OVERDETERMINED HARDY INEQUALITIES

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Abstract. Necessary and sufficient conditions on the weights w and w_0 are given for the higher order Hardy inequality

$$\left(\int_0^1 |u|^q w_0\right)^{1/q} \le C\left(\int_0^1 |u^{(k+1)}|^p w\right)^{1/p}$$

to hold for all solutions u of certain overdetermined boundary value problems.

1. Introduction

A function whose derivative is not too large cannot grow fast enough to become too large itself. This simple observation is of fundamental importance in many areas of analysis and its appearance in various, more precise, forms has provided basic tools in Harmonic Analysis, Differential Equations, Interpolation Theory and others.

In this paper we apply this principle to solutions of certain overdetermined, two-point boundary value problems in order to characterize weighted Lebesgue norm inequalities involving higher order derivatives. We extend results of Gurka for weighted inequalities involving solutions of the first-order overdetermined problem

$$u' = f \text{ in } (0,1), \quad u(0) = u(1) = 0, \tag{1.1}$$

which can be found in [8, Chapter 1, Section 8], and improve results of Kufner and Simader [6] for the higher-order overdetermined problem

$$u^{(k+1)} = f \text{ in } (0,1), \quad u(0) = u'(0) = \cdots = u^{(k)}(0) = u^{(k)}(1) = 0. \tag{1.2}$$

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Our main result is to give a characterization of weighted inequalities for the more general higher-order overdetermined problem

$$\begin{aligned} u^{(k+1)} &= f \text{ in } (0,1), \\ u^{(i)}(0) &= 0 \text{ for } i \in M_0, \\ u^{(i)}(1) &= 0 \text{ for } i \in M_1, \\ u^{(k)}(0) &= u^{(k)}(1) = 0, \end{aligned} \tag{1.3}$$

where M_0 and M_1 are appropriate subsets of $\{0, 1, \ldots, k-1\}$.

For each of these boundary value problems (1.1), (1.2), and (1.3), we give easily verified necessary and sufficient conditions which answer the question: For which weights w_0 and w and indices p and q does there exist a constant C such that

$$\left(\int_0^1 |u|^q w_0 \right)^{1/q} \leq C \left(\int_0^1 |f|^p w \right)^{1/p}$$

for all functions f and u satisfying (1.1) (or (1.2) or (1.3)).

Our approach draws on known Hardy-type inequalities, see [8], on recent results which provide weighted inequalities for integral operators with fairly general positive kernels, [11], and on higher-order Hardy inequalities, [2, 3, 4, 5, 7, 9]. After introducing some notation we begin with a simple lemma based on an idea of R. Oinarov, mentioned in [6], which shows that boundedness of a positive operator on a certain hyperplane in L^p_w is equivalent to boundedness on the whole space.

A weight is a non-negative, measurable function. If w is a weight and $0 < p < \infty$ we denote the collection of functions f for which

$$\|f\|_{p\,w} \equiv \left(\int_0^1 |f|^p w \right)^{1/p}$$

is finite by L^p_w. If $p \geq 1$ this is a Banach space. We define p' by $1/p + 1/p' = 1$ even when $p < 1$. The notation $A \approx B$ means that there are positive constants c_1 and c_2 such that $A \leq c_1 B$ and $B \leq c_2 A$. A non-negative operator on functions is one that maps non-negative functions to non-negative functions. The characteristic function of the set E, denoted χ_E, takes the value 1 on the set E and the value 0 otherwise.

Lemma 1.1. *Let $1 < p < \infty$, $0 < q < \infty$ and let w and w_0 be weights. Suppose that z satisfies*

$$\int_0^z w^{1-p'} \approx \int_z^1 w^{1-p'} < \infty \quad \textit{or} \quad \int_0^z w^{1-p'} = \int_z^1 w^{1-p'} = \infty \tag{1.4}$$

and set

$$H = \left\{ g : \int_0^z g = \int_z^1 g \right\}.$$

If T is a non-negative linear operator then $T : H \cap L^p_w \to L^q_{w_0}$ if and only if $T : L^p_w \to L^q_{w_0}$.

Proof. The "if" part of the theorem is trivial. To prove the other direction, suppose that $T : H \cap L^p_w \to L^q_{w_0}$. Since $L^1 \cap L^p_w$ is dense in L^p_w it is enough to show that $T : L^1 \cap L^p_w \to L^q_{w_0}$. Fix g in $L^1 \cap L^p_w$ and suppose, without loss of generality, that $\int_0^z |g| \le \int_z^1 |g|$.

Case 1. Suppose that $\int_0^z w^{1-p'} \approx \int_z^1 w^{1-p'} < \infty$. Set

$$h = \alpha w^{1-p'} \left(\int_z^1 |g| - \int_0^z |g| \right) \chi_{(0,z)}$$

where $1/\alpha = \int_0^z w^{1-p'}$. Clearly $h \ge 0$ and a simple calculation shows that $|g| + h \in H$. Since $g \le |g| + h$, we have

$$\|Tg\|_{q\,w_0} \le \|T(|g| + h)\|_{q\,w_0} \le C\||g| + h\|_{p\,w} \le C\|g\|_{p\,w} + C\|h\|_{p\,w}$$

so to complete the first case we have only to show that $\|h\|_{p\,w} \le C'\|g\|_{p\,w}$ for some constant C'. We estimate the norm of h using Hölder's inequality and property (1.4) of z.

$$\begin{aligned}\|h\|_{p\,w} =&\alpha \left(\int_0^z w^{(1-p')p} w \right)^{1/p} \left(\int_z^1 |g| - \int_0^z |g| \right) \le \alpha^{1/p'} \int_z^1 |g| \\ \le&\alpha^{1/p'} \left(\int_z^1 w^{1-p'} \right)^{1/p'} \left(\int_0^1 |g|^p w \right)^{1/p} \le C'\|g\|_{p\,w}\end{aligned}$$

as required.

