# SPACES DEFINED BY THE LEVEL FUNCTION AND THEIR DUALS

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Abstract. The classical level function construction of Halperin and Lorentz is extended to Lebesgue spaces with general measures. The construction is also carried farther. In particular, the level function is considered as a monotone map on its natural domain, a superspace of $L^p$. These domains are shown to be Banach spaces which, although closely tied to $L^p$ spaces, are not reflexive. A related construction is given which characterizes their dual spaces.

## 1. Introduction

The familiar Hölder inequality is (for a measure $\lambda$ on $\mathbf{R}$)

$$\left|\int_{\mathbf{R}} fg\,d\lambda\right| \le \|f\|_{p,\lambda}\|g\|_{p',\lambda}$$

where $1 \le p \le \infty$, $1/p+1/p' = 1$ and $\|h\|_{r,\lambda} = (\int_{\mathbf{R}} |h|^r\,d\lambda)^{1/r}$ is the norm on the Lebesgue space $L^r_\lambda$. The inequality is sharp, in the sense that

$$\sup\left|\int_{\mathbf{R}} fg\,d\lambda\right| = \|f\|_{p,\lambda}$$

where the supremum is taken over all functions $g$ such that $\|g\|_{p',\lambda} \le 1$. If $g$ is not free to range over the whole unit ball of $L^{p'}_\lambda$, but is constrained in some way, the sharpness of Hölder's inequality may be lost. The problem which has motivated this work is that of determining a sharp inequality to substitute for Hölder's inequality when $g$ is constrained to be positive and decreasing. (Actually it will be more convenient to require that $g$ be non-negative and non-increasing.) The substitute inequality is easy enough to write down

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but some work is required to understand how to use the result profitably. The substitute is this: If $g$ is non-negative and non-increasing then

$$\left|\int_{\mathbf{R}} fg\,d\lambda\right| \le \|f\|_{p\downarrow\lambda}\|g\|_{p',\lambda} \tag{1.1}$$

where

$$\|f\|_{p\downarrow\lambda} = \sup\left\{\int_{\mathbf{R}} |f|g\,d\lambda \,:\, g \text{ non-negative and non-increasing, and } \|g\|_{p',\lambda} \le 1\right\}.$$

The proof of this inequality is trivial but to use it effectively we must understand the expression $\|\cdot\|_{p\downarrow\lambda}$. That is the purpose of this paper.

This approach to Hölder's inequality has been considered before. In [3], Halperin introduces what he calls "D-type Hölder inequalities" which are similar to (1.1) but in which the measure $\lambda$ is assumed to be just a weight function times Lebesgue measure. Lorentz, in [4, §3.6], gives an account of Halperin's work and provides a new approach to the basic result, the construction of the level function. Our approach will be similar to Lorentz'. Halperin's results have been used recently to prove weighted Hardy inequalities [9]. Our generalisation here enables us to prove Hardy inequalities with general norms. (See Section 7.) In particular, inequalities for series and integrals can be proved simultaneously.

In Section 3 we show that $\|\cdot\|_{p\downarrow\lambda}$ is a norm and defines a space $L^{p\downarrow}_\lambda$ containing $L^p_\lambda$. Sections 4 and 5 are devoted to the construction of the level function and its extension to all of $L^{p\downarrow}_\lambda$. The dual space, $L^{p'*}_\lambda$, is constructed in Section 6 and in Section 7 some application of these ideas are outlined. We complete this introduction by introducing some notation and then proceed to Section 2 where we clarify what is meant by a non-increasing function in Lebesgue space.

Most of the notation used here is either standard or defined within the paper. Hopefully, the remainder is discussed here. A Borel measure on $\mathbf{R}$ is a non-negative measure defined on the Borel sets (the $\sigma$-algebra generated by the open sets) which is finite on intervals. Note that it is automatically $\sigma$-finite. If $f$ and $g$ are $\lambda$-measurable functions we say "$f$ majorises $g$" or "$f$ is a majorant of $g$" provided $f(x) \ge g(x)$ for $\lambda$-almost every $x$. We adopt the convention that integrals written with limits include the limit points in the range of integration except when the limits are $\pm\infty$. Thus $\int_a^b$ means $\int_{[a,b]}$ but $\int_{-\infty}^b$ means $\int_{(-\infty,b]}$. The notation $p'$ for the conjugate index of $p$ is used throughout so that $1/p + 1/p' = 1$ when $1 \le p \le \infty$.

## 2. Non-increasing Functions and Concavity

The definition of a non-increasing function is straightforward.

**Definition 2.1.** *Suppose $S \subset \mathbf{R}$. A function $g : S \to [0,\infty)$ is non-increasing on $S$ provided $g(x) \ge g(y)$ whenever $x \le y$.*

In this paper, however, we will be concerned with Lebesgue spaces in which functions are identified when they agree almost everywhere with respect to the measure $\lambda$. It is less straightforward to identify non-increasing equivalence classes of functions. The object of this section is to formulate a suitable definition of a non-increasing function in Lebesgue space. In addition we introduce the notion of $\lambda$-concavity and illustrate the principle connections between the two concepts.

To begin we must carefully define the essential supremum and the essential infimum.

**Definition 2.2.** *If $(X, \mu)$ is a measure space and $g : X \to [0, \infty)$ is $\mu$-measurable then*

$$\operatorname{ess\,sup}_\mu(g, X) = \sup\{\alpha : \mu\{x \in X : g(x) > \alpha\} > 0\}, \quad \textit{and}$$
$$\operatorname{ess\,inf}_\mu(g, X) = \inf\{\alpha : \mu\{x \in X : g(x) < \alpha\} > 0\}.$$

*If $\mu X = 0$ then $\operatorname{ess\,sup}_\mu(g, X) = 0$ and $\operatorname{ess\,inf}_\mu(g, X) = \infty$.*

It is immediate that if $Y \subseteq X$ then $\operatorname{ess\,sup}_\mu(g, Y) \leq \operatorname{ess\,sup}_\mu(g, X)$ and $\operatorname{ess\,inf}_\mu(g, Y) \geq \operatorname{ess\,inf}_\mu(g, X)$.

The next lemma contains an obvious (but not trivial) property of the essential supremum and essential infimum.

**Lemma 2.3.** *Suppose $\mu$ is a Borel measure and $g : \mathbf{R} \to [0, \infty)$ is Borel measurable. Then $g(x) \leq \operatorname{ess\,sup}_\mu(g, [x, \infty))$ and $g(x) \geq \operatorname{ess\,inf}_\mu(g, (-\infty, x])$ for $\mu$-almost every $x \in \mathbf{R}$.*

*Proof.* We prove the first statement only. Let $\bar{g}(x) = \operatorname{ess\,sup}_\mu(g, [x, \infty))$. Since $\bar{g}$ is non-increasing, it is Borel measurable. Choose sets $E_1, E_2, E_3, \dots$ of finite $\mu$-measure, whose union is all of $\mathbf{R}$. Fix $\varepsilon > 0$ and set

$$S_{m,n} = \{x \in E_m : g(x) - \bar{g}(x) > \varepsilon, \varepsilon n \leq g(x) < \varepsilon(n + 1)\}.$$

Certainly $\mu S_{m,n} < \infty$ and $\cup_{m=1}^\infty \cup_{n=0}^\infty S_{m,n} = \{x \in \mathbf{R} : g(x) - \bar{g}(x) > \varepsilon\}$. To complete the proof we show that $\mu S_{m,n} = 0$ for each fixed $m$ and $n$.

Suppose $\mu S_{m,n} > 0$ for some $m$ and $n$. If $x \in S_{m,n}$ then $g(x) > \varepsilon + \operatorname{ess\,sup}_\mu(g, [x, \infty))$ so $x$ is not an atom for $\mu$. Hence, if $\chi$ is the characteristic function of $S_{m,n}$, $\int_{-\infty}^y \chi \, d\mu$ is a continuous function of $y$. Thus there exists a $y \in \mathbf{R}$ such that $\int_{-\infty}^y \chi \, d\mu = (\mu S_{m,n})/2$. Choose $x \in S_{m,n}$ with $x < y$. Since $\mu([x, \infty) \cap S_{m,n}) > 0$, we have

$$\bar{g}(x) = \operatorname{ess\,sup}_\mu(g, [x, \infty)) \geq \operatorname{ess\,sup}_\mu(g, [x, \infty) \cap S_{m,n}) \geq \varepsilon n.$$

But since $x \in S_{m,n}$, $\bar{g}(x) < g(x) - \varepsilon < \varepsilon(n + 1) - \varepsilon = \varepsilon n$. This contradiction completes the proof.

There are many ways to approach the notion of a function being non-increasing almost everywhere. The next theorem shows that five of the most tempting are equivalent.

**Theorem 2.4.** *If $\mu$ is a regular, Borel measure and $g : \mathbf{R} \to [0, \infty)$ is Borel measurable, then the following are equivalent.*

(1) *For some non-increasing function $\bar{g}$ on $\mathbf{R}$, $g = \bar{g}$ $\mu$-almost everywhere.*
(2) *$g$ is non-increasing on some subset $S \subset \mathbf{R}$ such that $\mu(\mathbf{R} \setminus S) = 0$.*
(3) $\mu \times \mu\{(x, y) : x \leq y, g(x) < g(y)\} = 0$.
(4) *Whenever $a \leq b \leq c \leq d$,*

$$\int_c^d d\mu \int_a^b g \, d\mu \geq \int_c^d g \, d\mu \int_a^b d\mu.$$

(5) $\operatorname{ess\,sup}_\mu(g, [x, \infty)) \leq \operatorname{ess\,inf}_\mu(g, (-\infty, x])$ *for all $x \in \mathbf{R}$.*

*Proof.* (5)⇒(1). Let $\bar{g}(x) = \operatorname{ess\,sup}_\mu(g, [x,\infty))$. $\bar{g}$ is non-increasing and by the lemma

$$g(x) \le \bar{g}(x) = \operatorname{ess\,sup}_\mu(g, [x,\infty)) \le \operatorname{ess\,inf}_\mu(g, (-\infty, x]) \le g(x)$$

for $\mu$-almost every $x \in \mathbf{R}$. Thus $g = \bar{g}$ $\mu$-almost everywhere.

