MASKED FACTORABLE MATRICES

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ABSTRACT. The class of masked factorable matrices is introduced and simple necessary and sufficient conditions are given for matrices in the class to represent bounded transformations between Lebesgue sequence spaces.

1. INTRODUCTION

Define the matrix $M = (m_{n\,j})$ by $m_{n\,j} = u_n v_j$ for fixed non-negative sequences $\mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_j\}$ with neither $\mathbf{u}$ nor $\mathbf{v}$ identically zero. As a transformation between Lebesgue spaces the action of M is easy to analyse using only the sharpness of Hölder's inequality. The matrix M is a bounded map from l^p to l^q if and only if $\mathbf{u} \in l^q$ and $\mathbf{v} \in l^{p'}$ where $1/p + 1/p' = 1$. It is a much more interesting question to ask about the boundedness of M if some of its entries are replaced by zeros. We call this "masking" the matrix M. One simple rule for masking M is to replace $m_{n\,j}$ by 0 when $j > n$. This gives the factorable matrices studied in [1]. The situation in this case is by no means as simple as the unmasked case but it has been completely resolved.

In this paper we look at more general masking schemes and give easily verified conditions on the sequences $\mathbf{u}$ and $\mathbf{v}$ which are necessary and sufficient for these masked factorable matrices to be bounded as maps from l^p to l^q. This work is closely connected with corresponding results for integral operators given in [2,3,4] but the many simplifications and occasional complications give the discrete case its own unique character.

Our masking scheme is described by two sequences $\{a_n\}$ and $\{b_n\}$ taking values in $\mathbf{Z}^+_\infty$ and satisfying $a_n \le b_n$ for all $n \in \mathbf{Z}^+$. Masking in the nth row of M is

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determined by leaving those entries $m_{n\,j}$ with $a_n \le j \le b_n$ undisturbed and setting the others to zero. Thus, from now on we use the redefined entries

$$m_{n\,j} = \begin{cases} u_n v_j & a_n \le j \le b_n \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

to define the matrix M.

This scheme gives us a great deal of flexibility since, for example, we may leave the nth row intact by setting $a_n = 1$ and $b_n = \infty$ or we may eliminate it entirely by setting $a_n = b_n = \infty$. Factorable matrices may be recovered by setting $a_n = 1$ and $b_n = n$ for all n and their duals obtained by setting $a_n = n$ and $b_n = \infty$ for all n. There is a further restriction we must impose on the sequences $\{a_n\}$ and $\{b_n\}$ which is that the sequences admit a normalizing set, see Definition 2.1, but a great many sequences are permitted including all those for which $b_n - a_n$ is bounded and all those for which $\{a_n\}$ and $\{b_n\}$ are non-decreasing.

Necessary and sufficient conditions for the boundedness of M as a map from l^p to l^q are given in the next section and in Section 3 we look at the existence of normalizing sets and give various examples and special cases.

We use the following notation throughout: For the positive integers we write $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$ and for the ordered set of positive integers with infinity we write $\mathbf{Z}^+_\infty = \mathbf{Z}^+ \cup \{\infty\}$. The characteristic function of the set E is denoted χ_E so that $\chi_E(x)$ takes the value 1 if $x \in E$ and the value 0 otherwise. The expression $Y \lessapprox Z$ means that there exists a positive constant N depending only on the indices p and q and the constant c of Definition 2.1 such that $Y \le NZ$. We write $Y \approx Z$ as a short form of $Y \lessapprox Z$ and $Z \lessapprox Y$.

2. Boundedness of Masked Factorable Matrices

Fix sequences $\{a_n\}$ and $\{b_n\}$ with values in $\mathbf{Z}^+_\infty$ which satisfy $a_n \le b_n$ for all $n \in \mathbf{Z}^+$. For each $k \in \mathbf{Z}^+_\infty$ we define N_k and J_k by

$$N_k = \{n \in \mathbf{Z}^+ : a_n \le k \le b_n\} \quad \text{and} \quad J_k = \{j \in \mathbf{Z}^+_\infty : N_k \cap N_j \ne \emptyset\}.$$

Definition 2.1. Suppose that $K \subset \mathbf{Z}^+_\infty$. We say that K *normalizes* the pair $(\{a_n\}, \{b_n\})$ provided there exists a finite constant c such that

$$1 \le |\{k \in K : a_n \le k \le b_n\}| \le c \tag{2.1}$$

for all $n \in \mathbf{Z}^+$.

