. Moreover,

$$\begin{aligned}\pi^{\text{mult}}_{S,1}(U) &\le \left(C_q(Y) \prod_{j=1}^{m} K_{Q_j,2}^{Q_j/(\mathcal{R}(q-Q_j))} \right) \prod_{j=1}^{m} \left(\|U^{\overline{\{j\}}}\|_{CW(Q_j,1)} \right)^{Q_j/(\mathcal{R}(q-Q_j))} \\ &= \left(C_q(Y) \prod_{j=1}^{m} K_{Q_j,2}^{R_j/((m-1)R)} \right) \prod_{j=1}^{m} \left(\|U^{\overline{\{j\}}}\|_{CW(Q_j,1)} \right)^{R_j/((m-1)R)}.\end{aligned}$$

But $S = Q$ and we have already established estimates for $\|U^{\overline{\{j\}}}\|_{CW(Q_j,1)}$ so,

$$\begin{aligned}\pi^{\text{mult}}_{Q,1}(U) \le C_q(Y)^{m-1} &\left(\prod_{k=1}^{m} K_{r_k,2}^{r_k/(R(q-r_k))} \right)^{m-2} \\ &\times \left(\prod_{k=1}^{m} K_{Q_k,2}^{R_k/((m-1)R)} \right) \prod_{k=1}^{m} \left(\|U^{\{k\}}\|_{CW(r_k,1)} \right)^{r_k/(R(q-r_k))}.\end{aligned}$$

This may be rearranged to yield the estimate given. □

Combining inequality (3.2) with the Bohnenblust-Hille theorem, we show that the composition of a bounded m-linear operator and a $\frac{2(m-1)}{m}$-summing operator with a cotype q codomain is multiple summing.

Theorem 3.5. *Let $X_1, \dots, X_m, Y, Z$ be Banach spaces, $m \ge 2$, and suppose Y has cotype $q \ge 2$. If $U : X_1 \times \cdots \times X_m \to Z$ is a bounded multilinear map and $T : Z \to Y$ is $\frac{2(m-1)}{m}$-summing, then $T \circ U$ is multiple $(\frac{2mq}{2+mq}, 1)$-summing and*

$$\pi^{\text{mult}}_{\frac{2mq}{2+mq},1}(T \circ U) \le C_q(Y) K_{\frac{2(m-1)}{m},2} \pi_{\frac{2(m-1)}{m}}(T) BH_{m-1} \|U\|.$$

Proof. Let $r = \frac{2(m-1)}{m}$ and note that $1 \le r < 2 \le q$. Our first step is to show that $T \circ U$ is multiple $(r, 1)$-summing in the coordinates of $\overline{\{k\}}$ for each $k = 1, \dots, m$. Fix a k and an $x_k \in X_k$, and suppose that $x_j = (x_j(1), \dots, x_j(N_j)) \in X_j^{N_j}$ satisfy $w_1(x_j) \le 1$ for $j \ne k$. Since T is r-summing,

$$\left(\sum_{i \in N^{\overline{\{k\}}}} \|T \circ U(x(i), x_k)\|_Y^r \right)^{1/r} \le \pi_r(T) \sup_{\|z^*\|_{Z^*} \le 1} \left(\sum_{i \in N^{\overline{\{k\}}}} |z^*(U(x(i), x_k))|_Y^r \right)^{1/r}.$$

The Bohnenblust-Hille theorem can be applied to the multilinear functional $\varphi : x \mapsto z^*(U(x, x_k))$ for $x \in X^{\overline{\{k\}}}$ to see that φ is multiple $(r, 1)$-summing, and

$$\pi^{\text{mult}}_{r,1}(\varphi) \le BH_{m-1} \|\varphi\| \le BH_{m-1} \|z^*\|_{Z^*} \|U\| \|x_k\|_{X_k}.$$

Thus,

$$\Big(\sum_{i \in N^{\overline{\{k\}}}} \|T \circ U(x(i), x_k)\|_Y^r \Big)^{1/r} \leq \pi_r(T) \sup_{\|z^*\|_{Z^*} \leq 1} BH_{m-1} \|z^*\|_{Z^*} \|U\| \|x_k\|_{X_k}.$$

It follows that $T \circ U$ is multiple $(r, 1)$-summing in all coordinates except k, for $k = 1, \ldots, m$, and

$$\|(T \circ U)^{\overline{\{k\}}}\|_{CW(r,1)} \leq \pi_r(T) BH_{m-1} \|U\|.$$

Take $r_1 = \cdots = r_m = r$ in inequality (3.2) and verify that $S = \frac{2mq}{2+mq}$. We conclude that $T \circ U$ is multiple $(\frac{2mq}{2+mq}, 1)$-summing and

$$\pi^{\text{mult}}_{\frac{2mq}{2+mq},1}(T \circ U) \leq C_q(Y) K_{r,2} \pi_r(T) BH_{m-1} \|U\|.$$

□

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