Case 2. Suppose that $\int_0^z w^{1-p'} = \int_z^1 w^{1-p'} = \infty$. For each positive integer n set

$$h_n = \alpha_n w_n^{1-p'} \left(\int_z^1 |g| - \int_0^z |g| \right) \chi_{(0,z)}$$

where $w_n^{1-p'} = w^{1-p'} \chi_{\{w^{1-p'} < n\}}$ and $1/\alpha_n = \int_0^z w_n^{1-p'}$. Again, $h_n \ge 0$, $|g| + h_n \in H$, and $g \le |g| + h_n$ so for each n we have

$$\|Tg\|_{q\,w_0} \le \|T(|g| + h_n)\|_{q\,w_0} \le C\||g| + h_n\|_{p\,w} \le C\|g\|_{p\,w} + C\|h_n\|_{p\,w}.$$

Now $w^{1-p'}$ is zero where $w \ne w_n$ so

$$\|h_n\|_{p\,w} = \alpha_n \left(\int_0^z w_n^{(1-p')p} w \right)^{1/p} \left(\int_z^1 |g| - \int_0^z |g| \right) \le \alpha_n^{1/p'} \int_z^1 |g|.$$

As $n \to \infty$ we see that $\alpha_n \to 0$ so we have $\|Tg\|_{q\,w_0} \le C\|g\|_{p\,w}$ which completes the second case and the proof.

Remark. Although we may choose z so that there is equality in the first part of (1.4), the weaker restriction is enough and the extra freedom may prove to be useful when verifying the conditions of Theorems 2.3, 3.7 and 3.8.

We note that for some weights w it is not possible to find a z satisfying (1.4).

2. The first-order, overdetermined problem.

In this section we characterize the weights w_0 and w for which there exists a constant C such that

$$\|u\|_{q\,w_0} \le C\|f\|_{p\,w}, \quad \text{for } f \text{ and } u \text{ satisfying (1.1).} \tag{2.1}$$

Gurka has solved this problem for indices p and q satisfying $1 < p \le q < \infty$ but our conditions, while still necessary and sufficient, are different in form than his. Gurka's work is presented in [8, Chapter 1, Section 8].

We also solve the problem in the case $0 < q < p$, $1 < p < \infty$.

Definition 2.1. *For fixed $z \in (0,1)$, let $S = S_1 + S_2$ where*

$$S_1 g(x) = \left(\int_0^x g\right)\chi_{(0,z)}(x) \text{ and } S_2 g(x) = \left(\int_x^1 g\right)\chi_{(z,1)}(x).$$

Note that S_1 and S_2, and hence S, are non-negative operators.

Lemma 2.2. *Suppose that f and u satisfy* (1.1) *and set $g = (\chi_{(0,z)} - \chi_{(z,1)})f$. Then $u = Sg$.*

Proof. Since $u(0) = u(1) = 0$ we have

$$u(x) = \int_0^x f = -\int_x^1 f$$

and hence

$$u(x) = \left(\int_0^x f\right)\chi_{(0,z)}(x) - \left(\int_x^1 f\right)\chi_{(z,1)}(x) = Sg(x).$$

Theorem 2.3. *Let $0 < q < \infty$ and $1 < p < \infty$. Suppose w_0 and w are weights and z satisfies* (1.4). *Then there exists a constant C such that* (2.1) *holds if and only if I or II below holds.*

I. $1 < p \le q < \infty$,

$$\sup_{0<x<z}\left(\int_x^z w_0\right)^{1/q}\left(\int_0^x w^{1-p'}\right)^{1/p'} < \infty, \tag{2.2}$$

and

$$\sup_{z<x<1}\left(\int_z^x w_0\right)^{1/q}\left(\int_x^1 w^{1-p'}\right)^{1/p'} < \infty. \tag{2.3}$$

II. $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$,

$$\left(\int_0^z\left(\int_x^z w_0\right)^{r/p}\left(\int_0^x w^{1-p'}\right)^{r/p'} w_0(x)\,dx\right)^{1/r} < \infty, \tag{2.4}$$

and

$$\left(\int_z^1 \left(\int_z^x w_0\right)^{r/p} \left(\int_x^1 w^{1-p'}\right)^{r/p'} w_0(x)\,dx\right)^{1/r} < \infty. \tag{2.5}$$