Observe that if K normalizes $(\{a_n\}, \{b_n\})$ then the condition (2.1) may be written as

$$1 \le |\{k \in K : n \in N_k\}| \le c \tag{2.2}$$

for all $n \in \mathbf{Z}^+$. That is, there exists at least one and at most c values of $k \in K$ for which $n \in N_k$.

Lemma 2.2. *If K normalizes $(\{a_n\},\{b_n\})$ then $|\{k \in K : j \in J_k\}| \le 2c$ for all $j \in \mathbf{Z}^+_\infty$. Here c is the constant from Definition 2.1.*

Proof. Fix $j \in \mathbf{Z}^+_\infty$. The symmetry in the definition of J_k shows that

$$\{k \in K : j \in J_k\} = K \cap J_j$$

and if $k \in J_j$ then $a_n \le k \le b_n$ for some $n \in N_j$ so

$$\inf\{a_n : n \in N_j\} \le k \le \sup\{b_n : n \in N_j\}.$$

Choose sequences $\{m_i\} \subset N_j$ and $\{n_i\} \subset N_j$ such that

$$\lim_{i\to\infty} a_{m_i} = \inf\{a_n : n \in N_j\} \quad \text{and} \quad \lim_{i\to\infty} b_{n_i} = \sup\{b_n : n \in N_j\}.$$

Since for each i, $a_{m_i} \le j \le b_{m_i}$ and $a_{n_i} \le j \le b_{n_i}$ we have $a_{n_i} \le b_{m_i}$ and so

$$\{k \in K : a_{m_i} \le k \le b_{n_i}\} \subset \{k \in K : a_{m_i} \le k \le b_{m_i}\} \cup \{k \in K : a_{n_i} \le k \le b_{n_i}\}.$$

Now

$$\begin{aligned}|K \cap J_j| &\le \lim_{i\to\infty} |\{k \in K : a_{m_i} \le k \le b_{n_i}\}| \\ &\le \lim_{i\to\infty} |\{k \in K : m_i \in N_k\}| + |\{k \in K : n_i \in N_k\}| \le 2c\end{aligned}$$

by (2.2)

We begin with two propositions that follow readily from known results.

Proposition 2.3. *Suppose that $1 < p < \infty$, $0 < q < \infty$, and $\{u_n\}$ and $\{v_j\}$ are non-negative sequences. Fix $k \in \mathbf{Z}^+_\infty$ and let C_k be the least constant, finite or infinite, such that the inequality*

$$\sum_{n\in N_k}\Big(u_n\sum_{j=k}^{b_n} v_j x_j\Big)^q \le C_k^q\Big(\sum_{j\in J_k} x_j^p\Big)^{q/p} \tag{2.3}$$

holds for all non-negative sequences $\{x_j\}$. Then $C_k \approx A_k$ when $1 < p \le q < \infty$ and $C_k \approx B_k$ when $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Here

$$A_k = \sup_{\{l : k \le l\}} \Big(\sum_{\substack{a_n \le k \\ l \le b_n}} u_n^q\Big)^{1/q}\Big(\sum_{j=k}^{l} v_j^{p'}\Big)^{1/p'}, \textit{ and}$$

$$B_k^r = \sum_{m\in N_k}\Big(\sum_{\substack{a_n \le k \\ b_m \le b_n}} u_n^q\Big)^{r/p}\Big(\sum_{j=k}^{b_m} v_j^{p'}\Big)^{r/p'} u_m^q.$$

