HARDY'S INEQUALITY AND MONOTONICITY

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Abstract. Simple and natural proofs are given for the characterization of embeddings of the cones of monotone functions between Lebesgue spaces with different indices and general measures. Also, Hardy inequalities with general measures are shown to break into two parts, the action of a Hardy operator on a single Lebesgue space followed by an embedding for monotone functions. This decomposition is used to provide simple new proofs of the known characterizations for the Hardy inequality and to give some new necessary and sufficient conditions for such inequalities to hold. The characterization extends to Geometric Mean Inequalities and to Hardy inequalities for negative indices.

1. Introduction: One Hardy Inequality

Fix a measure λ on $\mathbf{R}$ for which intervals are measurable and define

$$\Lambda(x) \equiv \int_{(-\infty,x]} d\lambda \quad \text{and} \quad \bar{\Lambda}(x) \equiv \int_{[x,\infty)} d\lambda, \quad x \in \mathbf{R}.$$

Theorem 1.1. *Let $1 < p < \infty$. For all λ-measurable f,*

$$\left(\int \left| \frac{1}{\Lambda(x)} \int_{(-\infty,x]} f \, d\lambda \right|^p d\lambda(x) \right)^{1/p} \leq \frac{p}{p-1} \left(\int |f|^p \, d\lambda \right)^{1/p}.$$

Before we proceed to the proof of this inequality, a discussion of its significance is in order. Between 1915 and 1934, G. H. Hardy proved this result for sequences, for functions on the half line, and for functions in Lebesgue spaces with power weights. From the early 1970's onward a great many related results were established under

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the general heading of Hardy-type inequalities. This expansion has been primarily concerned with providing easily verified necessary and sufficient conditions under which the various increasingly general and complicated inequalities should hold.

I hope that the inequality above, which we will prove presently using only techniques available in 1934, will help to simplify some of the work on Hardy-type inequalities. There are no conditions to check, the inequality holds for all measures for which the inner integral makes sense. Moreover, as we show in Section 3, this is in some sense the only Hardy inequality that is needed since all others factor through this one, with the other factor being an embedding for a cone of monotone functions.

The other factor is not neglected. Section 2 contains a new and simple approach to embeddings for such cones. As illustrations of the simplicity of the method, related inequalities for the Geometric Mean Operator and Hardy-type inequalities with negative indices are given in Section 4.

In such a large field it is difficult to give comprehensive references. We refer the reader to the monographs [3] and [4] and the references therein.

Notation is quite standard. For $1 < p < \infty$ we let $p' = p/(p-1)$ and let $L^p(\lambda)$ be the Banach space of all λ-measurable functions f for which

$$\|f\|_{L^p(\lambda)} \equiv \left(\int |f|^p \, d\lambda \right)^{1/p} < \infty.$$

All integrals are over the real line unless otherwise specified. We write $F \downarrow$ or $F \uparrow$ to indicate that the function F is in the collection of all non-negative, non-increasing functions or all non-negative, non-decreasing functions, respectively. The constant C may be different at different occurrences. The notation $A \approx B$ indicates that there are constants c and C such that $cA \le B \le CA$.

Proof of Theorem 1.1. Let f^* denote the non-increasing rearrangement of f with respect to the measure λ. That is,

$$f^*(t) = \inf\{\alpha > 0 : \lambda\{y : |f(y)| > \alpha\} \le t\}.$$

Observe that for each $x \in \mathbf{R}$, $\left(\chi_{(-\infty,x]}\right)^* = \chi_{(0,\Lambda(x)]}$. It is well known (see [1]) that for λ-measurable functions f and h,

$$\int |f|^p \, d\lambda = \int_0^\infty (f^*)^p \quad \text{and} \quad \left| \int fh \, d\lambda \right| \le \int_0^\infty f^* h^*.$$

Applying the second of these with $h = \chi_{(-\infty,x]}$ yields

$$\left| \frac{1}{\Lambda(x)} \int_{(-\infty,x]} f \, d\lambda \right| \le \frac{1}{\Lambda(x)} \int_0^{\Lambda(x)} f^* = (F \circ \Lambda)(x) \tag{1.1}$$

where $F(t) = \frac{1}{t} \int_0^t f^*$. Note that F is a non-negative, non-increasing function on $[0, \infty)$. Since

$$\lambda\{y : F(\Lambda(y)) > F(t)\} \le \lambda\{y : \Lambda(y) < t\} \le t$$

we have $(F\circ\Lambda)^*(t)\le F(t)$. Hardy's inequality [2, Theorem 330] shows that

$$\int (F\circ\Lambda)^p\,d\lambda \le \int_0^\infty \left(\frac{1}{t}\int_0^t f^*\right)^p dt \le (p')^p\int_0^\infty (f^*)^p = (p')^p\int |f|^p\,d\lambda.$$

Combining this with (1.1) completes the proof.

The Hardy averaging operator P and its dual Q, defined by

$$Pf(x)=\frac{1}{\Lambda(x)}\int_{(-\infty,x]} f\,d\lambda \quad\text{and}\quad Qf(x)=\int_{[x,\infty)} f\frac{d\lambda}{\Lambda},$$

will be central to our discussion so the remainder of the section will be devoted to their properties, beginning with two obvious but important ones.

$$\text{If } f\ge 0 \text{ then } Qf\ge 0 \text{ and } Qf \text{ is non-increasing.}$$

$$\text{If } f\ge 0 \text{ is non-increasing then } Pf\ge 0 \text{ is non-increasing and } f\le Pf.$$

Next we note that the Hardy inequality of Theorem 1.1 expresses the boundedness of P and Q on the Lebesgue spaces $L^p(\lambda)$.