Proof. We begin by showing that (2.1) holds if and only if $S : L^p_w \to L^q_{w_0}$. Suppose first that $S : L^p_w \to L^q_{w_0}$ and that f and u satisfy (1.1). Now, with $g = (\chi_{(0,z)} - \chi_{(z,1)})f$, we use Lemma 2.2 and the boundedness of S to get

$$\|u\|_{q\,w_0} = \|Sg\|_{q\,w_0} \le C\|g\|_{p\,w} = C\|f\|_{p,w}.$$

Conversely, suppose that (2.1) holds. According to Lemma 1.1, it is enough to prove that $\|Sg\|_{q\,w_0} \le C\|g\|_{p\,w}$ for functions $g \in L^p_w$ satisfying $\int_0^z g = \int_z^1 g$ in order to conclude that $S : L^p_w \to L^q_{w_0}$. Fix such a g and define f and u by

$$f = (\chi_{(0,z)} - \chi_{(z,1)})g, \quad u(x) = \int_0^x f.$$

Since $u(1) = \int_0^1 f = \int_0^z g - \int_z^1 g = 0$ it is clear that f and u satisfy (1.1). Thus, using Lemma 2.2 again,

$$\|Sg\|_{q\,w_0} = \|u\|_{q\,w_0} \le C\|f\|_{p,w} = C\|g\|_{p\,w}.$$

To complete the proof, we show that the boundedness of S is equivalent to the conditions in I and II.

Since S is the sum of the two non-negative operators S_1 and S_2, it is bounded if and only if both S_1 and S_2 are bounded. The boundedness of $S_1 : L^p_w \to L^q_{w_0}$ means that there exists a constant C such that

$$\left(\int_0^1 \left|\left(\int_0^x g\right)\chi_{(0,z)}(x)\right|^q w_0(x)\,dx\right)^{1/q} \le C\left(\int_0^1 |g|^p w\right)^{1/p}$$

for all functions g on $[0,1]$. Since the left hand side does not depend on the values of g on $[z,1]$, the above inequality is clearly equivalent to the inequality

$$\left(\int_0^z \left|\int_0^x g\right|^q w_0(x)\,dx\right)^{1/q} \le C\left(\int_0^z |g|^p w\right)^{1/p} \tag{2.6}$$

for all functions g on $[0,z]$. The weights for which this type of Hardy inequality holds have been completely characterized. See [8, Theorems 1.14, 1.15, and 9.3]. The inequality (2.6) holds if and only if $1 < p \le q < \infty$ and (2.2) holds or $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$, and (2.4) holds.

A similar analysis shows that the boundedness of S_2 reduces to a conjugate Hardy inequality which yields the conditions (2.3), (2.5). This completes the proof.

Remark. Techniques are available for dealing with the endpoint cases $0 < q < p = \infty$, $1 < p \le q = \infty$, $p = 1 \le q \le \infty$, and $0 < q < p = 1$. In particular, weighted Hardy inequalities have been characterized in these cases. See [8, Chapter 1, Section 5] and [10]. Our methods produce results in these cases with only minor modifications.

3. Higher-order, overdetermined problems

We begin this section with a discussion of boundary value problems which have the "right" number of boundary conditions, that is, the number of boundary conditions is the same as the order of the problem.

Let $N_i = \{0, 1, \ldots, i-1\}$ and fix subsets M_0 and M_1 of N_k such that $|M_0|+|M_1| = k$. We consider the boundary value problem

$$u^{(k)} = f; \quad u^{(i)}(0) = 0 \text{ for } i \in M_0, \quad u^{(i)}(1) = 0 \text{ for } i \in M_1 \tag{3.1}$$

for some locally integrable function f. Drábek and Kufner [2] have shown that it has a unique solution for every locally integrable function f if and only if (M_0, M_1) satisfies the Pólya condition:

$$|M_0 \cap N_i| + |M_1 \cap N_i| \geq i, \quad i = 1, 2, \ldots, k. \tag{3.2}$$

To better understand this condition we introduce the $2 \times k$ incidence matrix $E = (e_{\alpha\, i})$ of (M_0, M_1) by setting $e_{\alpha\, i} = 1$ if $i - 1 \in M_\alpha$ and $e_{\alpha\, i} = 0$ otherwise. The condition (3.2) states that there are at least i 1's in the first i columns of E for $i = 1, 2, \ldots, k$.

For a pair (M_0, M_1) satisfying the Pólya condition there is a Green's function $G(x, s)$ for the boundary value problem (3.1) (see, for example, [1, p162ff]) so that for any locally integrable function f, the solution of (3.1) is given by

$$u(x) = \int_0^1 G(x, s) f(s)\, ds.$$

These Green's functions are well understood. If $M_0 = N_k$ and M_1 is empty, then

$$G(x, s) = \frac{(x-s)^{k-1}}{(k-1)!} \chi_{(0,x)}(s). \tag{3.3}$$

If $M_1 = N_k$ and M_0 is empty then

$$G(x, s) = -\frac{(x-s)^{k-1}}{(k-1)!} \chi_{(x,1)}(s).$$

For any other pair, (M_0, M_1), Sinnamon, in [9], has verified a conjecture of Kufner showing that the associated Green's functions are equivalent to functions of a particularly simple form. This result is reproduced in Proposition 3.2 below.