Proof. First note that if $n \in N_k$ and $k \le j \le b_n$ then $n \in N_j$ so $j \in J_k$. Therefore, the value of x_j has no effect on the inequality (2.3) unless $j \in J_k$. We may rewrite the left hand side of (2.3) as

$$\sum_{n\in N_k}\left(u_n\sum_{j=k}^{b_n}v_jx_j\right)^q=\sum_{i=1}^{\infty}\sum_{\substack{n\in N_k\\ b_n=i}}\left(u_n\sum_{j=k}^{i}v_j\chi_{J_k}(j)x_j\right)^q=\sum_{i=1}^{\infty}\left(U_i\sum_{j=1}^{i}V_jx_j\right)^q$$

where $U_i=(\sum_{\substack{n\in N_k\\ b_n=i}}u_n^q)^{1/q}$ and $V_j=v_j\chi_{J_k}(j)\chi_{[k,\infty]}(j)$.

If the inequality (2.3) holds for some non-negative sequence $\{x_j\}$ then we also have

$$\sum_{i=1}^{\infty}\left(U_i\sum_{j=1}^{i}V_jx_j\right)^q\le C_k^q\left(\sum_{j=1}^{\infty}x_j^p\right)^{q/p} \tag{2.4}$$

since we have just extended the range of summation on the right hand side. Conversely, if (2.4) holds for all non-negative sequences $\{x_j\}$ then it holds for those non-negative sequences for which $x_j=0$ when $j\notin J_k$. Thus (2.3) holds. The best constant C_k in the inequality (2.4) and hence in (2.3) is known to satisfy

$$C_k\approx\sup_{l\ge1}\left(\sum_{i=l}^{\infty}U_i^q\right)^{1/q}\left(\sum_{j=1}^{l}V_j^{p'}\right)^{1/p'}$$

when $1<p\le q<\infty$ and

$$C_k^r\approx\sum_{l=1}^{\infty}\left(\sum_{i=l}^{\infty}U_i^q\right)^{r/p}\left(\sum_{j=1}^{l}V_j^{p'}\right)^{r/p'}U_l^q$$

when $0<q<p$ and $1<p<\infty$. For this result refer to [1, Theorem 1] or [5, Theorem 7.1]. Replacing U_i and V_j by their definitions and simplifying yields the conclusion.

The next proposition is proved in a similar fashion. We omit the details.

Proposition 2.4. *Suppose that $1<p<\infty$, $0<q<\infty$, and $\{u_n\}$ and $\{v_j\}$ are non-negative sequences. Fix $k\in\mathbf{Z}_\infty^+$ and let C_k' be the least constant, finite or infinite, such that the inequality*

$$\sum_{n\in N_k}\left(u_n\sum_{j=a_n}^{k}v_jx_j\right)^q\le {C_k'}^q\left(\sum_{j\in J_k}x_j^p\right)^{q/p}$$

holds for all non-negative sequences $\{x_j\}$. Then $C_k'\approx A_k'$ when $1<p\le q<\infty$ and $C_k'\approx B_k'$ when $0<q<p$, $1<p<\infty$ and $1/r=1/q-1/p$. Here

$$A_k'=\sup_{\{l:l\le k\}}\left(\sum_{\substack{a_n\le l\\ k\le b_n}}u_n^q\right)^{1/q}\left(\sum_{j=l}^{k}v_j^{p'}\right)^{1/p'},\ \textit{and}$$

$${B_k'}^r=\sum_{m\in N_k}\left(\sum_{\substack{a_n\le a_m\\ k\le b_n}}u_n^q\right)^{r/p}\left(\sum_{j=a_m}^{k}v_j^{p'}\right)^{r/p'}u_m^q.$$

We are now ready to state and prove the main result.