Corollary 1.2. *Suppose $1<p<\infty$. Then*

$$\|Pf\|_{L^p(\lambda)}\le p'\|f\|_{L^p(\lambda)} \quad\textit{and}\quad \|Qf\|_{L^p(\lambda)}\le p\|f\|_{L^p(\lambda)}$$

for all $f\in L^p(\lambda)$.

Proof. The first statement is just Theorem 1.1 and we deduce the second from the first, with p replaced by p', by employing a standard duality argument. If $\|h\|_{L^{p'}(\lambda)}\le 1$ then

$$\left|\int h(Qf)\,d\lambda\right| = \left|\int (Ph)f\,d\lambda\right| \le \|Ph\|_{L^{p'}(\lambda)}\|f\|_{L^p(\lambda)}\le p\|f\|_{L^p(\lambda)}.$$

Taking the supremum over all such h completes the proof.

A simple calculation shows that $PQ=P+Q$ but for some measures it may not be the case that $QP=P+Q$. Indeed, we easily check that

$$QPf(x)=\left(\int_{[x,\infty)}\frac{d\lambda}{\Lambda^2}\right)\int_{(-\infty,x]} f(t)\,d\lambda(t)+\int_{[x,\infty)} f(t)\left(\int_{[t,\infty)}\frac{d\lambda}{\Lambda^2}\right)d\lambda(t) \tag{1.2}$$

but the hoped-for reduction to $Pf(x)+Qf(x)$ requires that Λ^{-2} integrate to $\Lambda(x)^{-1}$ on $[x,\infty)$. This property obviously fails for finite measures, even for absolutely continuous measures where the Fundamental Theorem of Calculus may be employed. However, it may also fail for certain infinite measures, for more subtle reasons. Of course, integration shows that

$$\int_{[x,\infty)}\frac{d\lambda}{\Lambda^2}=\frac{1}{\Lambda(x)},$$

and hence $QP=P+Q$, when λ is an infinite, absolutely continuous measure.

Definition 1.3. *Let $1 < p < \infty$. We say $\lambda \in \mathcal{I}_p(\infty)$ provided*

$$\Lambda(x)^{1-p} - \Lambda(\infty)^{1-p} \le C \int_{[x,\infty)} \Lambda^{-p}\, d\lambda$$

for some constant C. Similarly, $\lambda \in \mathcal{I}_p(-\infty)$ provided

$$\bar{\Lambda}(x)^{1-p} - \bar{\Lambda}(-\infty)^{1-p} \le C \int_{(-\infty,x]} \bar{\Lambda}^{-p}\, d\lambda$$

for some constant C.

The class $\mathcal{I}_p(\infty)$ includes all non-atomic measures and a great many others. However, as we show in Example 4.4, not all measures are in $\mathcal{I}_p(\infty)$.

The remark that motivated Definition 1.3 now yields

Lemma 1.4. *If $\lambda \in \mathcal{I}_2(\infty)$ then for all $f \ge 0$,*

$$Pf + Qf \le C(QPf) + \frac{1}{\Lambda(\infty)} \int f\, d\lambda$$

for some constant $C \ge 1$.

Proof. Using Definition 1.3 to continue the calculation (1.2) we have

$$\begin{aligned}
&C(QPf(x)) \\
&\ge \left(\frac{1}{\Lambda(x)} - \frac{1}{\Lambda(\infty)}\right) \int_{(-\infty,x]} f(t)\, d\lambda(t) + \int_{[x,\infty)} f(t) \left(\frac{1}{\Lambda(t)} - \frac{1}{\Lambda(\infty)}\right) d\lambda(t) \\
&= Pf(x) + Qf(x) - \frac{1}{\Lambda(\infty)} \int f\, d\lambda.
\end{aligned}$$

Clearly, we may take $C \ge 1$ if desired.

We have seen that the operator Q produces non-increasing functions from non-negative ones. Now we observe that the image under Q of the non-negative functions is a large subset of the cone of non-increasing functions. The result follows from Lemma 1.2 of [7] where it was given for non-decreasing functions.

Proposition 1.5. *If F is a non-negative, non-increasing function then there exist non-negative functions $f_1, f_2, \ldots$ such that $Qf_n(x)$ increases to $F(x)$ for λ-almost every $x \in \mathbf{R}$.*

Finally, we will need some properties of the level function f^o. See [6, 7] for details.

Proposition 1.6. *If f is a non-negative λ-measurable function which is bounded and compactly supported then there is a non-negative, non-increasing function f^o such that $Pf \le P(f^o)$ and $\|f^o\|_{L^p(\lambda)} \le \|f\|_{L^p(\lambda)}$ for $1 < p < \infty$.*

2. Embedding the Cones of Monotone Functions

The first three theorems in this section give a complete description of when the cone of non-increasing functions of one Lebesgue space is embedded in another Lebesgue space. These results are known for absolutely continuous measures and for sequences but the proofs given here are more general and much simpler. Theorems 2.1 and 2.2 apply for all measures on **R** and Theorem 2.3 applies in almost as great a generality and thus gives a widely applicable version of the most popular and most useful form of the embedding characterization.

The fact that Theorem 2.3 does not apply for all measures shows that certain difficulties encountered in the known embeddings for sequence spaces but not in those for weighted spaces are essential features of the theory. It also explains why these problems are not encountered in the case of weighted spaces.

In the remaining three theorems of the section we collect the corresponding results for the cone of non-decreasing functions.