Definition 3.1. *For a pair (M_0, M_1) we define non-negative integers a, b, c, and d, as follows: Let a be the number of consecutive 1's beginning the top row of E; b be the number of consecutive 1's beginning the bottom row of E; c be the number of consecutive 0's ending the top row of E; and d be the number of consecutive 0's ending the bottom row of E. Also define A, B, C, and D by*

$$A = \begin{cases} a-1, & \text{if } a+c = k \\ a, & \text{if } a+c < k \end{cases}, \qquad C = \begin{cases} c-1, & \text{if } a+c = k \\ c, & \text{if } a+c < k \end{cases},$$

$$B = \begin{cases} b-1, & \text{if } b+d=k \\ b, & \text{if } b+d<k \end{cases}, \qquad D = \begin{cases} d-1, & \text{if } b+d=k \\ d, & \text{if } b+d<k \end{cases}.$$

To illustrate the definition we offer an example. Take $k=6$, $M_0=\{0,1,2\}$, and $M_1=\{1,3,4\}$. Then we have

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$a=3$, $b=0$, $c=3$, $d=1$, $A=2$, $B=0$, $C=2$, and $D=1$. Notice that (M_0,M_1) satisfies the Pólya condition.

Proposition 3.2. *Suppose that (M_0,M_1) satisfies $|M_0|+|M_1|=k$ and the Pólya condition* (3.2) *and that neither M_0 nor M_1 is empty. Then the Green's function, $G(x,s)$, of the boundary value problem* (3.1) *satisfies*

$$|G(x,s)| \approx x^a(1-x)^B s^C(1-s)^d, \quad \text{for } 0<x<s<1, \tag{3.4}$$

and

$$|G(x,s)| \approx x^A(1-x)^b s^c(1-s)^D, \quad \text{for } 0<s<x<1. \tag{3.5}$$

Note that since the Green's function $G(x,s)$ is continuous on $(0,1)\times(0,1)$ it follows that G does not change sign on $(0,1)\times(0,1)$, a remark which includes the function G from (3.3) as well.

Before we turn to the overdetermined case, we pause to introduce the weight conditions that arise.

Definition 3.3. *Suppose that $1<p<\infty$, $0<q<\infty$, $\lambda>0$, v_0 and v are weights on $[a,b]$, and ϕ is a non-negative, continuous function on $[a,b]$. Set $r=pq/(p-q)$.*

Define $B([a,b],v_0(t),v(t))$ to be

$$\sup_{a<x<b}\left(\int_x^b v_0(t)\,dt\right)^{1/q}\left(\int_a^x v(t)^{1-p'}\,dt\right)^{1/p'} \quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_x^b v_0(t)\,dt\right)^{r/p}\left(\int_a^x v(t)^{1-p'}\,dt\right)^{r/p'} v_0(x)\,dx\right)^{1/r} \quad \text{if } q<p,$$

and $B'([a,b],v_0(t),v(t))$ to be

$$\sup_{a<x<b}\left(\int_a^x v_0(t)\,dt\right)^{1/q}\left(\int_x^b v(t)^{1-p'}\,dt\right)^{1/p'} \quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_a^x v_0(t)\,dt\right)^{r/p}\left(\int_x^b v(t)^{1-p'}\,dt\right)^{r/p'} v_0(x)\,dx\right)^{1/r} \quad \text{if } q<p.$$

Define $B_1([a,b],\phi(s),\lambda,v_0(t),v(t))$ *to be*

$$\sup_{a<x<b}\left(\int_x^b\left(\int_x^t\phi\right)^{\lambda q}v_0(t)\,dt\right)^{1/q}\left(\int_a^x v(t)^{1-p'}\,dt\right)^{1/p'}\quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_x^b\left(\int_x^t\phi\right)^{\lambda q}v_0(t)\,dt\right)^{r/q}\left(\int_a^x v(t)^{1-p'}\,dt\right)^{r/q'}v(x)^{1-p'}\,dx\right)^{1/r}\quad \text{if } q<p,$$

and $B_1'([a,b],\phi(s),\lambda,v_0(t),v(t))$ *to be*

$$\sup_{a<x<b}\left(\int_a^x\left(\int_t^x\phi\right)^{\lambda q}v_0(t)\,dt\right)^{1/q}\left(\int_x^b v(t)^{1-p'}\,dt\right)^{1/p'}\quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_a^x\left(\int_t^x\phi\right)^{\lambda q}v_0(t)\,dt\right)^{r/q}\left(\int_x^b v(t)^{1-p'}\,dt\right)^{r/q'}v(x)^{1-p'}\,dx\right)^{1/r}\quad \text{if } q<p.$$

Define $B_2([a,b],\phi(s),\lambda,v_0(t),v(t))$ *to be*

$$\sup_{a<x<b}\left(\int_x^b v_0(t)\,dt\right)^{1/q}\left(\int_a^x\left(\int_t^x\phi\right)^{\lambda p'}v(t)^{1-p'}\,dt\right)^{1/p'}\quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_x^b v_0(t)\,dt\right)^{r/p}\left(\int_a^x\left(\int_t^x\phi\right)^{\lambda p'}v(t)^{1-p'}\,dt\right)^{r/p'}v_0(x)\,dx\right)^{1/r}\quad \text{if } q<p,$$

and $B_2'([a,b],\phi(s),\lambda,v_0(t),v(t))$ *to be*

$$\sup_{a<x<b}\left(\int_a^x v_0(t)\,dt\right)^{1/q}\left(\int_x^b\left(\int_x^t\phi\right)^{\lambda p'}v(t)^{1-p'}\,dt\right)^{1/p'}\quad \text{if } p\le q, \text{ and}$$

$$\left(\int_a^b\left(\int_a^x v_0(t)\,dt\right)^{r/p}\left(\int_x^b\left(\int_x^t\phi\right)^{\lambda p'}v(t)^{1-p'}\,dt\right)^{r/p'}v_0(x)\,dx\right)^{1/r}\quad \text{if } q<p.$$