Theorem 2.5. *Let $1 < p < \infty$, $0 < q < \infty$, $\{u_n\}$ and $\{v_j\}$ be non-negative sequences, and $\{a_n\}$ and $\{b_n\}$ be sequences taking values in $\mathbf{Z}^+_\infty$ which satisfy $a_n \le b_n$ for all $n \in \mathbf{Z}^+$. Suppose that K normalizes $(\{a_n\},\{b_n\})$. Define C to be the least constant, finite or infinite, such that the inequality*

$$\sum_{n=1}^{\infty}\Big(u_n\sum_{j=a_n}^{b_n} v_j x_j\Big)^q \le C^q\Big(\sum_{j=1}^{\infty} x_j^p\Big)^{q/p} \tag{2.5}$$

holds for all non-negative sequences $\{x_j\}$. Then $C \approx A$ when $1 < p \le q < \infty$, and $C \approx B + \bar{B}$ when $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Here

$$A = \sup_{\{(k,l):k\le l\}}\Big(\sum_{\substack{a_n\le k\\ l\le b_n}} u_n^q\Big)^{1/q}\Big(\sum_{j=k}^{l} v_j^{p'}\Big)^{1/p'},$$

$$B^r = \sum_{k\in K}\sum_{m\in N_k}\Big(\sum_{\substack{a_n\le k\\ b_m\le b_n}} u_n^q\Big)^{r/p}\Big(\sum_{j=k}^{b_m} v_j^{p'}\Big)^{r/p'} u_m^q, \text{ and}$$

$$\bar{B}^r = \sum_{k\in K}\sum_{m\in N_k}\Big(\sum_{\substack{a_n\le a_m\\ k\le b_n}} u_n^q\Big)^{r/p}\Big(\sum_{j=a_m}^{k} v_j^{p'}\Big)^{r/p'} u_m^q.$$

Proof. Throughout the proof we use the definitions of A_k, B_k, C_k, A'_k, B'_k, and C'_k given in Propositions 2.3 and 2.4.
(Sufficiency) Fix a non-negative sequence x_j with $\sum_{j=1}^\infty x_j^p \le 1$ and define y_k by $y_k^p = \sum_{j\in J_k} x_j^p$. By Lemma 2.2,

$$\sum_{k\in K} y_k^p = \sum_{k\in K}\sum_{j\in J_k} x_j^p = \sum_{j=1}^{\infty} x_j^p |\{k\in K : j\in J_k\}| \le 2c.$$

Let I^q denote the left hand side of (2.5). We decompose I into two parts. According to (2.2), $|\{k \in K : n\in N_k\}| \ge 1$ for each n, so we have

$$\begin{aligned}
I =&\Big(\sum_{n=1}^{\infty}\Big(u_n\sum_{j=a_n}^{b_n} v_j x_j\Big)^q\Big)^{1/q}\\
\le&\Big(\sum_{n=1}^{\infty}\sum_{\{k\in K:n\in N_k\}}\Big(u_n\sum_{j=k}^{b_n} v_j x_j + u_n\sum_{j=a_n}^{k} v_j x_j\Big)^q\Big)^{1/q}\\
\le& c_1\left[\Big(\sum_{k\in K}\sum_{n\in N_k}\Big(u_n\sum_{j=k}^{b_n} v_j x_j\Big)^q\Big)^{1/q} + \Big(\sum_{k\in K}\sum_{n\in N_k}\Big(u_n\sum_{j=a_n}^{k} v_j x_j\Big)^q\Big)^{1/q}\right].
\end{aligned}$$

Here $c_1 = \max(1, 2^{(1-q)/q})$.

Now we apply Propositions 2.3 and 2.4 to get

$$I \le c_1 \left[\Big(\sum_{k\in K} C_k^q y_k^q \Big)^{1/q} + \Big(\sum_{k\in K} {C'_k}^q y_k^q \Big)^{1/q} \right]. \tag{2.6}$$

If $1 < p \le q < \infty$ then $C_k \approx A_k$ and $C'_k \approx A'_k$ and it is easy to see that $A_k \le A$ and $A'_k \le A$ for each k so, using the fact that $q/p \ge 1$, we have

$$I \lessapprox A \Big(\sum_{k\in K} y_k^q \Big)^{1/q} \le A \Big(\sum_{k\in K} y_k^p \Big)^{1/p} \le A(2c)^{1/p}.$$

It follows that $C \lessapprox A$.