(1)⇒(2). Set $S = \{x : g(x) = \bar{g}(x)\}$.

(2)⇒(3). If $x \le y$ and $x, y \in S$ then $g(x) \ge g(y)$. Hence

$$\mu \times \mu\{(x,y) : x \le y, g(x) < g(y)\} \le \mu \times \mu(((\mathbf{R} \setminus S) \times \mathbf{R}) \cup (\mathbf{R} \times (\mathbf{R} \setminus S))) = 0.$$

(3)⇒(4). If $a \le b \le c \le d$ then for $\mu \times \mu$-almost every pair $(x,y)$ with $a \le x \le b$ and $c \le y \le d$ we have $g(x) \ge g(y)$ so

$$\begin{aligned}\int_c^d d\mu(y) \int_a^b g(x)\,d\mu(x) &= \int_{a\le x\le b, c\le y\le d} g(x)\,d(\mu\times\mu)(x,y) \\ &\ge \int_{a\le x\le b, c\le y\le d} g(y)\,d(\mu\times\mu)(x,y) = \int_c^d g(y)\,d\mu(y)\int_a^b d\mu(x).\end{aligned}$$

(4)⇒(5). Fix $x \in \mathbf{R}$. If $U$ is any Borel subset of $(-\infty, x]$ and $V$ is any Borel subset of $[x, \infty)$ then the hypothesis of (4), together with the regularity of $\mu$ yields

$$\int_V d\mu \int_U g\,d\mu \ge \int_V g\,d\mu \int_U d\mu.$$

Set $\alpha = \operatorname{ess\,sup}_\mu(g, [x,\infty))$ and $\beta = \operatorname{ess\,inf}_\mu(g, (-\infty, x])$ and fix $\varepsilon > 0$. Choose $U \subset \{y \le x : g(y) < \beta + \varepsilon\}$ such that $0 < \mu U < \infty$. Choose $V \subset \{y \ge x : g(y) > \alpha - \varepsilon\}$ such that $0 < \mu V < \infty$. Now

$$(\beta+\varepsilon)\mu V \mu U \ge \int_V d\mu \int_U g\,d\mu \ge \int_V g\,d\mu \int_U d\mu \ge (\alpha - \varepsilon)\mu V \mu U$$

so $\alpha - \varepsilon \le \beta + \varepsilon$. Since $\varepsilon$ was arbitrary, $\alpha \le \beta$ as required.

**Definition 2.5.** *Suppose $\mu$ is a regular, Borel measure. A Borel measurable function $g : \mathbf{R} \to [0,\infty)$ is non-increasing $\mu$-almost everywhere, or just $\mu$-non-increasing, provided one and hence all of the conditions* (1)–(5) *of Theorem 2.4 are satisfied.*

Condition (1) shows that we may now speak of $\mu$-non-increasing functions in $L^p_\mu$ since an equivalence class (modulo equality $\mu$-almost everywhere) is $\mu$-non-increasing precisely when (at least) one representative of the class is non-increasing on $\mathbf{R}$. Indeed, given a $\mu$-non-increasing function in some Lebesgue space we are free to suppose that we have a non-increasing function which represents the same equivalence class. Conditions (4) and (5) will prove to be the most useful in the sequel, not only as properties of known $\mu$-non-increasing functions but also as means of showing that a given function is itself $\mu$-non-increasing.

Integrating a non-increasing function with respect to Lebesgue measure gives a concave function. We will make use of an analogue of concavity defined here. (See also [4].)

**Definition 2.6.** *Suppose $\lambda$ is a regular, Borel measure such that $\Lambda(x) = \lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. A $\lambda$-measurable function $G$ on $\mathbf{R}$ is said to be $\lambda$-concave provided*

$$(\Lambda(b) - \Lambda(x))(G(x) - G(a)) \geq (G(b) - G(x))(\Lambda(x) - \Lambda(a)) \tag{2.1}$$

*whenever $a \leq x \leq b$.*

It is occasionally convenient to use (2.1) in the form

$$G(x)(\Lambda(b) - \Lambda(a)) \geq G(a)(\Lambda(b) - \Lambda(x)) + G(b)(\Lambda(x) - \Lambda(a)). \tag{2.2}$$

It is not difficult to see that the definition of $\lambda$-concavity implies the following, seemingly stronger statement. If $a \leq b \leq d$, and $a \leq c \leq d$ then

$$(\Lambda(d) - \Lambda(c))(G(b) - G(a)) \geq (G(d) - G(c))(\Lambda(b) - \Lambda(a)). \tag{2.3}$$

The next theorem relates the notions of $\lambda$-non-increasing and $\lambda$-concave.

**Theorem 2.7.** *Again let $\lambda$ be a regular, Borel measure on $\mathbf{R}$ such that $\lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. Suppose that $f$ is a non-negative, $\lambda$-measurable function on $\mathbf{R}$ such that $F(x) = \int_{-\infty}^{x} f\, d\lambda < \infty$ for all $x \in \mathbf{R}$. Then $F$ is $\lambda$-concave if and only if $f$ is $\lambda$-non-increasing.*

*Proof.* Suppose that $f$ is $\lambda$-non-increasing and that $a \leq x \leq b$. By part (4) of Theorem 2.4, for each $\varepsilon > 0$,

$$\int_{x+\varepsilon}^{b+\varepsilon} d\lambda \int_{a+\varepsilon}^{x+\varepsilon} f\, d\lambda \geq \int_{x+\varepsilon}^{b+\varepsilon} f\, d\lambda \int_{a+\varepsilon}^{x+\varepsilon} d\lambda.$$

As $\varepsilon \to 0$ this becomes

$$(\Lambda(b) - \Lambda(x))(F(x) - F(a)) \geq (F(b) - F(x))(\Lambda(x) - \Lambda(a))$$

so $F$ is $\lambda$-concave.

Conversely, if $F$ is $\lambda$-concave then suppose $a \leq b \leq c \leq d$ and $\varepsilon > 0$. We have $a - \varepsilon \leq b \leq d$ and $a - \varepsilon \leq c - \varepsilon \leq d$ so by (2.3)

$$(\Lambda(d) - \Lambda(c - \varepsilon))(F(b) - F(a - \varepsilon)) \geq (F(d) - F(c - \varepsilon))(\Lambda(b) - \Lambda(a - \varepsilon)).$$

As $\varepsilon \to 0$ this becomes

$$\int_{c}^{d} d\lambda \int_{a}^{b} f\, d\lambda \geq \int_{c}^{d} f\, d\lambda \int_{a}^{b} d\lambda$$

so by part (4) of Theorem 2.4 $f$ is $\lambda$-non-increasing.

**Theorem 2.8.** *Suppose $\lambda$ is a regular, Borel measure on $\mathbf{R}$ such that $\Lambda(x) = \lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. If $F$ is a non-negative, $\lambda$-measurable function on $\mathbf{R}$ which has a $\lambda$-concave majorant then $F$ has a unique least $\lambda$-concave majorant denoted by $F^\flat$.*

*Proof.* Let $F^\flat(x) = \inf\{G(x) : G$ is a $\lambda$-concave majorant of F$\}$. By hypothesis the infimum is not empty so $F^\flat(x) < \infty$ for $\lambda$-almost every $x$. It remains to prove that $F^\flat$ is $\lambda$-concave since uniqueness follows from minimality. Fix $a, x, b \in \mathbf{R}$ with $a \le x \le b$. If $\Lambda(a) = \Lambda(b)$ then $\Lambda(a) = \Lambda(x) = \Lambda(b)$ so (2.1) holds trivially for $F^\flat$. If $\Lambda(a) < \Lambda(b)$ then for each $\varepsilon > 0$ there exists a $\lambda$-concave majorant $G$ of $F$ such that $G(x) - F^\flat(x) \le \varepsilon/(\Lambda(b) - \Lambda(a))$. Using the $\lambda$-concavity of $G$ in the form (2.2) we have

$$\begin{aligned} F^\flat(x)(\Lambda(b) - \Lambda(a)) + \varepsilon &\ge G(x)(\Lambda(b) - \Lambda(a)) \\ &\ge G(a)(\Lambda(b) - \Lambda(x)) + G(b)(\Lambda(x) - \Lambda(a)) \\ &\ge F^\flat(a)(\Lambda(b) - \Lambda(x)) + F^\flat(b)(\Lambda(x) - \Lambda(a)) \end{aligned}$$

and since $\varepsilon$ was arbitrary,

$$F^\flat(x)(\Lambda(b) - \Lambda(a)) \ge F^\flat(a)(\Lambda(b) - \Lambda(x)) - F^\flat(b)(\Lambda(x) - \Lambda(a)).$$

Therefore $F^\flat$ is $\lambda$-concave.