We remark that $B([a,b],v_0(t),v(t))$ is finite if and only if the Hardy inequality

$$\left(\int_a^b\left|\int_a^x f(t)\,dt\right|^q v_0(x)\,dx\right)^{1/q}\le C\left(\int_a^b|f(x)|^p v(x)\,dx\right)^{1/p}$$

holds for all f. A history of this problem may be found in [8] and a simple proof of the case $0<q<p$, $1<p<\infty$ was given recently in [10]. A change of variable shows that $B'([a,b],v_0(t),v(t))$ is finite if and only if the conjugate Hardy inequality

$$\left(\int_a^b\left|\int_x^b f(t)\,dt\right|^q v_0(x)\,dx\right)^{1/q}\le C\left(\int_a^b|f(x)|^p v(x)\,dx\right)^{1/p}$$

holds for all f.

Both $B_1([a,b],\phi(s),\lambda,v_0(t),v(t))$ and $B_2([a,b],\phi(s),\lambda,v_0(t),v(t))$ are finite if and only if, by Theorem 1.1 in [11], the inequality

$$\left(\int_a^b \left|\int_a^x \left(\int_t^x \phi(s)\,ds\right)^\lambda f(t)\,dt\right|^q v_0(x)\,dx\right)^{1/q} \le C\left(\int_a^b |f(x)|^p v(x)\,dx\right)^{1/p}.$$

holds. Also, $B_1'([a,b],\phi(s),\lambda,v_0(t),v(t))$ and $B_2'([a,b],\phi(s),\lambda,v_0(t),v(t))$ are both finite if and only if the inequality

$$\left(\int_a^b \left|\int_x^b \left(\int_x^t \phi(s)\,ds\right)^\lambda f(t)\,dt\right|^q v_0(x)\,dx\right)^{1/q} \le C\left(\int_a^b |f(x)|^p v(x)\,dx\right)^{1/p}$$

holds.

Now we return to the boundary value problems (1.2) and (1.3). We solve these overdetermined problems by successively solving (1.1) and then (3.1).

Definition 3.4. *Suppose M_0 and M_1 are subsets of N_k, with $|M_0|+|M_1|=k$, that satisfy the Pólya condition and let $G(x,s)$ be the Green's function of the problem* (3.1)*. Define T by*

$$Tg(x)=\int_0^z g(t)\left[\int_t^z |G(x,s)|\,ds\right]dt+\int_z^1 g(t)\left[\int_z^t |G(x,s)|\,ds\right]dt.$$

Lemma 3.5. *Let $0<q<\infty$ and $1<p<\infty$, let w_0 and w be weights and let z satisfy* (1.4)*. Suppose M_0 and M_1 are subsets of N_k, with $|M_0|+|M_1|=k$, that satisfy the Pólya condition. Then there exists a constant C such that*

$$\|u\|_{q\,w_0}\le C\|f\|_{p\,w},\quad \textit{for } f \textit{ and } u \textit{ satisfying } (1.3) \tag{3.6}$$

if and only if $T: L^p_w\to L^q_{w_0}$.

Proof. Suppose $T: L^p_w\to L^q_{w_0}$ and f and u satisfy (1.3). Set $g=(\chi_{(0,z)}-\chi_{(z,1)})f$. Since f and $u^{(k)}$ satisfy (1.1) we may apply Lemma 2.2 to get $u^{(k)}=Sg$, where S is the operator of Definition 2.1. Since $u^{(k)}$ and u satisfy (3.1) we also get

$$u(x)=\int_0^1 G(x,s)u^{(k)}(s)\,ds.$$

Combining these, and using the fact that G does not change sign on $(0,1)\times(0,1)$, we have

$$\begin{aligned}
\pm u(x)&=\int_0^1 |G(x,s)|Sg(s)\,ds\\
&=\int_0^z |G(x,s)|\int_0^s g(t)\,dt\,ds+\int_z^1 |G(x,s)|\int_s^1 g(t)\,dt\,ds\\
&=\int_0^z g(t)\left[\int_t^z |G(x,s)|\,ds\right]dt+\int_z^1 g(t)\left[\int_z^t |G(x,s)|\,ds\right]dt=Tg(x).
\end{aligned}$$

Thus,

$$\|u\|_{q\,w_0} = \|Tg\|_{q\,w_0} \le C\|g\|_{p\,w} = C\|f\|_{p,w}.$$

Conversely, suppose that (3.6) holds. Since T is a non-negative operator, Lemma 1.1 shows that it is enough to prove that $\|Tg\|_{q\,w_0} \le C\|g\|_{p\,w}$ for functions $g \in L^p_w$ satisfying $\int_0^z g = \int_z^1 g$ in order to conclude that $S : L^p_w \to L^q_{w_0}$. Fix such a g and define f and u by

$$f = (\chi_{(0,z)} - \chi_{(z,1)})g, \quad u = \int_0^1 G(x,s)Sg(s)\,ds.$$

Calculating as above we see that $\pm u(x) = Tg(x)$. The definition of u shows that Sg and u satisfy (3.1) so we have the endpoint conditions

$$u^{(i)}(0) = 0 \text{ for } i \in M_0 \text{ and } u^{(i)}(1) = 0 \text{ for } i \in M_1.$$