If $0 < q < p$ and $1 < p < \infty$ then we have $C_k \approx B_k$ and $C'_k \approx B'_k$ and we use Hölder's inequality with indicies r/q and p/q in (2.6) to obtain

$$I \lessapprox \left(\Big(\sum_{k\in K} B_k^r \Big)^{1/r} + \Big(\sum_{k\in K} {B'_k}^r \Big)^{1/r} \right) \Big(\sum_{k\in K} y_k^p \Big)^{1/p} \le (B + \bar{B})(2c)^{1/p}.$$

In this case it follows that $C \lessapprox B + \bar{B}$.

(Necessity) In the case $1 < p \le q < \infty$ we suppose that (2.5) holds with some finite constant C. Our object is to show that $A \lessapprox C$. Fix k and suppose that $\{x_j\}$ is a non-negative sequence with $x_j = 0$ for $j \notin J_k$. It is easy to see that (2.5) implies that (2.3) holds with C_k replaced by C. Since C_k is the least constant in (2.3) we have $C_k \le C$. By Proposition 2.3, $A_k \approx C_k \le C$. Since $A = \sup_{k\in K} A_k$ we have $A \lessapprox C$ as desired.

In the case $0 < q < p$, $1 < p < \infty$ we again suppose that the inequality (2.5) holds for some finite constant C and we make it our object to show that $B \lessapprox C$ and $\bar{B} \lessapprox C$. Let $\{X_k\}$ be a non-negative sequence such that $\sum_{k\in K} X_k^p < \infty$ and $X_k < C_k^{r/p}$. (If $C_k = 0$ we take $X_k = 0$ as well.)

For each k choose a non-negative sequence $\{x_{k\,j}\}$ such that $\sum_{j\in J_k} x_{k\,j} \le 1$, $x_{k\,j} = 0$ for $j \notin J_k$, and

$$\sum_{n\in N_k} \Big(u_n \sum_{j=k}^{b_n} v_j x_{k\,j} \Big)^q \ge X_k^{qp/r}$$

Then we may use the definition of N_k to get

$$\begin{aligned} \sum_{k\in K} X_k^p &= \sum_{k\in K} X_k^{qp/r} X_k^q \le \sum_{k\in K} \sum_{n\in N_k} \Big(u_n \sum_{j=k}^{b_n} v_j x_{k\,j} \Big)^q X_k^q \\ &= \sum_{n=1}^{\infty} \sum_{\substack{k=a_n \\ k\in K}}^{b_n} \Big(u_n \sum_{j=k}^{b_n} v_j x_{k\,j} X_k \Big)^q \le \sum_{n=1}^{\infty} \sum_{\substack{k=a_n \\ k\in K}}^{b_n} \Big(u_n \sum_{j=a_n}^{b_n} v_j x_{k\,j} X_k \Big)^q. \end{aligned}$$

Now the hypothesis on K shows that the last sum over k is a sum of at most c terms. Hence the last expression is no greater than

$$c\sum_{n=1}^{\infty}\Big(\sum_{\substack{k=a_n\\k\in K}}^{b_n} u_n \sum_{j=a_n}^{b_n} v_j x_{k\,j}X_k\Big)^q \le c\sum_{n=1}^{\infty}\Big(u_n\sum_{j=a_n}^{b_n} v_j \sum_{k\in K} x_{k\,j}X_k\Big)^q.$$

We are now in a position to apply the hypothesis that $C<\infty$ by applying the inquality (2.5) with x_j replaced by $\sum_{k\in K} x_{k\,j}X_k$. We obtain

$$\sum_{k\in K} X_k^p \le cC^q\Big(\sum_{j=1}^{\infty}\Big(\sum_{k\in K} x_{k\,j}X_k\Big)^p\Big)^{q/p}. \tag{2.7}$$

Because $x_{k\,j}=0$ for $j\notin J_k$ we may apply Hölder's inequality and Lemma 2.2 to see that the right hand side of (2.7) is no greater than

$$\begin{aligned}&cC^q\Big(\sum_{j=1}^{\infty}\Big(\sum_{k\in K} x_{k\,j}^pX_k^p\Big)|\{k\in K : j\in J_k\}|^{p/p'}\Big)^{q/p}\\ &\qquad\le c(2c)^{q/p'}C^q\Big(\sum_{k\in K}\Big(\sum_{j\in J_k} x_{k\,j}^p\Big)X_k^p\Big)^{q/p}\le c(2c)^{q/p'}C^q\Big(\sum_{k\in K}X_k^p\Big)^{q/p}\end{aligned}$$

where the last inequality follows from the choice of $\{x_{k\,j}\}$.