Theorem 2.1. *If* $0 < p \le q < \infty$ *then*

$$\sup_{F\downarrow} \frac{\left(\int F^q \, d\mu\right)^{1/q}}{\left(\int F^p \, d\lambda\right)^{1/p}} = \sup_x \frac{\left(\int_{(-\infty,x]} d\mu\right)^{1/q}}{\left(\int_{(-\infty,x]} d\lambda\right)^{1/p}} \tag{2.1}$$

Proof. Replacing F^p by F in (2.1) readily reduces the assertion to the case $p = 1 \le q$. For each $x \in \mathbf{R}$, $\chi_{(-\infty,x]}$ is non-increasing so one inequality is obvious. For the other, fix a non-negative, non-increasing F and apply Proposition 1.5 to find $f_n \ge 0$ such that Qf_n increases to F λ-almost everywhere. If A denotes the right-hand side of (2.1) then by the Monotone Convergence Theorem and Minkowski's integral inequality,

$$\begin{aligned}
\left(\int F^q \, d\mu\right)^{1/q} &= \lim_{n\to\infty} \left(\int \left(\int_{[t,\infty)} f_n(x) \, \frac{d\lambda(x)}{\Lambda(x)}\right)^q d\mu(t)\right)^{1/q} \\
&\le \lim_{n\to\infty} \int f_n(x) \left(\int_{(-\infty,x]} d\mu(t)\right)^{1/q} \frac{d\lambda(x)}{\Lambda(x)} \\
&\le A \lim_{n\to\infty} \int f_n(x) \int_{(-\infty,x]} d\lambda(t) \, \frac{d\lambda(x)}{\Lambda(x)} \\
&= A \lim_{n\to\infty} \int \int_{[t,\infty)} f_n(x) \, \frac{d\lambda(x)}{\Lambda(x)} \, d\lambda(t) \\
&= A \int F \, d\lambda.
\end{aligned}$$

Theorem 2.2. *If* $0 < q < p < \infty$ *and* $1/r = 1/q - 1/p$ *then*

$$\sup_{F\downarrow} \frac{\left(\int F^q\,d\mu\right)^{1/q}}{\left(\int F^p\,d\lambda\right)^{1/p}} \approx \left(\int \left(\int_{[x,\infty)} \frac{d\mu}{\Lambda}\right)^{r/q} d\lambda(x)\right)^{1/r}. \tag{2.2}$$

Proof. We can extend the operators P and Q to measures. If μ and ν are measures on $\mathbf{R}$ define

$$\dot{P}\mu(x) = \frac{1}{\Lambda(x)} \int_{(-\infty,x]} d\mu \quad \text{and} \quad \dot{Q}\nu(t) = \int_{[x,\infty)} \frac{d\nu}{\Lambda}.$$

Clearly

$$\dot{P}(f\lambda) = Pf, \quad \dot{Q}(g\lambda) = Qg, \quad \text{and} \quad \int (\dot{P}\mu)\nu = \int (\dot{Q}\nu)\mu.$$

Replacing F^q by F in (2.2) reduces the assertion to the case $q = 1 \le p$. In this case $r/q = p'$. If F is non-increasing and $\left(\int F^p\,d\lambda\right)^{1/p} \le 1$ then $F \le PF$ so

$$\int F\,d\mu \le \int PF\,d\mu = \int F(\dot{Q}\mu)\,d\lambda \le \left(\int (\dot{Q}\mu)^{p'}\,d\lambda\right)^{1/p'}$$

and taking the supremum over all such F yields

$$\sup_{F\downarrow} \frac{\int F\,d\mu}{\left(\int F^p\,d\lambda\right)^{1/p}} \le \left(\int \left(\int_{[x,\infty)} \frac{d\mu}{\Lambda}\right)^{p'} d\lambda(x)\right)^{1/p'}$$

For the converse, suppose that $h \ge 0$ with $\left(\int h^p\,d\lambda\right)^{1/p} \le 1$ and observe that $\dot{Q}\mu \le P\dot{Q}\mu$ to get

$$\int (\dot{Q}\mu)h\,d\lambda \le \int (P\dot{Q}\mu)h\,d\lambda = \int (PQh)\,d\mu$$

Now PQh is non-increasing and, by Corollary 1.2, $\|PQh\|_{L^p(\lambda)} \le pp'$ so taking the supremum over all h yields

$$\left(\int \left(\int_{[x,\infty)} \frac{d\mu}{\Lambda}\right)^{p'} d\lambda(x)\right)^{1/p'} \le pp' \sup_{F\downarrow} \frac{\int F\,d\mu}{\left(\int F^p\,d\lambda\right)^{1/p}}$$

and completes the proof.

For all absolutely continuous measures and a great many others we have an alternate characterization of the embedding. Notice that the term below involving $\Lambda(\infty)$ is absent when $\Lambda(\infty) = \infty$.