We also have $u^{(k)}(x) = Sg$ so, using Definition 2.1,

$$u^{(k)}(0) = Sg(0) = 0 \text{ and } u^{(k)}(1) = Sg(1) = 0.$$

Finally, differentiation yields $u^{(k+1)} = f$ and we have shown that f and u satisfy (1.3). Thus

$$\|Tg\|_{q\,w_0} = \|u\|_{q\,w_0} \le C\|f\|_{p,w} = C\|g\|_{p\,w}$$

which completes the proof.

Theorem 3.6. *Suppose that $p, q \in (1,\infty)$, w_0 and w are weights and z satisfies* (1.4)*. Then there exists a constant C such that*

$$\|u\|_{q\,w_0} \le C\|f\|_{p\,w}, \quad \textit{for } f \textit{ and } u \textit{ satisfying } (1.2) \tag{3.7}$$

if and only if

$$B_1([0,z], 1, k, w_0(t), w(t)) < \infty, \tag{3.8}$$
$$B_2([0,z], 1, k, w_0(t), w(t)) < \infty, \tag{3.9}$$
$$B([z,1], (t-z)^{(k-1)q}w_0(t), (t-z)^{-p}w(t)) < \infty, \tag{3.10}$$
$$B'([z,1], (t-z)^{kq}w_0(t), w(t)) < \infty, \textit{ and} \tag{3.11}$$
$$\sup_{j=1,\dots,k} \left(\int_z^1 (t-z)^{(k-j)q} w_0(t)\,dt\right)^{1/q} \left(\int_0^z (z-t)^{jp'} w(t)^{1-p'}\,dt\right)^{1/p'} < \infty. \tag{3.12}$$

Proof. Note that the boundary value problem (1.2) is a special case of the problem (1.3), we just take $M_0 = N_k$ and let M_1 be empty. By Lemma 3.5 it is enough the prove the equivalence of the conditions (3.8)–(3.12) and the boundedness of $T : L^p_w \to L^q_{w_0}$. Using the expression (3.3) for $G(x, s)$, Definition 3.4 reduces to

$$Tg(x) = \int_0^z g(t) \left[\int_t^z \frac{(x-s)^{k-1}}{(k-1)!} \chi_{(0,x)}(s)\, ds \right] dt \\ + \int_z^1 g(t) \left[\int_z^t \frac{(x-s)^{k-1}}{(k-1)!} \chi_{(0,x)}(s)\, ds \right] dt.$$

If $x < z$ the second term drops out and, performing the inner integration in the first term we have

$$Tg(x) = \int_0^x \frac{(x-t)^k}{k!} g(t)\, dt.$$

If $x > z$ some careful simplification yields

$$Tg(x) = \int_0^z \frac{(x-t)^k - (x-z)^k}{k!} g(t)\, dt \\ + \int_z^x \frac{(x-z)^k - (x-t)^k}{k!} g(t)\, dt + \frac{(x-z)^k}{k!} \int_x^1 g(t)\, dt.$$

Thus $k!Tg(x) = T_1g(x) + T_2g(x) + T_3g(x) + T_4g(x)$, where

$$T_1g(x) = \int_0^x (x-t)^k g(t)\, dt \chi_{(0,z)}(x),$$

a Riemann-Liouville operator on $(0, z)$;

$$T_2g(x) = \int_0^z [(x-t)^k - (x-z)^k] g(t)\, dt \chi_{(z,1)}(x) \\ = \sum_{j=1}^k \binom{k}{j} (x-z)^{k-j} \int_0^z (z-t)^j g(t)\, dt \chi_{(z,1)}(x),$$

a sum of rank one operators;

$$T_3g(x) = \int_z^x [(x-z)^k - (x-t)^k] g(t)\, dt \chi_{(z,1)}(x) \\ \approx (x-z)^{k-1} \int_z^x (t-z) g(t)\, dt \chi_{(z,1)}(x),$$

which is equivalent to a Hardy operator on $(z, 1)$; and

$$T_4g(x) = (x-z)^k \int_x^1 g(t)\, dt \chi_{(z,1)}(x),$$

a conjugate Hardy operator on $(z, 1)$.

Note that the Binomial Theorem gives the equation

$$(x-t)^k - (x-z)^k = [(x-z)+(z-t)]^k - (x-z)^k = \sum_{j=1}^{k} \binom{k}{j} (x-z)^{k-j}(z-t)^j$$

used to simplify T_2 and the straightforward estimate

$$(t-z)(x-z)^{k-1} \le (x-z)^k - (x-t)^k \le k(t-z)(x-z)^{k-1}, \quad 0 < z < t < x < 1,$$

was used to simplify T_3.

Since T_1, T_2, T_3, and T_4 are non-negative operators their separate boundedness is necessary and sufficient for the boundedness of their sum. T_1 is bounded if and only if (3.8) and (3.9) hold. The Hardy operators T_3 and T_4 are bounded if and only if (3.10) and (3.11) hold repectively. The sharpness of Hölder's inequality can be used to show that T_2 is bounded if and only if (3.12) holds.