Using this estimate for the right hand side of (2.7), taking qth roots, and dividing by the pth root of $\sum_{k\in K} X_k^p$ we conclude that

$$\Big(\sum_{k\in K} X_k^p\Big)^{1/r}\le c^{1/q}(2c)^{1/p'}C.$$

Since this last inequality holds whenever $0\le X_k<C_k^{r/p}$ and $\sum_{k\in K}X_k^p<\infty$ we have

$$\Big(\sum_{k\in K} C_k^r\Big)^{1/r}\lessapprox C.$$

Now $B^r=\sum_{k\in K}B_k^r$ and $B_k\approx C_k$ so $B\lessapprox C$ as required. A similar argument shows that $\bar B\lessapprox C$. This completes the proof.

The inequality (2.5) expresses the boundedness of the matrix M so Theorem 2.5 has the following corollary.

Corollary 2.6. *Let $1<p<\infty$, $0<q<\infty$, $\{u_n\}$ and $\{v_j\}$ be non-negative sequences, and $\{a_n\}$ and $\{b_n\}$ be sequences taking values in $\mathbf{Z}^+_\infty$ which satisfy $a_n\le b_n$ for all $n\in\mathbf{Z}^+$. Suppose that K normalizes $(\{a_n\},\{b_n\})$. The matrix $M=(m_{n\,j})$ defined by* (1.1) *is a bounded map from l^p to l^q if and only if either $p\le q$ and $A<\infty$ or $q<p$ and $B+\bar B<\infty$.*

We conclude this section by giving necessary and sufficient conditions without proof for various endpoint cases. The proofs are quite simple and do not require the hypothesis that $(\{a_n\},\{b_n\})$ has a normalizing set.

Corollary 2.7. *Let $\{u_n\}$ and $\{v_j\}$ be non-negative sequences and $\{a_n\}$ and $\{b_n\}$ be sequences taking values in $\mathbf{Z}^+_\infty$ which satisfy $a_n \le b_n$ for all $n \in \mathbf{Z}^+$. Let $M = (m_{n\,j})$ be defined by* (1.1). *Then $M : l^\infty \to l^q$ for $0 < q \le \infty$ if and only if*

$$\left\{ u_n \sum_{j=a_n}^{b_n} v_j \right\}_n \in l^q,$$

$M : l^p \to l^\infty$ for $0 < p < \infty$ if and only if

$$\left\{ u_n^{p'} \sum_{j=a_n}^{b_n} v_j^{p'} \right\}_n \in l^\infty,$$

$M : l^1 \to l^q$ for $1 \le q \le \infty$ if and only if

$$\left\{ v_j^q \sum_{n \in N_j} u_n^q \right\}_j \in l^\infty,$$

and $M : l^1 \to l^\infty$ if and only if

$$\left\{ u_n v_j \chi_{N_j}(n) \right\}_{n\,j} \in l^\infty.$$

3. Normalizing Sets and Examples

The requirement that $(\{a_n\}, \{b_n\})$ admit a normalizing set K is satisfied for a great many pairs of sequences. Moreover, it is often a simple matter to discover such a set K for a given pair of sequences. We begin this section with several examples of normalizing sets and a fairly general existence result. An application of Theorem 2.5 to embeddings of weighted sequential amalgams is also given.

The proofs of the first two examples are left to the reader.