Theorem 2.3. *Suppose that $\lambda \in \mathcal{I}_2(\infty)$. If $0 < q < p < \infty$ and $1/r = 1/q - 1/p$ then*

$$\text{(2.3)}\quad \sup_{F\downarrow} \frac{\left(\int F^q\, d\mu\right)^{1/q}}{\left(\int F^p\, d\lambda\right)^{1/p}} \approx \left(\int \left(\frac{1}{\Lambda(x)}\int_{(-\infty,x]} d\mu + \frac{1}{\Lambda(\infty)}\int d\mu\right)^{r/q} d\lambda(x)\right)^{1/r}.$$

Proof. As before, reduce to the case $q = 1 < p$ and note that r/q becomes p'. We introduce the constant-valued operator I and its extension to measures by

$$If(x) = \frac{1}{\Lambda(\infty)}\int f\, d\lambda \quad \text{and} \quad \dot{I}\mu(x) = \frac{1}{\Lambda(\infty)}\int d\mu.$$

Note that $I \equiv 0$ if $\Lambda(\infty) = \infty$. By Hölder's inequality

$$\int |If|^p\, d\lambda = \Lambda(\infty)^{1-p}\left|\int f\right|^p \le \int |f|^p\, d\lambda$$

so I is a bounded operator on $L^p(\lambda)$ for $1 < p < \infty$ and its norm is at most 1. It is easy to check that

$$\int (\dot{I}\mu) f\, d\lambda = \int If\, d\mu.$$

If $h \ge 0$ with $\|h\|_{L^p(\lambda)} \le 1$ then

$$\int ((\dot{P} + \dot{I})\mu) h\, d\lambda = \int ((Q + I)h)\, d\mu.$$

Since $(Q+I)h$ is non-increasing and $\|(Q+I)h\|_{L^p(\lambda)} \le p + 1$ we may take the supremum over all such h to get "$\ge$" in (2.3).

If F is non-increasing and $\|F\|_{L^p(\lambda)} \le 1$ then $F \le PF$ and Lemma 1.4 shows that for some $C \ge 1$,

$$PF \le PF + QF \le C(QPF) + IF \le C(QPF) + IPF \le C((Q+I)PF).$$

Therefore

$$\int F\, d\mu \le C\int (Q+I)PF\, d\mu = C\int (PF)((\dot{P} + \dot{I})\mu)\, d\lambda.$$

Since $\|PF\|_{L^p(\lambda)} \le p'$, applying Hölder's inequality and taking the supremum over all such F gives us "$\le$" in (2.3) and completes the proof.

Theorem 2.3 does not hold without some restriction like $\lambda \in \mathcal{I}_2(\infty)$. If we have the equivalence (2.3) for some measure λ, then by taking μ to be a single atom at the point y we find that the supremum is easy to evalute directly and we obtain, for $q = 1$,

$$\Lambda(y)^{-1/p} \approx \left(\int_{[y,\infty)} \Lambda^{-p'}\, d\lambda\right)^{1/p'} + \Lambda(\infty)^{-1/p}.$$

This implies that $\lambda \in \mathcal{I}_{p'}(\infty)$. Example 4.4 shows that this condition is not vacuous.

Embeddings for non-decreasing functions follow in a completely analogous fashion or by simply by applying the above results to the measure $\lambda(-x)$. We record the results below.

Theorem 2.4. *If $0 < p \le q < \infty$ then*

$$\sup_{F\uparrow} \frac{\left(\int F^q\, d\mu\right)^{1/q}}{\left(\int F^p\, d\lambda\right)^{1/p}} = \sup_x \frac{\left(\int_{[x,\infty)} d\mu\right)^{1/q}}{\left(\int_{[x,\infty)} d\lambda\right)^{1/p}}.$$

Theorem 2.5. *If $0 < q < p < \infty$ and $1/r = 1/q - 1/p$ then*

$$\sup_{F\uparrow} \frac{\left(\int F^q\, d\mu\right)^{1/q}}{\left(\int F^p\, d\lambda\right)^{1/p}} \approx \left(\int \left(\int_{(-\infty,x]} \frac{d\mu}{\bar{\Lambda}}\right)^{r/q} d\lambda(x)\right)^{1/r}.$$

Theorem 2.6. *Suppose that $\lambda \in \mathcal{I}_2(-\infty)$. If $0 < q < p < \infty$ and $1/r = 1/q - 1/p$ then*

$$\sup_{F\uparrow} \frac{\left(\int F^q\, d\mu\right)^{1/q}}{\left(\int F^p\, d\lambda\right)^{1/p}} \approx \left(\int \left(\frac{1}{\bar{\Lambda}(x)} \int_{[x,\infty)} d\mu + \frac{1}{\Lambda(\infty)} \int d\mu\right)^{r/q} d\lambda(x)\right)^{1/r}.$$

3. Hardy Inequalities with Two Measures and Two Indices

Considerable effort has been devoted to understanding under what conditions the Hardy operators are bounded from one Lebesgue space to another. The problem has been quite well resolved, not only for spaces of functions and for sequence spaces, but also for Lebesgue spaces with general measures. We recover most of these results in this section but once again with very simple proofs. Some of the equivalent conditions given are new in this generality.

The main contribution of this section, however, is to show that every Hardy inequality may be viewed as a combination of the Hardy inequality of Theorem 1.1 and an embedding of a cone of monotone functions.

Let μ be a measure on $\mathbf{R}$ and consider the inequality

$$\left(\int \left|\int_{-\infty}^x f\, d\lambda\right|^q d\mu(x)\right)^{1/q} \le C \left(\int |f|^p\, d\lambda\right)^{1/p}. \tag{3.1}$$

It is routine matter to check that (3.1) holds for all f if and only if it holds for non-negative f. Therefore we will generally restrict ourselves to non-negative f and drop the absolute value signs. Proposition 1.6 yields a further reduction by showing that (3.1) holds for all non-negative functions if and only if it holds for all non-negative, non-increasing functions.