## 3. The Spaces $L^{p\downarrow}_\lambda$.

Suppose that $\lambda$ is a regular, Borel measure on $\mathbf{R}$ which satisfies $\lambda(-\infty, x] < \infty$ for all $x \in \mathbf{R}$. We make the definition

$$\|f\|_{p\downarrow\lambda} = \sup \int_{\mathbf{R}} |f| g \, d\lambda$$

where the supremum is taken over all non-negative, $\lambda$-non-increasing functions $g$ satisfying $\|g\|_{p',\lambda} \le 1$. Note that our assumption $\lambda(-\infty, x] < \infty$ ensures that there are non-trivial, non-negative, non-increasing functions in $L^{p'}_\lambda$. For $1 \le p \le \infty$ we have $\|f\|_{p\downarrow\lambda} \le \|f\|_{p,\lambda}$ by Hölder's inequality. We define $L^{p\downarrow}_\lambda$ to be the collection of $\lambda$-measurable functions $f$ for which $\|f\|_{p\downarrow\lambda} < \infty$. It is easy to see that $L^{p\downarrow}_\lambda$ is a vector space. Since $\|f\|_{p\downarrow\lambda} \le \|f\|_{p,\lambda}$ we see that $L^{p\downarrow}_\lambda$ contains the Lebesgue space $L^p_\lambda$.

**Proposition 3.1.** *$\|\cdot\|_{p\downarrow\lambda}$ is a norm on $L^{p\downarrow}_\lambda$ for $1 \le p \le \infty$.*

*Proof.* Suppose $\|f\|_{p\downarrow\lambda} < \infty$. Certainly $\|f\|_{p\downarrow\lambda} \ge 0$ and if $\|f\|_{p\downarrow\lambda} = 0$ then since the characteristic function of $(-\infty, x]$ is in $L^{p'}_\lambda$ for all $x \in \mathbf{R}$ we have $\int_{-\infty}^x |f| \, d\lambda = 0$ for all $x \in \mathbf{R}$. Hence $f$ is zero $\lambda$-almost everywhere. It is immediate from the definition that $\|\cdot\|_{p\downarrow\lambda}$ is homogeneous and satisfies the triangle inequality.

**Theorem 3.2.** *If $f \in L^p_\lambda$ with $1 < p < \infty$ then $\|f\|_{p\downarrow\lambda} = \|f\|_{p,\lambda}$ if and only if $|f|$ is $\lambda$-non-increasing.*

*Proof.* If $|f|$ is $\lambda$-non-increasing then $(|f|/\|f\|_{p,\lambda})^{p-1}$ is also $\lambda$-non-increasing. (We may clearly assume that $f$ is not identically zero.) Moreover, $(|f|/\|f\|_{p,\lambda})^{p-1}$ has unit $L^{p'}_\lambda$-norm. Hence

$$\|f\|_{p,\lambda} \ge \|f\|_{p\downarrow\lambda} \ge \int_{\mathbf{R}} |f| (|f|/\|f\|_{p,\lambda})^{p-1} \, d\lambda = \|f\|_{p,\lambda}$$

so the two norms coincide.

Conversely, suppose that $\|f\|_{p,\lambda} = \|f\|_{p\downarrow\lambda}$. For $n = 1, 2, 3, \ldots$, choose non-negative, $\lambda$-non-increasing functions $g_n$ such that $\|g_n\|_{p',\lambda} \le 1$, and $\int |f| g_n \, d\lambda \ge \|f\|_{p,\lambda} - 1/n$. By the Banach-Alaoglu Theorem there is a subsequence $g_{n_k}$ converging weak* to some non-negative function $g$ with $\|g\|_{p',\lambda} \le 1$. Since each of the functions $g_{n_k}$ is $\lambda$-non-increasing we have, by Theorem 2.4, part (4),

$$\int_c^d d\lambda \int_a^b g_{n_k} \, d\lambda \ge \int_c^d g_{n_k} \, d\lambda \int_a^b d\lambda$$

whenever $a \le b \le c \le d$. Together with the fact that the characteristic functions of the intervals $[a, b]$ and $[c, d]$ are in $L^p_\lambda$ this yields, as $k \to \infty$,

$$\int_c^d d\lambda \int_a^b g \, d\lambda \ge \int_c^d g \, d\lambda \int_a^b d\lambda$$

so $g$ is also $\lambda$-non-increasing. Now for each $k$

$$\|f\|_{p,\lambda} \ge \int_{\mathbf{R}} |f| g_{n_k} \, d\lambda \ge \|f\|_{p,\lambda} - 1/n_k$$

so, as $k \to \infty$, we have equality in Hölder's inequality

$$\|f\|_{p,\lambda} = \int_{\mathbf{R}} |f| g \, d\lambda \le \|f\|_{p,\lambda} \|g\|_{p',\lambda} \le \|f\|_{p,\lambda}$$

It follows that $|f|^p$ is a constant multiple of $g^{p'}$ and in particular $|f|$ is $\lambda$-non-increasing.

Since $g \equiv 1$ is non-negative, $\lambda$-non-increasing and in $L^\infty_\lambda$, $\|f\|_{1,\lambda} = \|f\|_{1\downarrow\lambda}$ for all $f \in L^1_\lambda$. Thus $L^{1\downarrow}_\lambda = L^1_\lambda$. Consequently the above proposition does not hold for $p = 1$.

Although the spaces $L^{\infty\downarrow}_\lambda$ and $L^\infty_\lambda$ are quite different, the above proposition does not hold for $p = \infty$ either. Consider a bounded function $f$ which takes the value $\|f\|_{\infty,\lambda}$ on some interval $(-\infty, b)$. Whatever the function does on the rest of the line we would have $\|f\|_{\infty\downarrow\lambda} = \|f\|_{\infty,\lambda}$.

**Example 3.3.** *For* $1 < p < \infty$, $L^{p\downarrow}_\lambda \not\subset L^p_\lambda$.

Let $\lambda(x) = x^{-2} \, dx$ on $(1, \infty)$ and let $\lambda(-\infty, 1] = 0$. Set $f(x) = x^{1/p}$ and notice that $f \notin L^p_\lambda$. We will show that $f \in L^{p\downarrow}_\lambda$. A calculation yields

$$\int_1^{(p')^p} (p' - x^{1/p}) \, d\lambda(x) = \int_{(p')^p}^\infty (x^{1/p} - p') \, d\lambda(x).$$

Suppose that $g$ is non-negative and $\lambda$-non-increasing and that $\|g\|_{p',\lambda} \le 1$. Using the above calculation and Theorem 2.4, part (5) we obtain

$$\begin{aligned}\int_1^{(p')^p} (p' - x^{1/p}) g(x) \, d\lambda(x) &\ge \operatorname{ess\,inf}_\lambda(g, (-\infty, (p')^p]) \int_1^{(p')^p} (p' - x^{1/p}) \, d\lambda(x) \\ &\ge \operatorname{ess\,sup}_\lambda(g, [(p')^p, \infty)) \int_{(p')^p}^\infty (x^{1/p} - p') \, d\lambda(x) \ge \int_{(p')^p}^\infty (x^{1/p} - p') g(x) \, d\lambda(x).\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} fg\,d\lambda = \int_1^\infty x^{1/p} g(x)\,d\lambda(x) \le p' \int_1^\infty g(x)\,d\lambda(x) \le p' \|g\|_{p',\lambda} \left( \int_1^\infty d\lambda \right)^{1/p} \le p'.$$

Thus $\|f\|_{p\downarrow\lambda} \le p'$ so $f \in L^{p\downarrow}_\lambda$.

In order to understand the norm $\|\cdot\|_{p\downarrow\lambda}$ we will represent $\|f\|_{p\downarrow\lambda}$ as the $L^p_\lambda$-norm of a function associated with $f$. This associated function—the level function of $f$—will be constructed in Sections 4 and 5. It will have the following properties.

**Definition 3.4.** *Let $1 \le p \le \infty$. Given $f \in L^{p\downarrow}_\lambda$ we say that $f^o$ is a $p$-level function of $f$ provided $\|f\|_{p\downarrow\lambda} = \|f^o\|_{p,\lambda}$ and $\int |f| g\,d\lambda \le \int f^o g\,d\lambda$ for all non-negative, $\lambda$-non-increasing functions $g \in L^{p'}_\lambda$.*

To show the existence of $p$-level functions we will require the construction of Section 4. Uniqueness, however, is straightforward.

**Proposition 3.5.** *If $1 < p < \infty$ and $f \in L^{p\downarrow}_\lambda$ then there is at most one $p$-level function of $f$.*

*Proof.* Suppose that $f^o$ and $\bar{f}$ are $p$-level functions of $f$. Using Minkowski's inequality and the definition above we estimate as follows

$$\begin{aligned} \|f^o + \bar{f}\|_{p,\lambda} &\le \|f^o\|_{p,\lambda} + \|\bar{f}\|_{p,\lambda} = 2\|f\|_{p\downarrow\lambda} \\ &= 2 \sup \int_{\mathbf{R}} |f| g\,d\lambda \le \sup \int_{\mathbf{R}} f^o g + \bar{f} g\,d\lambda \le \|f^o + \bar{f}\|_{p,\lambda}. \end{aligned}$$

Here the suprema are over all non-negative, $\lambda$-non-increasing functions $g$ with $\|g\|_{p',\lambda} \le 1$. Equality in Minkowski's inequality above yields $f^o = c\bar{f}$ $\lambda$-almost everywhere for some $c \ge 0$. Since $\|f^o\|_{p,\lambda} = \|\bar{f}\|_{p,\lambda}$ it follows that $c = 1$ and $f^o = \bar{f}$ $\lambda$-almost everywhere as required.