If $M_1 = N_k$ and M_0 is empty it is easy to formulate and prove a similar theorem. We omit the details.

Theorem 3.7. *Suppose $p, q \in (1, \infty)$, w_0 and w are weights, z satisfies* (1.4)*, and M_0 and M_1 are non-empty subsets of $\{0, 1, \dots, k-1\}$ such that $|M_0| + |M_1| = k$ which satisfy the Pólya condition. Then there exists a constant C such that*

$$\|u\|_{q\,w_0} \le C\|f\|_{p\,w}, \quad \textit{for } f \textit{ and } u \textit{ satisfying } (1.3) \tag{3.13}$$

if and only if

$$B'\Big([0,z], t^{aq}(1-t)^{Bq} w_0(t), \Big(\int_t^z s^C(1-s)^d\,ds\Big)^{-p} w(t)\Big) < \infty,$$
$$B_1([0,z], s^c(1-s)^D, 1, t^{Aq}(1-t)^{bq} w_0(t), w(t)) < \infty,$$
$$B_2([0,z], s^c(1-s)^D, 1, t^{Aq}(1-t)^{bq} w_0(t), w(t)) < \infty,$$
$$B\Big([0,z], t^{aq}(1-t)^{Bq} \Big(\int_t^z s^C(1-s)^d\,ds\Big)^{q} w_0(t), w(t)\Big) < \infty,$$
$$\Big(\int_0^z t^{aq}(1-t)^{Bq} w_0(t)\,dt\Big)^{1/q} \Big(\int_z^1 \Big(\int_z^t s^C(1-s)^d\,ds\Big)^{p'} w(t)^{1-p'}\,dt\Big)^{1/p'} < \infty,$$
$$B\Big([z,1], t^{Aq}(1-t)^{bq} w_0(t), \Big(\int_z^t s^c(1-s)^D\,ds\Big)^{-p} w(t)\Big) < \infty,$$
$$B_1'([z,1], s^C(1-s)^d, 1, t^{aq}(1-t)^{Bq} w_0(t), w(t)) < \infty,$$
$$B_2'([z,1], s^C(1-s)^d, 1, t^{aq}(1-t)^{Bq} w_0(t), w(t)) < \infty,$$
$$B'\Big([z,1], t^{Aq}(1-t)^{bq} \Big(\int_z^t s^c(1-s)^D\,ds\Big)^{q} w_0(t), w(t)\Big) < \infty, \textit{ and}$$
$$\Big(\int_z^1 t^{Aq}(1-t)^{bq} w_0(t)\,dt\Big)^{1/q} \Big(\int_0^z \Big(\int_t^z s^c(1-s)^D\,ds\Big)^{p'} w(t)^{1-p'}\,dt\Big)^{1/p'} < \infty.$$

Proof. Again we begin by applying Lemma 3.5 to establish the equivalence of (3.13) with the boundedness of $T : L^p_w \to L^q_{w_0}$. Since neither M_0 nor M_1 is empty we can use the estimates (3.4) and (3.5) for $G(x,s)$ provided by Proposition 3.2 to get

$$\begin{aligned}
\int_t^z G(x,s)\,ds \approx{}& x^a(1-x)^B \int_t^z s^C(1-s)^d\,ds, \qquad \text{if } x \le t;\\
\int_t^z G(x,s)\,dt \approx{}& x^A(1-x)^b \int_t^x s^c(1-s)^D\,ds\\
&+ x^a(1-x)^B \int_x^z s^C(1-s)^d\,ds, \qquad \text{if } t < x \le z;\\
\int_t^z G(x,s)\,ds \approx{}& x^A(1-x)^b \int_t^z s^c(1-s)^D\,ds, \qquad \text{if } z < x;\\
\int_z^t G(x,s)\,ds \approx{}& x^a(1-x)^B \int_z^t s^C(1-s)^d\,ds, \qquad \text{if } x \le z;\\
\int_z^t G(x,s)\,dt \approx{}& x^A(1-x)^b \int_z^x s^c(1-s)^D\,ds\\
&+ x^a(1-x)^B \int_x^t s^C(1-s)^d\,ds, \qquad \text{if } z < x \le t; \text{ and}\\
\int_z^t G(x,s)\,ds \approx{}& x^A(1-x)^b \int_z^t s^c(1-s)^D\,ds, \qquad \text{if } t < x.
\end{aligned}$$

Making these substitutions in the expression for T from Definition 3.4 we obtain $Tg(x) \approx T_1g(x) + \cdots + T_8g(x)$ where