Lemma 3.1. *Inequality* (3.1) *holds for all f if and only if it holds for all non-negative, non-increasing f.*

Proof. Suppose (3.1) holds for all non-negative, non-increasing f. For an arbitrary function f we define $f_n = \min(n, |f|)\chi_{[-n,n]}$, $n = 1, 2, \dots$, and note that the functions f_n increase to $|f|$ pointwise as $n \to \infty$. Applying Proposition 1.6 we have

$$\left(\int \left(\int_{(-\infty,x]} f_n \, d\lambda\right)^q d\mu(x)\right)^{1/q} \le \left(\int \left(\int_{(-\infty,x]} f_n^o \, d\lambda\right)^q d\mu(x)\right)^{1/q}$$
$$\le C\left(\int (f_n^o)^p \, d\lambda\right)^{1/p} \le C\left(\int f_n^p \, d\lambda\right)^{1/p}$$

and the Monotone Convergence Theorem completes the proof.

With this in hand we may prove inequality (3.1) by combining Theorem 1.1 with the inequality

$$\left(\int \left(\int_{-\infty}^{x} f \, d\lambda\right)^q d\mu(x)\right)^{1/q} \le C\left(\int \left(\frac{1}{\Lambda(x)}\int_{-\infty}^{x} f \, d\lambda\right)^p d\lambda(x)\right)^{1/p}. \tag{3.2}$$

Inquality (3.2) may be viewed as an embedding of monotone functions in two different ways: If f is non-increasing and we take F to be Pf then F is also non-increasing so (3.2) holds for all non-increasing f provided

$$\left(\int F^q \Lambda^q \, d\mu\right)^{1/q} \le C\left(\int F^p \, d\lambda\right)^{1/p}, \quad F\downarrow. \tag{3.3}$$

If we take F to be ΛPf instead then F is non-decreasing for every $f \ge 0$ so (3.2) holds provided

$$\left(\int F^q \, d\mu\right)^{1/q} \le C\left(\int F^p \Lambda^{-p} \, d\lambda\right)^{1/p}, \quad F\uparrow. \tag{3.4}$$

As we see in Theorem 3.2 below, nothing is lost in using this two-step approach to inequality (3.1) when the second step is chosen to be the embedding (3.3) of non-increasing functions. (The first step is always Theorem 1.1.) This is not really surprising since the Hardy inequality of Theorem 1.1 is easily seen to be reversible for non-increasing functions.

When the second step is the embedding (3.4), of non-decreasing functions, nothing is lost in the two-step approach provided $\lambda \in \mathcal{I}_p(\infty)$. There may be some loss for badly behaved measures. Specifically, it is possible to construct a pair of measures λ and μ for which (3.1) holds but (3.4) fails. See Example 4.5.

Since there is no need to check conditions on the measure, the first approach is preferred. We nevertheless, take the second approach as well, in Theorem 3.3, because it leads to conditions that are always sufficient for (3.1). For a very large class of measures, including all non-atomic measures, the conditions are also necessary.

Theorem 3.2. *Suppose* $1 < p < \infty$ *and* $0 < q < \infty$. *Inequality* (3.1) *holds if and only if* (3.3) *does.*

Proof. If (3.3) holds then (3.2) holds for all non-increasing functions f. By Theorem 1.1 we have (3.1) for all non-increasing f and Lemma 3.1 shows that (3.1) holds for all f. Conversely, if (3.1) holds then, because any non-increasing F satisfies $F \le PF$, (3.3) follows immediately.

Theorem 3.3. *Suppose* $1 < p < \infty$ *and* $0 < q < \infty$. *Then* (3.1) *holds whenever* (3.4) *does. If* $\lambda \in \mathcal{I}_p(\infty)$ *and* $\Lambda(\infty) = \infty$ *then condition* (3.4) *is also necessary for* (3.1).

Proof. If (3.4) holds then (3.2) holds for all $f \ge 0$ and in view of Theorem 1.1 we have (3.1). For the converse we suppose that $\lambda \in \mathcal{I}_p(\infty)$ and $\Lambda(\infty) = \infty$ and define the measure $\nu = \Lambda^{-p}\lambda$. Set $\bar{N}(x) = \int_{[x,\infty)} d\nu$,

$$\bar{P}_\nu f(x) = \frac{1}{\bar{N}(x)} \int_{[x,\infty)} f\, d\nu \quad \text{and} \quad \bar{Q}_\nu f(x) = \int_{(-\infty,x]} f \frac{d\nu}{\bar{N}}.$$

By Definition 1.3 and the comments that precede Example 4.4 we have

$$\bar{N}(x) \approx \Lambda(x)^{1-p}.$$

It follows that $\bar{N}(-\infty) = \infty$ and

$$\int_{(-\infty,x]} \bar{N}^{-2}\, d\nu \approx \int_{(-\infty,x]} \Lambda^{p-2}\, d\lambda \approx \Lambda(x)^{p-1} \approx \bar{N}(x)^{-1}.$$

Therefore $\nu \in \mathcal{I}_2(-\infty)$ and by Lemma 1.4 every non-decreasing function F satisfies

$$F \le \bar{P}_\nu F \le C\bar{Q}_\nu \bar{P}_\nu F$$

for some constant C. Therefore

$$\left(\int F^q\, d\mu\right)^{1/q} \le \left(\int (\bar{Q}_\nu \bar{P}_\nu F)^q\, d\mu\right)^{1/q} = \left(\int \left(\int_{(-\infty,x]} \bar{P}_\nu F \bar{N}^{-1}\, d\nu\right)^q d\mu(x)\right)^{1/q}$$

and since $\bar{N}^{-1}\, d\nu \approx \Lambda^{-1}\, d\lambda$ we may use (3.1) to see that this is dominated by a multiple of

$$\left(\int \left(\bar{P}_\nu F \Lambda^{-1}\right)^p\, d\lambda\right)^p = \left(\int (\bar{P}_\nu F)^p\, d\nu\right)^{1/p}.$$

Since $\bar{P}_\nu$ is a bounded operator on $L^p(\nu)$ this is dominated in turn by a multiple of

$$\left(\int F^p\, d\nu\right)^{1/p} = \left(\int F^p \Lambda^{-p}\, d\lambda\right)^{1/p}.$$

This completes the proof.