**Proposition 3.6.** *Let $1 < p < \infty$ and suppose that $f \in L^{p\downarrow}_\lambda$. If $f^o$ is a $p$-level function of $f$ then $f^o$ is non-negative and $\lambda$-non-increasing.*

*Proof.* It is easy to see that if $f^o$ is a $p$-level function of $f$ then $|f^o|$ is also. By the previous proposition $f^o = |f^o|$ so $f^o$ is non-negative. Now the definition of $\|\cdot\|_{p\downarrow\lambda}$ together with the properties of the $p$-level function yields

$$\|f^o\|_{p,\lambda} = \|f\|_{p\downarrow\lambda} = \sup \int_{\mathbf{R}} |f| g\,d\lambda \le \sup \int_{\mathbf{R}} f^o g\,d\lambda = \|f^o\|_{p\downarrow\lambda} \le \|f^o\|_{p,\lambda}$$

where the suprema are taken over all non-negative, $\lambda$-non-increasing functions $g$ with $\|g\|_{p',\lambda} \le 1$. It follows that $\|f^o\|_{p\downarrow\lambda} = \|f^o\|_{p,\lambda}$. By Theorem 3.2 $f^o$ is $\lambda$-non-increasing.

## 4. The Level Function of a Bounded Function

Let $\lambda$ be our regular, Borel measure and fix a non-negative, essentially bounded, $\lambda$-measurable function $f$. For convenience in stating results we make the following definitions: $M = \|f\|_{\infty,\lambda}$ is the best essential bound for $f$; $F(x) = \int_{-\infty}^{x} f\,d\lambda$; and $\Lambda(x) = \int_{-\infty}^{x} d\lambda$. Recall that we have assumed that $\Lambda(x) < \infty$ for $x \in \mathbf{R}$. Since $0 \le f \le M$ $\lambda$-almost everywhere we also have $F(x) \le M\Lambda(x) < \infty$ $\lambda$-almost everywhere. Recall also our convention that $\int_{-\infty}^{x} = \int_{(-\infty,x]}$ which implies that the non-increasing functions $\Lambda$ and $F$ are right continuous on $\mathbf{R}$.

In order to construct $f^o$ we first construct $F^\flat(x) = \int_{-\infty}^{x} f^o\,d\lambda$. Since $f^o$ is to be $\lambda$-non-increasing the function $F^\flat$ should be $\lambda$-concave .

**Definition 4.1.** *$F^\flat$ is the unique least concave majorant of $F$.*

The existence of $F^\flat$ is guaranteed by Theorem 2.8 once we note that $F$ does have the $\lambda$-concave majorant $M\Lambda$. Several useful properties of the function $F^\flat$ follow immediately from the definition.

**Theorem 4.2.** *Suppose that*

$$
\begin{cases}
\lambda \text{ is a regular, Borel measure such that } \Lambda(x) = \lambda(-\infty, x] < \infty \text{ for all } x \in \mathbf{R} \\
f \text{ is a non-negative, } \lambda\text{-measurable function such that } M \equiv \|f\|_{\infty,\lambda} < \infty \\
F(x) = \displaystyle\int_{-\infty}^{x} f\,d\lambda \text{ and } F^\flat \text{ is the least } \lambda\text{-concave majorant of } F.
\end{cases}
\tag{4.1}
$$

*Then*

(1) $F^\flat(b) - F^\flat(a) \le M(\Lambda(b) - \Lambda(a))$ *whenever* $a \le b$.
(2) $\lim_{x\to-\infty} F^\flat(x) = 0$.
(3) $F^\flat$ *is non-decreasing.*
(4) $F^\flat$ *is right continuous.*

*Proof.* We begin with (1). Fix $a, b \in \mathbf{R}$ with $a \le b$. If $\Lambda(a) = 0$ then $0 \le F^\flat(b) - F^\flat(a) \le F^\flat(b) \le M\Lambda(b) = M(\Lambda(b) - \Lambda(a))$ and (1) follows. If $\Lambda(a) > 0$ then take $x < a$ and use the $\lambda$-concavity of $F^\flat$ in the form (2.1) to see that

$$
\begin{aligned}
(\Lambda(a) - \Lambda(x))(F^\flat(b) - F^\flat(a)) &\le (F^\flat(a) - F^\flat(x))(\Lambda(b) - \Lambda(a)) \\
&\le F^\flat(a)(\Lambda(b) - \Lambda(a)) \le M\Lambda(a)(\Lambda(b) - \Lambda(a))
\end{aligned}
$$

Now allow $x \to -\infty$ and divide by $\Lambda(a)$ to conclude that $F^\flat(b) - F^\flat(a) \le M(\Lambda(b) - \Lambda(a))$ completing the proof of (1).

Since $0 \le F^\flat(x) \le M\Lambda(x)$ and $\lim_{x\to-\infty} \Lambda(x) = 0$ we have $\lim_{x\to-\infty} F^\flat(x) = 0$, the conclusion of (2).

We prove (3) by contradiction. Suppose that $F^\flat$ is not non-decreasing. Then there exist $a, b \in \mathbf{R}$ such that $a < b$ and $F^\flat(a) > F^\flat(b)$. Either $F$ is bounded, in which case the constant function with value $\lim_{x\to\infty} F(x)$ is a $\lambda$-concave majorant of $F$, or else $F$ is unbounded, in which case $\lim_{x\to\infty} F(x) = \infty$. In either case we have $\lim_{x\to\infty} F(x) \ge$

$F^\flat(a) > F^\flat(b)$ so there exists some $y > b$ such that $F(y) > F^\flat(b)$. Since $F^\flat$ is concave and $a \le b \le y$ we have

$$(\Lambda(y) - \Lambda(b))(F^\flat(b) - F^\flat(a)) \ge (F^\flat(y) - F^\flat(b))(\Lambda(b) - \Lambda(a)). \tag{4.2}$$

Now $F^\flat(y)-F^\flat(b) \ge F(y)-F^\flat(b) > 0$ and $\Lambda(b)-\Lambda(a) \ge 0$ so the right hand side of (4.2) is non-negative. The left hand side is therefore non-negative as well. However $F^\flat(b) - F^\flat(a)$ is negative by assumption and $\Lambda(y) - \Lambda(b)$ is non-negative. It follows that $\Lambda(y) - \Lambda(b) = 0$. We have

$$0 = M(\Lambda(y) - \Lambda(b)) = M \int_{(b,y]} d\lambda \ge \int_{(b,y]} f\, d\lambda = F(y) - F(b) \ge F(y) - F^\flat(b).$$

This contradicts the choice of $y$ and completes the proof of (3).

To prove (4) fix $x \in \mathbf{R}$. By part (1) and the right continuity of $\Lambda$,

$$0 \le \lim_{y\to x^+} F^\flat(y) - F^\flat(x) \le M \lim_{y\to x^+} \Lambda(y) - \Lambda(x) = 0.$$

Thus $F^\flat$ is right continuous at $x$. This completes the proof of the theorem.

To determine the function $f^o$ from $F^\flat$ we require the following differentiation lemma together with the Radon-Nikodým theorem.

**Lemma 4.3** ([7, p262])**.** *If $G$ is a non-decreasing function which is right continuous then there is a unique Borel measure $\mu$ such that for all $a, b \in \mathbf{R}$ with $a \le b$ we have*

$$\mu(a, b] = G(b) - G(a).$$

**Theorem 4.4.** *There is a non-negative, $\lambda$-non-increasing, $\lambda$-measurable function $f^o$ satisfying*

$$F^\flat(x) = \int_{-\infty}^x f^o\, d\lambda$$

*for all $x \in \mathbf{R}$. $f^o$ is called the level function of $f$ with respect to $\lambda$.*

*Proof.* Apply Lemma 4.3 to $F^\flat$ producing a Borel measure $\mu$ satisfying $F^\flat(x) - F^\flat(a) = \int_{(a,x]} d\mu$ whenever $a \le x$. In particular, allowing $a \to -\infty$ yields $F^\flat(x) = \int_{-\infty}^x d\mu$. The assumption of regularity on $\lambda$, together with part (1) of Theorem 4.2 shows that $\int_E d\mu \le M \int_E d\lambda$ for all Borel sets $E$. Thus $\mu$ is absolutely continuous with respect to $\lambda$ and we may define $f^o$ to be the Radon-Nikodým derivative of $\mu$ with respect to $\lambda$. $f^o$ is non-negative and $\lambda$-measurable and we have

$$F^\flat(x) = \int_{-\infty}^x d\mu = \int_{-\infty}^x f^o\, d\lambda.$$

Since $F^\flat$ is $\lambda$-concave we apply Theorem 2.7 to conclude that $f^o$ is $\lambda$-non-increasing.

Although we now have a definition of the function $f^o$ the relationship between $f$ and its level function has not been fully examined. In the remainder of this section we will first explore and then exploit the close connection between $f$ and $f^o$. We will show that $f$ and $f^o$ agree $\lambda$-almost everywhere except on a collection of disjoint intervals where $f^o$ is constant. The corresponding intervals in Halperin's construction were called the level intervals for the function $f$ and prompted the name "level function" for (a variant of) $f^o$.

It will be convenient to use the notation $G(x-) = \lim_{y\to x, y\le x} G(y)$. Since $\Lambda$, $F$, and $F^\flat$ are all non-decreasing the limits $\Lambda(x-)$, $F(x-)$, and $F^\flat(x-)$ exist for all $x \in \mathbf{R}\cup\{\infty\}$. The corresponding limit from the right will not be needed since $\Lambda$, $F$, and $F^\flat$ are all continuous from the right. In keeping with this, $G(-\infty)$ will be taken to mean $\lim_{x\to-\infty} G(x)$.