Example 3.1. *Suppose a and b are positive integers with $a < b$. Then $\{bi : i = 1, 2, \dots\}$ is a normalizing set for $(\{an\}, \{an + b\})$ and $\{\lfloor (b/a)^i \rfloor : i = 0, 1, 2, \dots\}$ is a normalizing set for $(\{an\}, \{bn\})$. Here $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.*

Example 3.2. *If $\{a_n\}$ and $\{b_n\}$ are sequences with values in $\mathbf{Z}^+$ and there exists a $Z > 0$ such that $0 \le b_n - a_n \le Z$ for all n then $K = \mathbf{Z}^+$ is a normalizing set for $(\{a_n\}, \{b_n\})$.*

Proposition 3.3. *If $\{a_n\}$ and $\{b_n\}$ are non-decreasing sequences with values in $\mathbf{Z}^+_\infty$ which satisfy $a_n \le b_n$ for all n then there exists a set $K \subset \mathbf{Z}^+_\infty$ which normalizes $(\{a_n\}, \{b_n\})$.*

The last proposition is a special case of the next one. We introduce a partial order on intervals $[l, r]$ with $1 \le l \le r \le \infty$ by writing $[l, r] \prec [L, R]$ provided $l \le L$ and $r \le R$. Theorem 5.2 of [2] easily implies the following.

Proposition 3.4. *If $\{a_n\}$ and $\{b_n\}$ are sequences with values in $\mathbf{Z}_\infty^+$ which satisfy $a_n \le b_n$ for all n and the set $\{[a_n, b_n] : n \in \mathbf{Z}^+\}$ is totally ordered with respect to $\prec$ then there exists a set $K \subset \mathbf{Z}_\infty^+$ which normalizes $(\{a_n\}, \{b_n\})$.*

We now turn to an application of Theorem 2.5. Fix a sequence $\mathbf{t} = \{t_m\}$ satisfying $1 = t_1 < t_2 < \dots$ and sequences $\mathbf{u}$ and $\mathbf{v}$ of positive terms. If $q, s \ge 1$ the *weighted sequential amalgam space* $l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}$ is the collection of those sequences $\mathbf{y} = \{y_j\}$ for which the norm

$$\|\mathbf{y}\|_{l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}} = \Bigg(\sum_{n=1}^{\infty} u_n \Bigg(\sum_{j=t_n}^{t_{n+1}-1} v_j |y_j|^s \Bigg)^{q/s} \Bigg)^{1/q}$$

is finite. Theorem 2.5 can be used to determine which weighted l^p spaces can be embedded in $l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}$ when $s < p$. Recall that the norm on the weighted space $l^p_{\mathbf{w}}$ is $\|\mathbf{y}\|_{l^p_{\mathbf{w}}} = (\sum_{j=1}^{\infty} w_j |y_j|^p)^{1/p}$ for any sequence $\mathbf{w}$ of non-negative terms.

Although the following result can be proved directly, it is instructive to see how the weight conditions of Theorem 2.5 simplify in this important special case.

Theorem 3.5. *Suppose $1 \le s < p$, $1 < q < \infty$ and $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are as above. Then $l^p_{\mathbf{w}}$ is embedded in $l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}$ when $p \le q$ if and only if*

$$\sup_{n \ge 1} u_n^{s/q} \Bigg(\sum_{j=t_n}^{t_{n+1}-1} v_j^{p/(p-s)} w_j^{s/(s-p)} \Bigg)^{(p-s)/p} < \infty. \tag{3.1}$$

Also, $l^p_{\mathbf{w}}$ is embedded in $l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}$ when $q < p$ if and only if

$$\sum_{n=1}^{\infty} u_n^{r/q} \Bigg(\sum_{j=t_n}^{t_{n+1}-1} v_j^{p/(p-s)} w_j^{s/(s-p)} \Bigg)^{r(p-s)/(ps)} < \infty. \tag{3.2}$$

Here $1/r = 1/q - 1/p$.