Theorems 3.2 and 3.3 reduce the Hardy inequality (3.1) to problems that we have solved in Section 2. It is nevertheless interesting to directly observe the conditions one obtains by this reduction and to compare them with known results.

First we look at the conditions obtained by factoring the Hardy inequality through an embedding of the cone of non-increasing functions. The next three results follow directly from Theorem 3.2 and Theorems 2.1, 2.2, and 2.3.

Corollary 3.4. *Suppose* $1 < p \le q < \infty$. *Inequality* (3.1) *holds if and only if*

$$\sup_x \left(\int_{(-\infty,x]} \Lambda^q \, d\mu \right)^{1/q} \left(\int_{(-\infty,x]} d\lambda \right)^{-1/p} < \infty.$$

Corollary 3.5. *Suppose* $0 < q < p < \infty$, $p > 1$ *and* $1/r = 1/q - 1/p$. *Inequality* (3.1) *holds if and only if*

$$\int \left(\int_{[x,\infty)} \Lambda^{q-1} \, d\mu \right)^{r/q} d\lambda < \infty.$$

Corollary 3.6. *Suppose* $0 < q < p < \infty$, $p > 1$ *and* $1/r = 1/q - 1/p$. *If* $\lambda \in \mathcal{I}_2(\infty)$ *then inequality* (3.1) *holds if and only if*

$$\int \left(\frac{1}{\Lambda(x)} \int_{(-\infty,x]} \Lambda^q \, d\mu + \frac{1}{\Lambda(\infty)} \int \Lambda^q \, d\mu \right)^{r/q} d\lambda < \infty.$$

Now we turn our attention to the conditions obtained by factoring the Hardy inequality through an embedding of the cone of non-decreasing functions. The next three results follow from Theorem 3.3 and Theorems 2.4, 2.5, and 2.6.

Here we apply the results of Section 2 with the measure λ replaced by $\Lambda^{-p}\lambda$. Note that the function

$$\bar{N}(x) \equiv \int_{[x,\infty)} \Lambda^{-p} \, d\lambda$$

is equivalent to $\Lambda(x)^{1-p}$ when $\lambda \in \mathcal{I}_p(\infty)$ and $\lambda(\infty) = \infty$.

Corollary 3.7. *Suppose* $1 < p \le q < \infty$. *Inequality* (3.1) *holds whenever*

$$\sup_x \left(\int_{[x,\infty)} d\mu \right)^{1/q} \left(\int_{[x,\infty)} \Lambda^{-p} \, d\lambda \right)^{-1/p} < \infty.$$

If $\lambda \in \mathcal{I}_p(\infty)$ *and* $\Lambda(\infty) = \infty$ *then* (3.1) *holds if and only if*

$$\sup_x \left(\int_{[x,\infty)} d\mu \right)^{1/q} \left(\int_{(-\infty,x]} d\lambda \right)^{1/p'} < \infty. \tag{3.5}$$

Corollary 3.8. *Suppose* $0 < q < p < \infty$, $p > 1$ *and* $1/r = 1/q - 1/p$. *Inequality* (3.1) *holds whenever*

$$\int \left(\int_{(-\infty,x]} \frac{d\mu}{\bar{N}} \right)^{r/q} \Lambda(x)^{-p} \, d\lambda(x) < \infty.$$

If $\lambda \in \mathcal{I}_p(\infty)$ *and* $\Lambda(\infty) = \infty$ *then* (3.1) *holds if and only if*

$$\int \left(\int_{(-\infty,x]} \Lambda^{p-1} \, d\mu \right)^{r/q} \Lambda(x)^{-p} \, d\lambda(x) < \infty.$$

Corollary 3.9. *Suppose* $0 < q < p < \infty$, $p > 1$ *and* $1/r = 1/q - 1/p$. *If* $\lambda \in \mathcal{I}_2(\infty)$ *then inequality* (3.1) *holds whenever*

$$\int \left(\frac{1}{\bar{N}(x)} \int_{[x,\infty)} d\mu + \frac{1}{\bar{N}(\infty)} \int d\mu \right)^{r/q} \Lambda(x)^{-p} \, d\lambda(x) < \infty.$$

If $\lambda \in \mathcal{I}_p(\infty) \cap \mathcal{I}_2(\infty)$ *and* $\Lambda(\infty) = \infty$ *then* (3.1) *holds if and only if*

$$\int \left(\int_{[x,\infty)} d\mu \right)^{r/q} \Lambda(x)^{r/q'} \, d\lambda(x) < \infty. \tag{3.6}$$

Our proofs require some mild assumptions on the measure λ to show the equivalence of (3.1) with either (3.5) or (3.6), depending on the range of indices. However, it is known that when $q > 1$ these conditions are necessary and sufficient for (3.1) for any measures. See [5] or [6]. This is not due to a lack of care in our proofs, but points to an essential feature of the approach. Although all Hardy inequalities factor through an embedding of the cone of non-increasing functions, not every Hardy inequality factors through an embedding of the cone of non-decreasing functions. See Example 4.5.