**Theorem 4.5.** *Suppose* (4.1) *holds and define the subset* $U$ *of* $\mathbf{R}$ *by* $U = \{x \in \mathbf{R} : F^\flat(x) > F(x) \text{ and } F^\flat(x-) > F(x-)\}$. *Then*

(1) *if* $F^\flat(x) > F(x)$ *for some* $x \in \mathbf{R}$ *then there exists some* $b > x$ *such that* $(x, b) \subset U$,
(2) *if* $F^\flat(x-) > F(x-)$ *for some* $x \in \mathbf{R}$ *then there exists some* $a < x$ *such that* $(a, x) \subset U$, *and*
(3) $U$ *is open.*

*Proof.* Suppose that $F^\flat(x) > F(x)$. Since $F$ is right continuous we may choose $b$ greater than $x$ such that $F(b) < F^\flat(x)$. If $y \in (x, b)$ then $F^\flat(y) \ge F^\flat(y-) \ge F^\flat(x) > F(b) \ge F(y) \ge F(y-)$ so $y \in U$. This proves (1). Now suppose that $F^\flat(x-) > F(x-)$. Choose $a$ less than $x$ such that $F^\flat(a) > F(x-)$. If $y \in (a, x)$ then $F^\flat(y) \ge F^\flat(y-) \ge F^\flat(a) > F(x-) \ge F(y) \ge F(y-)$ so $y \in U$ and (2) is proved. (3) is immediate.

**Definition 4.6.** *Suppose* (4.1) *holds. Define* $a_i$, $b_i$, *and* $I_i$ *by*

$$\tag{4.3} \begin{cases} U = \{x \in \mathbf{R} : F^\flat(x) > F(x) \textit{ and } F^\flat(x-) > F(x-)\}, \\ U = \bigcup_i (a_i, b_i) \qquad \textit{(disjoint union)}, \\ (a_i, b_i) \subset I_i \subset [a_i, b_i], \\ a_i \in I_i \textit{ if and only if } F^\flat(a_i) > F(a_i), \quad \textit{and} \\ b_i \in I_i \textit{ if and only if } F^\flat(b_i-) > F(b_i-). \end{cases}$$

Since $U$ is an open subset of $\mathbf{R}$ it is a (finite or countable) disjoint union of open intervals. This defines the points $a_i, b_i \in [-\infty, \infty]$. Adding in one or both endpoints as specified to the interval $(a_i, b_i)$ gives the interval $I_i$.

The intervals $I_i$ are subsets of $\mathbf{R}$ since if for some $i$, $a_i = -\infty$ then $F^\flat(-\infty) = 0 = F(-\infty)$ so $-\infty \notin I_i$. Also, if $b_i = \infty$ for some $i$ then either $F(\infty-) = \infty$ or else the constant function with value $F(\infty-)$ is a $\lambda$-concave majorant of $F$ and hence of $F^\flat$. In either case $F^\flat(\infty-) = F(\infty-)$ so $\infty \notin I_i$.

The intervals $I_i$ are disjoint since their interiors are the disjoint components of the open set $U$ and if $b_i = a_j$ for some $i$ and $j$ with $b_i \in I_i$ and $a_j \in I_j$ then we would have $F^\flat(a_j) > F(a_j)$ and $F^\flat(b_i-) > F(b_i-)$ so $b_i = a_j \in U$ which is impossible.

**Theorem 4.7.** *Suppose that* (4.1) *and* (4.3) *hold. $f^o$ is constant $\lambda$-almost everywhere on each interval $I_i$.*

*Proof.* Fix $i$ and drop the subscripts so that $I = I_i$, $a = a_i$, and $b = b_i$. It is sufficient to prove the following three statements:

(1) $f^o$ is constant $\lambda$-almost everywhere on $(c, d)$ whenever $(c, d) \subset I$ and $0 < \lambda(c, d) < \infty$;
(2) $f^o$ is constant $\lambda$-almost everywhere on $[a, d)$ whenever $[a, d) \subset I$ and $0 < \lambda[a, d) < \infty$; and
(3) $f^o$ is constant $\lambda$-almost everywhere on $(c, b]$ whenever $(c, b] \subset I$ and $0 < \lambda(c, b] < \infty$.

We begin with (1). Let $m = (F^\flat(d-)-F^\flat(c))/(\Lambda(d-)-\Lambda(c))$, $C_I = F^\flat(d-)-m\Lambda(d-) = F^\flat(c) - m\Lambda(c)$, and $C_m = \sup\{F(x) - m\Lambda(x)\colon x \in \mathbf{R}\}$. We will show that $C_I = C_m$. $m\Lambda + C_m$ is $\lambda$-concave and majorises $F$ so by the minimality of $F^\flat$ we have

$$F^\flat \leq m\Lambda + C_m. \tag{4.4}$$

In particular $C_I = F^\flat(c) - m\Lambda(c) \leq C_m$.

To show that $C_I \geq C_m$ we take a sequence $\{y_n\} \subset \mathbf{R}$ such that $\lim_{n\to\infty} F(y_n) - m\Lambda(y_n) = C_m$. For each $n$ either $y_n > d$, $y_n < c$, or $c \leq y_n \leq d$ and at least one of these conditions must hold for infinitely many $n$. We distinguish three cases based on this observation.

First suppose that $y_n > d$ for infinitely many $n$. Since $F^\flat$ is $\lambda$-concave we have

$$(\Lambda(y_n) - \Lambda(d-))(F^\flat(d-) - F^\flat(c)) \geq (F^\flat(y_n) - F^\flat(d-))(\Lambda(d-) - \Lambda(c)),$$

or equivalently, $C_I \geq F^\flat(y_n) - m\Lambda(y_n)$, for infinitely many $n$. This implies that $C_I \geq C_m$.

Next suppose that $y_n < c$ for infinitely many $n$. By the $\lambda$-concavity of $F^\flat$,

$$(\Lambda(d-) - \Lambda(c))(F^\flat(c) - F^\flat(y_n)) \geq (F^\flat(d-) - F^\flat(c))(\Lambda(c) - \Lambda(y_n)).$$

That is, $C_I \geq F^\flat(y_n) - m\Lambda(y_n)$, for infinitely many $n$ so again $C_I \geq C_m$.

In the remaining case, $c \leq y_n \leq d$ for infinitely many $n$, let $y$ be a limit point of $\{y_n\}$ in $[c, d]$. If $y$ is a right limit point then $C_m = F(y) - m\Lambda(y) \leq F^\flat(y) - m\Lambda(y) \leq C_m$ by (4.4). Similarly, if y is a left limit point then $C_m = F(y-) - m\Lambda(y-) \leq F^\flat(y-) - m\Lambda(y-) \leq C_m$. Thus either $F^\flat(y) = F(y)$ or $F^\flat(y-) = F(y-)$ so $y \notin U$. We are left with two possibilities, either $y = c$ or $y = d$. If $y = c$ then $y$ must be a right limit point so $C_m = F(c) - m\Lambda(c) = C_I$ and if $y = d$ then $y$ is a left limit point and therefore $C_m = F(d-) - m\Lambda(d-) = C_I$.

Now we use the fact that $C_I = C_m$ to show that $f^o$ is constant $\lambda$-almost everywhere on $(c, d)$. Let $x \in (c, d)$. Using the form (2.2) of the $\lambda$-concavity of $F^\flat$ we have

$$F^\flat(x)(\Lambda(d-) - \Lambda(c)) \geq F^\flat(c)(\Lambda(d-) - \Lambda(x)) + F^\flat(d-)(\Lambda(x) - \Lambda(c)),$$

which can be written in the form $F^\flat(x)-m\Lambda(x) \geq C_I$. We now apply (4.4) to obtain $C_m \geq F^\flat(x)-m\Lambda(x) \geq C_I$ for all $x \in (c, d)$. This implies that $\int_{-\infty}^x (f^o - m)\, d\lambda = F^\flat(x) - m\Lambda(x)$ is constant on $(c, d)$ and hence $f^o = m$ $\lambda$-almost everywhere on $(c, d)$ as required.

The proof of (2) is similar. This time let $m = (F^\flat(d-) - F^\flat(a-))/(\Lambda(d-) - \Lambda(a-))$, $C_I = F^\flat(d-) - m\Lambda(d-) = F^\flat(a-) - m\Lambda(a-)$, and $C_m = \sup\{F(x) - m\Lambda(x)\colon x \in \mathbf{R}\}$. Since $C_m$ has not changed we still have (4.4). Thus $C_I \le C_m$. To show that $C_I \ge C_m$ we again take a sequence $\{y_n\} \subset \mathbf{R}$ such that $\lim_{n\to\infty} F(y_n) - m\Lambda(y_n) = C_m$ and split the argument into cases.

If $y_n > d$ for infinitely many $n$ or $y_n < a$ for infinitely many $n$ then the $\lambda$-concavity of $F^\flat$ implies that $C_I \ge F^\flat(y_n) - m\Lambda(y_n)$ for infinitely many $n$ so $C_I \ge C_m$. If $a \le y_n \le d$ for infinitely many $n$, let $y$ be a limit point of the $y_n$'s in $[a, d]$. As before $y \notin U$ so either $y = a$ or $y = d$. The case $y = a$ cannot occur since if $y = a$ then $y$ is a right limit point and by (4.4), $F(a) - m\Lambda(a) = C_m \ge F^\flat(a) - m\Lambda(a)$. It follows that $F(a) = F^\flat(a)$ so by the hypothesis (4.3) $a \notin I$ which is contrary to assumption. If $y = d$ then $y$ is a left limit point and therefore $C_m = F(d-) - m\Lambda(d-) = C_I$.