Proof. The embedding $l^p_{\mathbf{w}} \hookrightarrow l^q_{\mathbf{u}}(l^s_{\mathbf{v}})_{\mathbf{t}}$ holds if and only if there exists a constant C such that

$$\Bigg(\sum_{n=1}^{\infty} u_n \Bigg(\sum_{j=t_n}^{t_{n+1}-1} v_j |y_j|^s \Bigg)^{q/s} \Bigg)^{1/q} \le C \Bigg(\sum_{j=1}^{\infty} w_j |y_j|^p \Bigg)^{1/p} \tag{3.3}$$

holds for all sequences $\mathbf{y}$. Setting $x_j = w_j^{s/p} |y_j|^s$ and raising both sides of (3.1) to the power s shows that (3.3) is equivalent to

$$\Bigg(\sum_{n=1}^{\infty} \Bigg(u_n^{s/q} \sum_{j=t_n}^{t_{n+1}-1} v_j w_j^{-s/p} x_j \Bigg)^{q/s} \Bigg)^{s/q} \le C^s \Bigg(\sum_{j=1}^{\infty} x_j^{p/s} \Bigg)^{s/p} \tag{3.4}$$

holding for all non-negative sequences $\mathbf{x}$. We apply Theorem 2.5 with $a_n = t_n$, $b_n = t_{n+1} - 1$, and p, q, u_n, and v_j replaced by p/s, q/s, $u_n^{s/q}$, and $v_j w_j^{-s/p}$ respectively.

It is easy to check that $K = \{t_m : m = 1, 2, \dots\}$ normalizes $(\{t_n\}, \{t_{n+1} - 1\})$. Our conclusion is that (3.4) is equivalent when $p \le q$ (or rather when $p/s \le q/s$) to the finiteness of

$$\sup_{\{(k,l):k \le l\}} \Bigg(\sum_{\substack{t_n \le k \\ l \le t_{n+1}-1}} u_n \Bigg)^{s/q} \Bigg(\sum_{j=k}^{l} [v_j w_j^{-s/p}]^{(p/s)'} \Bigg)^{1/(p/s)'}.$$

This expression simplifies because the intervals $[t_n, t_{n+1} - 1]$ do not overlap as n varies. Thus, for fixed k and l with $k \le l$ there is at most one value of n for which $t_n \le k \le t_{n+1} - 1$. Moreover, if there is such an n then the second factor is largest when $k = t_n$ and $l = t_{n+1} - 1$. We can, therefore, replace the supremum over k and l with a supremum over n to get

$$\sup_{n \ge 1} u_n^{s/q} \Bigg(\sum_{j=t_n}^{t_{n+1}-1} [v_j w_j^{-s/p}]^{(p/s)'} \Bigg)^{1/(p/s)'} < \infty.$$

This becomes (3.1) once we check that $(p/s)' = p/(p - s)$.

In the case $q < p$ the conclusion of Theorem 2.5 is that (3.4) holds if and only if both

$$\sum_{k \in K} \sum_{m \in N_k} \Bigg(\sum_{\substack{t_n \le k \\ m \le n}} u_n \Bigg)^{r/p} \Bigg(\sum_{j=k}^{t_{m+1}-1} v_j^{p/(p-s)} w_j^{s/(s-p)} \Bigg)^{r(p-s)/(ps)} u_m \tag{3.5}$$

and

$$\sum_{k \in K} \sum_{m \in N_k} \Bigg(\sum_{\substack{n \le m \\ k \le t_{n+1}-1}} u_n \Bigg)^{r/p} \Bigg(\sum_{j=t_m}^{k} v_j^{p/(p-s)} w_j^{s/(s-p)} \Bigg)^{r(p-s)/(ps)} u_m \tag{3.6}$$

are finite. (Note that if $1/r = 1/q - 1/p$ then $s/r = s/q - s/p$ so the r of Theorem 2.5 is appropriately replaced by r/s here.) Once again, these simplify because the intervals $[t_n, t_{n+1} - 1]$ are disjoint. Because $k \in K$, k must be of the form t_i for some i and the condition $m \in N_k$ is $t_m \le t_i \le t_{m+1} - 1$ which forces $m = i$. The double sum becomes a single sum and (3.5) simplifies to (3.2). If we simplify (3.6) in the same way the resulting expression is clearly dominated by (3.2).

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