4. Related Results

Fix a function $\varphi : (0, \infty) \to \mathbf{R}$ that is either concave and non-decreasing or else convex and non-increasing and define the operator T on $f \geq 0$ by

$$Tf(x) = (\varphi^{-1} \circ P(\varphi \circ f))(x) = \varphi^{-1} \left(\frac{1}{\Lambda(x)} \int_{(-\infty,x]} \varphi(f(t)) \, d\lambda(t) \right).$$

Theorem 4.1. *Suppose* $1 < p < \infty$ *and* $0 < q < \infty$. *Then*

$$\left(\int (Tf)^q \, d\mu\right)^{1/q} \leq C \left(\int f^p \, d\lambda\right)^{1/p}, \quad f \geq 0, \tag{4.1}$$

if and only if

$$\left(\int (Pf)^q \, d\mu\right)^{1/q} \leq C \left(\int f^p \, d\lambda\right)^{1/p}, \quad f \geq 0, \tag{4.2}$$

if and only if

$$\left(\int F^q \, d\mu\right)^{1/q} \leq C \left(\int F^p \, d\lambda\right)^{1/p}, \quad F \downarrow . \tag{4.3}$$

(The constants C *need not be the same.)*

Proof. Jensen's inequality shows that $Tf \leq Pf$ so (4.2) implies (4.1). Theorem 3.2, with μ replaced by $\Lambda^q \mu$ shows that (4.3) implies (4.2). To prove the remaining implication, fix a non-negative, non-increasing function F.

In the case that φ is concave and non-decreasing, $\varphi \circ F$ is also non-increasing so $P(\varphi \circ F)$ is non-increasing and $\varphi \circ F \leq P(\varphi \circ F)$. Applying φ^{-1} we conclude that TF is non-increasing and $F \leq TF$.

If instead, φ is convex and non-increasing, then $\varphi \circ F$ is non-decreasing so $P(\varphi \circ F)$ is non-decreasing and $\varphi \circ F \geq P(\varphi \circ F)$. Applying φ^{-1} we again conclude that TF is non-increasing and $F \leq TF$.

Now it is evident that (4.1) implies (4.3). This completes the proof.

The function $\varphi(s) = \log s$ is non-decreasing and concave so Theorem 4.1 includes the following.

Corollary 4.2. *Suppose* $1 < p < \infty$ *and* $0 < q < \infty$. *The inequality*

$$\left(\int \left(\exp\left(\frac{1}{\Lambda(x)} \int_{(-\infty, x]} \log(f(t)) \, d\lambda(t)\right)\right)^q d\mu(x)\right)^{1/q} \leq C \left(\int f^p \, d\lambda\right)^{1/p}$$

holds for all $f \geq 0$ *if and only if*

$$\sup_{F\downarrow} \frac{\left(\int F^q \, d\mu\right)^{1/q}}{\left(\int F^p \, d\lambda\right)^{1/p}} < \infty.$$

This corollary, together with Theorems 2.1, 2.2, and 2.3, yields a weight characterization for the Geometric Mean Operator.

The function $\varphi(s) = 1/s$ is non-increasing and convex so Theorem 4.1 also includes the next assertion. We have replaced f by $1/f$ in the statement and raised both sides to the power -1.

Corollary 4.3. *Suppose* $1 < p < \infty$ *and* $0 < q < \infty$. *The inequality*

$$\left(\int f^{-p}\,d\lambda\right)^{-1/p} \le C\left(\int\left(\frac{1}{\Lambda(x)}\int_{(-\infty,x]} f(t)\,d\lambda(t)\right)^{-q} d\mu(x)\right)^{-1/q}, \quad f \ge 0,$$

holds if and only if

$$\sup_{F\downarrow}\frac{\left(\int F^q\,d\mu\right)^{1/q}}{\left(\int F^p\,d\lambda\right)^{1/p}} < \infty.$$

Using this corollary, Theorems 2.1, 2.2, and 2.3 characterize Hardy inequalities with negative indices.

We return briefly to the classes $\mathcal{I}_p(\infty)$ to make some comments and provide two examples.

Firstly, it is not clear that the classes $\mathcal{I}_p(\infty)$ are different for different p. The measure constructed in Example 4.4 is not in $\mathcal{I}_p(\infty)$ for any $p > 1$. However, we do know that if $1 < p < q$ then $\mathcal{I}_q(\infty) \subset \mathcal{I}_p(\infty)$ because if $\lambda \in \mathcal{I}_q(\infty)$ then

$$\begin{aligned}\Lambda(x)^{1-p} - \Lambda(\infty)^{1-p} &\le \Lambda(x)^{q-p}(\Lambda(x)^{1-q} - \Lambda(\infty)^{1-q})\\ &\le C\Lambda(x)^{q-p}\int_{[x,\infty)}\Lambda^{-q}\,d\lambda \le C\int_{[x,\infty)}\Lambda^{-p}\,d\lambda\end{aligned}$$

so $\lambda \in \mathcal{I}_p(\infty)$. For the last inequality above we have used the monotonicity of Λ to take Λ^{q-p} inside the integral.