To show that $f^o$ is constant $\lambda$-almost everywhere on $[a, d)$ take $x \in [a, d)$. The $\lambda$-concavity of $F^\flat$ yields $F^\flat(x) - m\Lambda(x) \ge C_I$ which combines with (4.4) to give $C_I = F^\flat(x) - m\Lambda(x)$ for all $x \in [a, d)$. This implies that $\int_{-\infty}^x (f^o - m)\, d\lambda = F^\flat(x) - m\Lambda(x)$ is constant on $[a, d)$ and hence $f^o = m$ $\lambda$-almost everywhere on $(a, d)$. Moreover,

$$(f^o(a) - m)\lambda\{a\} = (F^\flat(a) - m\Lambda(a)) - (F^\flat(a-) - m\Lambda(a-)) = C_I - C_I = 0$$

so either $\lambda\{a\} = 0$ or $f^o(a) = m$. It follows that $f^o = m$ $\lambda$-almost everywhere on $[a, d)$.

The proof of (3) will complete the theorem. Let $m = (F^\flat(b) - F^\flat(c))/(\Lambda(b) - \Lambda(c))$, $C_I = F^\flat(b) - m\Lambda(b) = F^\flat(c) - m\Lambda(c)$, and $C_m = \sup\{F(x) - m\Lambda(x)\colon x \in \mathbf{R}\}$. Again we have (4.4) so $C_I \le C_m$. To show that $C_I \ge C_m$ we take a sequence $\{y_n\} \subset \mathbf{R}$ such that $\lim_{n\to\infty} F(y_n) - m\Lambda(y_n) = C_m$ and split the argument into cases.

If $y_n > b$ for infinitely many $n$ or $y_n < c$ for infinitely many $n$ then the $\lambda$-concavity of $F^\flat$ implies that $C_I \ge F^\flat(y_n) - m\Lambda(y_n)$ for infinitely many $n$ so $C_I \ge C_m$. If $c \le y_n \le b$ for infinitely many $n$, let $y$ be a limit point of the $y_n$'s in $[c, b]$. As before $y \notin U$ so either $y = b$ or $y = c$. The case $y = b$ cannot occur since if $y = b$ then $y$ is a left limit point and by (4.4), $F(b-) - m\Lambda(b-) = C_m \ge F^\flat(b-) - m\Lambda(b-)$. It follows that $F(b-) = F^\flat(b-)$ so by the hypothesis (4.3) $b \notin I$ which is contrary to assumption. If $y = c$ then $y$ is a right limit point and therefore $C_m = F(c) - m\Lambda(c) = C_I$.

To show that $f^o$ is constant $\lambda$-almost everywhere on $(c, b]$ take $x \in (c, b]$. The $\lambda$-concavity of $F^\flat$ yields $F^\flat(x) - m\Lambda(x) \ge C_I$ which combines with (4.4) to give $C_I = F^\flat(x) - m\Lambda(x)$ for all $x \in (c, b]$. This implies that $\int_{-\infty}^x (f^o - m)\, d\lambda = F^\flat(x) - m\Lambda(x)$ is constant on $(c, b]$ and hence $f^o = m$ $\lambda$-almost everywhere on $(c, b]$.

Knowing that $f^o$ is constant on the intervals $I_i$ allows us to compute its value on each $I_i$.

**Corollary 4.8.** *Suppose* (4.1) *and* (4.3) *hold. If $\lambda I_i < \infty$ then*

$$f^o(t) = (1/\lambda I_i) \int_{I_i} f\, d\lambda$$

*for $\lambda$-almost every $t \in I_i$. If $\lambda I_i = \infty$ then*

$$f^o(t) = \limsup_{x\to\infty} (1/\lambda(I_i \cap (-\infty, x])) \int_{I_i \cap (-\infty, x]} f\, d\lambda$$

*for $\lambda$-almost every $t \in I_i$.*

*Proof.* Drop the subscript $i$ as before and suppose without loss of generality that $\lambda I > 0$. First suppose that $\lambda I < \infty$. By the theorem,

$$f^o(t) = (1/\lambda I) \int_I f^o \, d\lambda$$

for $\lambda$-almost every $t \in I$. The first statement of the theorem will follow if we show that $\int_I f \, d\lambda = \int_I f^o \, d\lambda$. Four simple consequences of Definition 4.6 will be useful: If $a \in I$ then $F^\flat(a-) = F(a-)$, if $a \notin I$ then $F^\flat(a) = F(a)$, if $b \notin I$ then $F^\flat(b-) = F(b-)$ and if $b \in I$ then $F^\flat(b) = F(b)$. There are four cases. If $I = (a, b)$ then

$$\int_I f \, d\lambda = F(b-) - F(a) = F^\flat(b-) - F^\flat(a) = \int_I f^o \, d\lambda.$$

If $I = (a, b]$ then

$$\int_I f \, d\lambda = F(b) - F(a) = F^\flat(b) - F^\flat(a) = \int_I f^o \, d\lambda.$$

If $I = [a, b)$ then

$$\int_I f \, d\lambda = F(b-) - F(a-) = F^\flat(b-) - F^\flat(a-) = \int_I f^o \, d\lambda.$$

If $I = [a, b]$ then

$$\int_I f \, d\lambda = F(b) - F(a-) = F^\flat(b) - F^\flat(a-) = \int_I f^o \, d\lambda.$$

Suppose now that $\lambda I = \infty$ and denote by $m$ the value that $f^o$ takes $\lambda$-almost everywhere on $I$. Since all intervals bounded on the right are $\lambda$-finite by assumption we must have either $I = (a, \infty)$ or $I = [a, \infty)$. The arguments for the two cases are similar so we consider only the case $I = (a, \infty)$. We have $F^\flat(a) = F(a)$ and

$$\begin{aligned}
\limsup_{x\to\infty} (1/\lambda(I_i \cap (-\infty, x])) \int_{I_i \cap (-\infty, x]} f \, d\lambda &= \limsup_{x\to\infty} \frac{F(x) - F(a)}{\Lambda(x) - \Lambda(a)} \\
&\le \limsup_{x\to\infty} \frac{F^\flat(x) - F^\flat(a)}{\Lambda(x) - \Lambda(a)} = m.
\end{aligned}$$

It remains to prove the inequality $m \le \limsup_{x\to\infty} (F(x) - F(a))/(\Lambda(x) - \Lambda(a))$.

For a fixed $c > a$ let $s = \sup\{(F(x) - F^\flat(c))/(\Lambda(x) - \Lambda(c)) : x \in (c, \infty),\ \Lambda(x) > \Lambda(c)\}$. If $y > c$ then $F(y) \le s(\Lambda(y) - \Lambda(c)) + F^\flat(c)$. If $y \le c$ then by the $\lambda$-concavity of $F^\flat$,

$$(\Lambda(x) - \Lambda(c))(F^\flat(c) - F^\flat(y)) \ge (F^\flat(x) - F^\flat(c))(\Lambda(c) - \Lambda(y)) \ge (F(x) - F^\flat(c))(\Lambda(c) - \Lambda(y))$$

for each $x > c$. It follows that $F(y) \le F^\flat(y) \le s(\Lambda(y) - \Lambda(c)) + F^\flat(c)$. We have shown that the $\lambda$-concave function $s(\Lambda(y)-\Lambda(c))+F^\flat(c)$ majorises $F(y)$. Since $F^\flat$ is the least $\lambda$-concave majorant of $F$ we have $F^\flat(y) \le s(\Lambda(y)-\Lambda(c))+F^\flat(c)$ for all $y \in \mathbf{R}$. In particular, if $y$ is chosen so that $\Lambda(y) > \Lambda(c)$, we have $s \ge (F^\flat(y) - F^\flat(c))/(\Lambda(y) - \Lambda(c)) = m$.

Now if $x > c$ and $\Lambda(c) > \Lambda(a)$, we have the trivial inequality $m \le s(\Lambda(x)-\Lambda(c))/(\Lambda(x)-\Lambda(a)) + m(\Lambda(c) - \Lambda(a))/(\Lambda(x) - \Lambda(a))$ so

$$m \le \lim_{c\to\infty} \sup_{\Lambda(x)>\Lambda(c)} \left( \frac{F(x) - F^\flat(c)}{\Lambda(x) - \Lambda(c)} \frac{\Lambda(x) - \Lambda(c)}{\Lambda(x) - \Lambda(a)} + m\frac{\Lambda(c) - \Lambda(a)}{\Lambda(x) - \Lambda(a)} \right)$$
$$= \lim_{c\to\infty} \sup_{\Lambda(x)>\Lambda(c)} \frac{F(x) - F^\flat(a)}{\Lambda(x) - \Lambda(a)} = \limsup_{x\to\infty} \frac{F(x) - F(a)}{\Lambda(x) - \Lambda(a)}.$$

This completes the proof.

As mentioned $f$ and $f^o$ coincide except on the intervals $I_i$.