$$\begin{aligned}
T_1g(x) &= x^a(1-x)^B \int_x^z \left[\int_t^z s^C(1-s)^d\,ds\right] g(t)\,dt\chi_{(0,z)}(x),\\
T_2g(x) &= x^A(1-x)^b \int_0^x \left[\int_t^x s^c(1-s)^D\,ds\right] g(t)\,dt\chi_{(0,z)}(x),\\
T_3g(x) &= x^a(1-x)^B \int_0^x \left[\int_x^z s^C(1-s)^d\,ds\right] g(t)\,dt\chi_{(0,z)}(x),\\
T_4g(x) &= x^a(1-x)^B \int_z^1 \left[\int_z^t s^C(1-s)^d\,ds\right] g(t)\,dt\chi_{(0,z)}(x),\\
T_5g(x) &= x^A(1-x)^b \int_z^x \left[\int_z^t s^c(1-s)^D\,ds\right] g(t)\,dt\chi_{(z,1)}(x),\\
T_6g(x) &= x^a(1-x)^B \int_x^1 \left[\int_x^t s^C(1-s)^d\,ds\right] g(t)\,dt\chi_{(z,1)}(x),\\
T_7g(x) &= x^A(1-x)^b \int_x^1 \left[\int_z^x s^c(1-s)^D\,ds\right] g(t)\,dt\chi_{(z,1)}(x),\\
T_8g(x) &= x^A(1-x)^b \int_0^z \left[\int_t^z s^c(1-s)^D\,ds\right] g(t)\,dt\chi_{(z,1)}(x).
\end{aligned}$$

Once again, the boundedness of these positive operators is equivalent to the boundedness of their sum so we may examine each one individually. T_1 is a conjugate Hardy operator on $[0,z]$, bounded if and only if

$$B'([0,z], t^{aq}(1-t)^{Bq}w_0(t), \left(\int_t^z s^C(1-s)^d\,ds\right)^{-p} w(t)) < \infty.$$

T_2 is bounded if and only if

$$B_1([0,z], s^c(1-s)^D, 1, t^{Aq}(1-t)^{bq}w_0(t), w(t))$$

and

$$B_2([0,z], s^c(1-s)^D, 1, t^{Aq}(1-t)^{bq}w_0(t), w(t))$$

are finite.

T_3 is a Hardy operator on $[0,z]$, bounded if and only if

$$B([0,z], t^{aq}(1-t)^{Bq}\left(\int_t^z s^C(1-s)^d\,ds\right)^{q} w_0(t), w(t)) < \infty.$$

The operator T_4 is of rank one so the sharpness of Hölder's inequality gives necessary and sufficient conditions for its boundedness:

$$\left(\int_0^z t^{aq}(1-t)^{Bq}w_0(t)\,dt\right)^{1/q}\left(\int_z^1\left(\int_z^t s^C(1-s)^d\,ds\right)^{p'} w(t)^{1-p'}\,dt\right)^{1/p'} < \infty.$$

In the same way as $T_1 \dots T_4$ give rise to the first five conditions, $T_5 \dots T_8$ give rise to the last five. This completes the proof.

Example 3.8. *Suppose p, q, M_0, and M_1 are as in Theorem 3.7 and set*

$$w(t)^{1-p'} = t^\alpha(1-t)^\beta \quad \text{and} \quad w_0(t) = t^\gamma(1-t)^\delta. \tag{3.14}$$

Then there exists a constant C such that (3.13) *holds provided $\alpha+1$, $\beta+1$, $\gamma+1+aq$ and $\delta+1+bq$ are positive. Here a and b depend on M_0 and M_1 as in Definition 3.1.*

Proof. Set $z = 1/2$. With $w(t)$ as above, it is immediate that (1.4) holds so it remains to verify the ten weight conditions of Theorem 3.7.

From Definition 3.1 we see that either $A = a$ or $A = a-1$ and either $B = b$ or $B = b-1$. Suppose for the moment that $A = a$ and $B = b$. To verify the weight conditions in this case we follow these three steps: 1. Each condition involves integrals over subintervals of $(0,z)$ or subintervals of $(z,1)$ or both so extend the range of integration in each case to either $(0,z)$ or $(z,1)$ as appropriate. 2. Use the fact that (positive or negative) powers of $1-x$, $1-t$, and $1-s$ are bounded above on the interval $(0,z)$ and powers of x, t and s are bounded above on the interval $(z,1)$. 3. Use the restrictions on α, β, γ, and δ to evaluate the remaining integrals.

We illustrate this procedure by showing that the third weight condition is satisfied when $p \le q$.

$$B_2([0,z], s^c(1-s)^D, 1, t^{Aq}(1-t)^{bq}w_0(t), w(t))$$

$$= \sup_{0<x<z} \left(\int_x^z t^{Aq+\gamma}(1-t)^{bq+\delta}\,dt\right)^{1/q} \left(\int_0^x \left(\int_t^x s^c(1-s)^D\,ds\right)^{p'} t^\alpha (1-t)^\beta\,dt\right)^{1/p'}$$

$$\le K \sup_{0<x<z} \left(\int_0^z t^{Aq+\gamma}\,dt\right)^{1/q} \left(\int_0^z \left(\int_0^z s^c\,ds\right)^{p'} t^\alpha\,dt\right)^{1/p'}$$

which is finite because $c \ge 0$, $\alpha + 1 > 0$, and $Aq + \gamma + 1 = aq + \gamma + 1 > 0$. ($K$ is the constant arising from Step 2. It depends on $bq + \delta$ and β.)

The case $1 < q < p$ of the third weight condition, as well as all the other weight conditions may be verified in this way. If $A = a - 1$ or $B = b - 1$ the result still holds but a bit more care must be taken in the estimates. We omit the details.

Remark. If the weights w and w_0 are of the form (3.14) and satisfy (1.4) then the restrictions on α, β, γ, and δ are also necessary for the inequality (3.13).

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