Secondly, it is worth pointing out that although the inequality defining $\mathcal{I}_p(\infty)$ may fail in exceptional cases, a reverse inequality always holds. To see this, first observe that for each $s > 0$, $\lambda\{t : \Lambda(t) < s\} \le s$ so for each $x \in \mathbf{R}$,

$$\lambda\{t \le x : \Lambda(t) < s\} \le \min(s, \Lambda(\infty)).$$

Now

$$\begin{aligned}\int_{[x,\infty)}\Lambda(t)^{-p}\,d\lambda(t) &= \int_{[x,\infty)}\int_{\Lambda(t)}^{\infty} ps^{-p-1}\,ds\,d\lambda(t)\\ &= \int_{\Lambda(x)}^{\infty}\int_{\{t\le x:\Lambda(t)<s\}} d\lambda(t)ps^{-p-1}\,ds\\ &\le \int_{\Lambda(x)}^{\infty}\min(s,\Lambda(\infty))ps^{-p-1}\,ds\\ &= p'\Lambda(x)^{1-p} - (p'-1)\Lambda(\infty)^{1-p}.\end{aligned}$$

Thirdly, an estimate much like the last one reveals that no problem of this sort arises when integrating powers greater than -1. One can show that for $p > -1$,

$$\int_{(-\infty,x]}\Lambda^p\,d\lambda \approx \Lambda(x)^{p+1}.$$

Example 4.4. *There exists a measure λ that is not in $\mathcal{I}_p(\infty)$ for any $p > 1$.*

Proof. The measure we construct is purely atomic with atoms of weight $\lambda\{n\} = b_n$ at each positive integer n. The b_n are defined below. The remainder of the example is phrased in terms of sequences and sums rather than functions and integrals.

Suppose $a_n \geq 1$ are rapidly increasing for $n \geq 1$ so that $2 \leq \frac{a_{n+1}}{a_n} \to \infty$. Then

$$a_n \leq \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{a_k}{a_{k+1}} \frac{a_{k+1}}{a_{k+2}} \dots \frac{a_{n-1}}{a_n} a_n \leq a_n \sum_{k=1}^{n} 2^{k-n} \leq 2a_n.$$

Also, for any $\alpha < 0$,

$$a_n^\alpha \leq \sum_{k=n}^{\infty} a_k^\alpha = \sum_{k=n}^{\infty} \left(\frac{a_k}{a_{k-1}} \frac{a_{k-1}}{a_{k-2}} \dots \frac{a_{n+1}}{a_n} \right)^\alpha a_n^\alpha \leq a_n^\alpha \sum_{k=n}^{\infty} 2^{\alpha(k-n)} \leq \frac{1}{1-2^\alpha} a_n^\alpha.$$

Define $b_n = a_n$ when n is even and $b_n = 1$ when n is odd. Then for any $n \geq 2$ we have

$$\sum_{k=1}^{n} b_k \geq b_n + b_{n-1} \geq a_{n-1}.$$

If $m \geq 2$ is odd then

$$\sum_{n=1}^{m} b_n \leq 1 + \sum_{n=1}^{m-1} a_n \leq 3a_{m-1}$$

and

$$\sum_{n=m}^{\infty} \left(\sum_{k=1}^{n} b_k \right)^{-p} b_n \leq a_{m-1}^{-p} + \sum_{n=m+1}^{\infty} a_n^{-p} a_n \leq a_{m-1}^{-p} + a_m^{1-p}/(1-2^{1-p}).$$

As $m \to \infty$ through the odd numbers we see that

$$\frac{\sum_{n=m}^{\infty} \left(\sum_{k=1}^{n} b_k \right)^{-p} b_n}{\left(\sum_{n=1}^{m} b_n \right)^{1-p}} \leq C(a_{m-1}^{-1} + (a_m/a_{m-1})^{1-p}) \to 0.$$

Example 4.5. *There exist measures λ and μ such that the Hardy inequality* (3.1) *holds but the embedding of non-increasing functions* (3.4) *does not.*

Proof. Fix $p > 1$ and let $q = p$. We use the measure λ constructed in Example 4.4 and define μ to be the purely atomic measure with atoms of weight

$$\mu\{n\} = \left(\sum_{k=1}^{n} b_k \right)^{1-p} - \left(\sum_{k=1}^{n+1} b_k \right)^{1-p}$$

at each positive integer n. Since the series $\sum b_n$ diverges,

$$\sum_{n=m}^{\infty} \mu\{n\} = \left(\sum_{k=1}^{m} b_k \right)^{1-p}.$$

Using this fact and Theorem 2.4, we see that the embedding (3.4) holds if and only if for all m,

$$\left(\sum_{k=1}^{m} b_k\right)^{1-p} \leq C \sum_{n=m}^{\infty} \left(\sum_{k=1}^{n} b_k\right)^{-p} b_n.$$

This is not the case, according to Example 4.4, and we conclude that the embedding (3.4) fails for these measures.

To show that (3.1) does hold we apply Theorem 3.4. We have

$$\sum_{n=1}^{m} \left(\sum_{k=1}^{n} b_k\right)^{-p} \mu\{n\} \leq \sum_{n=1}^{m} \left(\sum_{k=1}^{n} b_k\right)^{-p} \left(\sum_{k=1}^{n} b_k\right)^{1-p} \leq \sum_{n=1}^{m} \sum_{k=1}^{n} b_k.$$

Using the estimates from Example 4.4 see that for $n \geq 2$

$$\sum_{k=1}^{n} b_k = b_n + \sum_{k=1}^{n-1} a_k \leq b_n + 2a_{n-1} \leq 2b_{n-1} + 3b_n$$

and so

$$\sum_{n=1}^{m} \left(\sum_{k=1}^{n} b_k\right)^{-p} \mu\{n\} \leq b_1 + \sum_{n=2}^{m} (2b_{n-1} + 3b_n) \leq 5 \sum_{n=1}^{m} b_n.$$

This verifies the condition of Theorem 3.4 and shows that inequality (3.1) holds for the measures μ and λ.

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