**Theorem 4.9.** *$f^o = f$ $\lambda$-almost everywhere off $\cup_i I_i$.*

*Proof.* Let $E = \mathbf{R} \setminus \cup_i I_i$, set

$$g(t) = \begin{Bmatrix} f(t), & \text{for } t \in E \\ f^o(t), & \text{for } t \notin E \end{Bmatrix} \quad \text{and} \quad G(x) = \int_{-\infty}^{x} g \, d\lambda.$$

It is enough to show that $G = F^\flat$ for then $g = f^o$ $\lambda$-almost everywhere and hence $f = f^o$ $\lambda$-almost everwhere on $E$. If $x \in E$ then each interval $I_i$ lies either entirely to the right or entirely to the left of $x$. Let $J_x$ be the collection of those indices $i$ for which $I_i$ lies entirely to the left of $x$. Note that for each $i \in J_x$ $I_i \subset (-\infty, x]$ so $\lambda I_i < \infty$. By Corollary 4.8,

$$G(x) = \sum_{i\in J_x} \int_{I_i} f^o \, d\lambda + \int_{(-\infty,x]\cap E} f \, d\lambda = \sum_{i\in J_x} \int_{I_i} f \, d\lambda + \int_{(-\infty,x]\cap E} f \, d\lambda = F(x).$$

For $x \in E$, however, we must have $F(x) = F^\flat(x)$ since if $F(x) < F^\flat(x)$ then by Theorem 4.5, (1), $x$ is the left endpoint of some connected component $(a_i, b_i)$ of $U$ and according to Definition 4.6 $x \in I_i$, contrary to assumption. Thus $G(x) = F^\flat(x)$.

If $a_i \in I_i$ for some $i$ then $F^\flat(a_i-) = F(a_i-)$ and an argument similar to the above shows that $G(a_i-) = F^\flat(a_i-)$.

If $x \notin E$ then $x \in I_i$ for some $i$. If $a_i \notin I_i$ then

$$G(x) = G(a_i) + \int_{(a_i,x]} g \, d\lambda = F^\flat(a_i) + \int_{(a_i,x]} f^o \, d\lambda = F^\flat(x).$$

If $a_i \in I_i$ then

$$G(x) = G(a_i-) + \int_{[a_i,x]} g \, d\lambda = F^\flat(a_i-) + \int_{[a_i,x]} f^o \, d\lambda = F^\flat(x).$$

This completes the proof.

The decomposition of $f^o$ now enables us to show that $f^o$ is indeed the $p$-level function of $f$ that we set out to construct. This fact is established in Theorems 4.11 and 4.12. Theorem 4.10 is needed in the proof of 4.12 but is also an interesting and useful result in its own right.

**Theorem 4.10.** *Suppose that* (4.1) *and* (4.3) *hold. For any* $\alpha \ge 0$,

$$\int_{-\infty}^{\infty} (f^o)^\alpha f \, d\lambda = \int_{-\infty}^{\infty} (f^o)^{\alpha+1} \, d\lambda.$$

*Proof.* By the previous theorem, $f^o = f$ $\lambda$-almost everywhere off $\cup_i I_i$ so it is enough to show that

$$\int_{I_i} (f^o)^\alpha f \, d\lambda = \int_{I_i} (f^o)^{\alpha+1} \, d\lambda \tag{4.5}$$

for each interval $I_i$. If $\lambda I_i < \infty$ then Corollary 4.8 shows that $f^o$ takes the value $m_i = (1/\lambda I_i) \int_{I_i} f \, d\lambda$ $\lambda$-almost everywhere on $I_i$. Thus,

$$\int_{I_i} (f^o)^\alpha f \, d\lambda = m_i^\alpha \int_{I_i} f \, d\lambda = m_i^{\alpha+1} \int_{I_i} d\lambda = \int_{I_i} (f^o)^{\alpha+1} \, d\lambda.$$

If $\lambda I_i = \infty$ then

$$f^o(t) = \limsup_{x \to \infty} (1/\lambda(I_i \cap (-\infty, x])) \int_{I_i \cap (-\infty, x]} f \, d\lambda \equiv m_i$$

$\lambda$-almost everywhere on $I_i$. If $m_i > 0$ then $\int_{I_i} f \, d\lambda = \infty$ and both sides of (4.5) are infinite. If $m_i = 0$ then it is enough to show that $f = 0$ $\lambda$-almost everywhere on $I_i$ since then (4.5) holds trivially. Either $I_i = (a_i, \infty)$ and $F^\flat(a_i) = F(a_i)$, or $I_i = [a_i, \infty)$ and $F^\flat(a_i-) = F(a_i-)$. Thus for $t \in I_i$ we have either

$$\int_{(a_i,t]} f \, d\lambda = F(t) - F(a_i) \le F^\flat(t) - F^\flat(a_i) = \int_{(a_i,t]} f^o \, d\lambda = 0, \quad \text{or}$$

$$\int_{[a_i,t]} f \, d\lambda = F(t) - F(a_i-) \le F^\flat(t) - F^\flat(a_i-) = \int_{[a_i,t]} f^o \, d\lambda = 0.$$

It follows that $f = 0$ $\lambda$-almost everywhere on $I_i$. This completes the proof.

**Theorem 4.11.** *Suppose that* (4.1) *and* (4.3) *hold. If $g$ is non-negative and $\lambda$-non-increasing then* $\int_{\mathbf{R}} fg \, d\lambda \le \int_{\mathbf{R}} f^o g \, d\lambda$.

*Proof.* By Theorem 2.4 part (1) we may supose that $g : \mathbf{R} \to [0, \infty]$ is non-increasing. Therefore, for each $s \ge 0$, $E_s = \{t \in \mathbf{R} : g(t) \ge s\}$ is (either empty or) an interval of one of the two forms $(-\infty, x)$, or $(-\infty, x]$. Since for all $x \in (-\infty, \infty]$,

$$\int_{(-\infty,x)} f \, d\lambda = F(x-) \le F^\flat(x-) = \int_{(-\infty,x)} f^o \, d\lambda$$

and for all $x \in (-\infty, \infty)$,

$$\int_{(-\infty,x]} f \, d\lambda = F(x) \le F^\flat(x) = \int_{(-\infty,x]} f^o \, d\lambda$$

we see that $\int_{E_s} f \, d\lambda \le \int_{E_s} f^o \, d\lambda$ for all $s \ge 0$.

The theorem now follows by Fubini's Theorem.

$$\begin{aligned}
\int_{\mathbf{R}} f(t) g(t) \, d\lambda(t) &= \int_{\mathbf{R}} \int_0^{g(t)} ds \, f(t) \, d\lambda(t) = \int_0^\infty \int_{E_s} f(t) \, d\lambda(t) \, ds \\
&\le \int_0^\infty \int_{E_s} f^o(t) \, d\lambda(t) \, ds = \int_{\mathbf{R}} f^o(t) g(t) \, d\lambda(t).
\end{aligned}$$

**Theorem 4.12.** *Suppose that* (4.1) *and* (4.3) *hold. If* $1 \le p \le \infty$ *and* $f \in L^1_\lambda \cap L^\infty_\lambda$ *then* $\|f\|_{p\downarrow\lambda} = \|f^o\|_{p,\lambda}$.

*Proof.* It follows from Corollary 4.8 and Theorem 4.9 that $\|f^o\|_{\infty,\lambda} \le \|f\|_{\infty,\lambda}$ and Theorem 4.10, with $\alpha = 0$, yields $\|f^o\|_{1,\lambda} = \|f\|_{1,\lambda}$. Thus $f^o \in L^1_\lambda \cap L^\infty_\lambda$ and hence $f^o \in L^p_\lambda$. Now by Theorem 4.11

$$\|f\|_{p\downarrow\lambda} = \sup \int_{\mathbf{R}} fg\, d\lambda \le \sup \int_{\mathbf{R}} f^o g\, d\lambda = \|f^o\|_{p\downarrow\lambda} \le \|f^o\|_{p,\lambda}$$

where the sup is taken over all non-negative, $\lambda$-non-increasing functions $g$ with $\|g\|_{p',\lambda} \le 1$.

To prove the opposite inequality note that since $f^o \in L^p_\lambda$, $(f^o/\|f^o\|_{p,\lambda})^{p-1}$ is non-negative, $\lambda$-non-increasing and has $L^{p'}_\lambda$-norm 1. (If $\|f^o\|_{p,\lambda} = 0$ then $f^o = 0$ $\lambda$-almost everywhere so $f = 0$ $\lambda$-almost everywhere and the result follows.) Thus, by Theorem 4.10,

$$\|f\|_{p\downarrow\lambda} \ge \|f^o\|_{p,\lambda}^{1-p} \int_{\mathbf{R}} f(f^o)^{p-1}\, d\lambda = \|f^o\|_{p,\lambda}^{1-p} \int_{\mathbf{R}} (f^o)^p\, d\lambda = \|f^o\|_{p,\lambda}.$$

## 5. The Level Function Extended to $L^{p\downarrow}_\lambda$.

In this section we consider the construction of the level function as a mapping $f \to f^o$. The first theorem of this section shows that the mapping preserves order which enables us to extend the construction of the previous section to a map from $L^{p\downarrow}_\lambda$ to $L^p_\lambda$. The notation introduced in Section 4 will be used freely throughout this section and since we must now consider the level function construction applied to more than one function we will use $g$, $G$, $G^\flat$, and $g^o$ to correspond to $f$, $F$, $F^\flat$, and $f^o$ in the obvious way.

We begin with a simple exercise in measure theory.

**Lemma 5.1.** *Suppose* $g \ge 0$ *is an essentially bounded* $\lambda$*-measurable function and* $g^o$ *is the level function of* $g$*. Then*

$$\lim_{x \to t-} \frac{G^\flat(t) - G^\flat(x)}{\Lambda(t) - \Lambda(x)} \tag{5.1}$$

*makes sense, exists, and equals* $g^o(t)$ *for* $\lambda$*-almost every* $t \in \mathbf{R}$*. Here* $G^\flat(x) = \int_{-\infty}^x g^o\, d\lambda$ *and* $\Lambda(x) = \lambda(-\infty, x]$.

*Proof.* To show that the limit makes sense we show that the denominator is non-zero for all $x < t$ for $\lambda$-almost every $t \in \mathbf{R}$. That is, that the set $T = \{t \in \mathbf{R} : \Lambda(t) = \Lambda(x) \text{ for some } x < t\}$ has $\lambda$-measure zero. To each $t \in T$ assign $x_t < t$ such that $\lambda(x_t, t] = \Lambda(t) - \Lambda(x_t) = 0$. Let $S$ be the open set $\cup_{t \in T}(x_t, t)$. Certainly $S \subset T$. If $t \in T \setminus S$ then $(x_t, t) \subset S$ so $t$ is an endpoint of a connected component (maximal open subinterval) of $S$. It follows that $T \setminus S$ is at most countable. Since $T$ clearly contains no atoms of $\lambda$ we see that $\lambda(T \setminus S) = 0$.

Now $S$ is a union of open intervals in $\mathbf{R}$ and hence is a countable subunion of those intervals. Since each interval $(x_t, t)$ has $\lambda$-measure zero it follows that $\lambda S = 0$. Therefore $\lambda T = \lambda S + \lambda(T \setminus S) = 0$.

To show that the limit exists we must consider the points of discontinuity of $g^o$. $g^o$ is $\lambda$-non-increasing so by Theorem 2.4 part (1) we may assume without loss of generality that $g^o$ is non-increasing. It follows that $g^o$ has at most countably many points of discontinuity. The set $T$ together with the set of points of discontinuity of $g^o$ which are not atoms for $\lambda$ has $\lambda$-measure zero. Suppose $t$ is outside this set. If $t$ is an atom for $\lambda$ then the limit clearly exists and equals $g^o(t)$. If t is not an atom then $g^o$ is continuous at $t$ and since $g^o$ is non-increasing,

$$g^o(t) \leq \frac{G^\flat(t) - G^\flat(x)}{\Lambda(t) - \Lambda(x)} \leq g^o(x).$$

Allowing $x \to t-$ we see that the limit (5.1) exists and equals $g^o(t)$.

**Theorem 5.2.** *Suppose $f$ and $g$ are non-negative, essentially bounded, $\lambda$-measurable functions with $f \leq g$ $\lambda$-almost everywhere. Then $f^o \leq g^o$ $\lambda$-almost everywhere. Here $f^o$ and $g^o$ are the level functions of $f$ and $g$ respectively.*

*Proof.* By Theorem 2.4 part (1) we may assume without loss of generality that both $f^o$ and $g^o$ are non-increasing. Moreover, since $f^o$ is constant $\lambda$-almost everywhere on its level intervals $I_i$ (Definition 4.6) we may also assume without loss of generality that $f^o$ is constant everywhere on each $I_i$. As usual let $F(x) = \int_{-\infty}^x f\,d\lambda$, $G(x) = \int_{-\infty}^x g\,d\lambda$, $F^\flat(x) = \int_{-\infty}^x f^o\,d\lambda$, $G^\flat(x) = \int_{-\infty}^x g^o\,d\lambda$, and $\Lambda(x) = \lambda(-\infty, x]$.

Choose $t \in \mathbf{R}$ such that the limit (5.1) equals $g^o(t)$. Set $m = f^o(t)$, $C_m = \sup\{F(x) - m\Lambda(x) : x \in \mathbf{R}\}$, and $D_m = \sup\{G(x) - m\Lambda(x) : x \in \mathbf{R}\}$. Our first task is to show that both $C_m$ and $D_m$ are finite. Since $m\Lambda + C_m$ is a $\lambda$-concave majorant of $F$ the minimality of $F^\flat$ implies that $F^\flat \leq m\Lambda + C_m$. Similarly $G^\flat \leq m\Lambda + D_m$. Since $f^o$ is non-increasing and $m = f^o(t)$ the function $F^\flat(x) - m\Lambda(x) = \int_{-\infty}^x f^o - m\,d\lambda$ is non-decreasing for $x \leq t$ and non-increasing for $x \geq t$. Hence

$$C_m = \sup\{F^\flat(x) - m\Lambda(x) : x \in \mathbf{R}\} = F^\flat(t) - m\Lambda(t) < \infty \tag{5.2}$$

where the first equality follows from

$$C_m = \sup\{F(x) - m\Lambda(x) : x \in \mathbf{R}\} \leq \sup\{F^\flat(x) - m\Lambda(x) : x \in \mathbf{R}\} \leq C_m.$$

If the function $G^\flat(x) - m\Lambda(x) = \int_{-\infty}^x g^o - m\,d\lambda$ is eventually non-increasing then $D_m < \infty$ as for $C_m$. If the function is not eventually non-increasing then $g^o(x) > m$ for all $x$ and in particular $g^o(t) > m$ and the theorem is proved. We may assume therefore that $D_m < \infty$.

The main step in the proof is the proof of

$$\text{there exist } x_n \geq t \text{ such that } \lim_{n\to\infty} G(x_n) - m\Lambda(x_n) = D_m. \tag{5.3}$$

Before we prove (5.3) we will show how (5.3) will complete the theorem. If $x < t$ then the proof of the Lemma 5.1 shows that $\Lambda(t) - \Lambda(x) > 0$ so $\Lambda(x_n) - \Lambda(x) > 0$ for each $n$. Since $g^o$ is non-increasing and $x_n \geq t$,

$$\begin{aligned} \frac{G^\flat(t) - G^\flat(x)}{\Lambda(t) - \Lambda(x)} &\geq \frac{G^\flat(x_n) - G^\flat(x)}{\Lambda(x_n) - \Lambda(x)} \geq \frac{G^\flat(x_n) - D_m - m\Lambda(x)}{\Lambda(x_n) - \Lambda(x)} \\ &= m - \frac{D_m - (G^\flat(x_n) - m\Lambda(x_n))}{\Lambda(x_n) - \Lambda(x)} \geq m - \frac{D_m - (G^\flat(x_n) - m\Lambda(x_n))}{\Lambda(t) - \Lambda(x)} \end{aligned}$$

As $n \to \infty$ this becomes $(G^\flat(t) - G^\flat(x))/(\Lambda(t) - \Lambda(x)) \geq m$. As $x \to t-$ we have $g^o(t) \geq m$ by the choice of $t$. This completes the proof subject to (5.3).

En route to (5.3) we show

$$\text{there exist } y_n \geq t \text{ such that } \lim_{n\to\infty} F(x_n) - m\Lambda(x_n) = C_m. \tag{5.4}$$

If $F^\flat(t) = F(t)$ then (5.4) follows from (5.2) by setting $y_n = t$ for all $n$. Otherwise, $F^\flat(t) > F(t)$ so by Lemma 4.5 there is some $b'$ such that $(t, b') \subset U$ (Definition 4.6). Let $I$ be the level interval of $f$ which contains $(t, b')$. Note that $t \in I$ since either $t$ is interior to $I$ or else $t$ is the left endpoint of $I$ and $F^\flat(t) > F(t)$ so that $t \in I$ by Definition 4.6. Now $f^o$ is constant on $I$ so $f^o$ takes the value $f^o(t) = m$ on $I$. Let $b$ be the right endpoint of $I$. If $b \in I$ then $F^\flat(b-) > F(b-)$ and since $b \notin U$ we have $F^\flat(b) = F(b)$. Thus $C_m = F^\flat(t) - m\Lambda(t) = F^\flat(b) - m\Lambda(b) = F(b) - m\Lambda(b)$ and (5.4) follows with $y_n = b$ for all $n$. If $b \notin I$ then $F^\flat(b) > F(b)$ and since $b \notin U$ we have $F^\flat(b-) = F(b-)$. Thus $C_m = F^\flat(t) - m\Lambda(t) = F^\flat(b-) - m\Lambda(b-) = F(b-) - m\Lambda(b-)$ and (5.4) follows with $\{y_n\}$ taken to be any sequence in $(t, b)$ which converges to $b$.

We are now ready to prove (5.3). Let $\{x_n\}$ be any sequence of real numbers such that $D_m = \lim_{n\to\infty} G(x_n) - m\lambda(x_n)$. If $x_n \geq t$ for infinitely many $n$ then (5.3) holds on dropping to a subsequence. Otherwise we may assume (after dropping finitely many terms) that $x_n < t$ for all $n$. In this case we show that $D_m = \lim_{n\to\infty} G(y_n) - m\Lambda(y_n)$ to complete the proof. Since $x_n < t \leq y_n$ we have

$$G(y_n) - G(x_n) = \int_{(x_n, y_n]} g\, d\lambda \geq \int_{(x_n, y_n]} f\, d\lambda = F(y_n) - F(x_n).$$

Thus

$$\begin{aligned} 0 \geq -G(y_n) - F(x_n) + G(x_n) + F(y_n) \\ &= D_m - (G(y_n) - m\Lambda(y_n)) + C_m - (F(x_n) - m\Lambda(x_n)) \\ &\quad - (D_m - (G(x_n) - m\Lambda(x_n))) - (C_m - (F(y_n) - m\Lambda(y_n))) \end{aligned}$$

and since $\lim_{n\to\infty} D_m - (G(x_n) - m\Lambda(x_n)) = \lim_{n\to\infty} C_m - (F(y_n) - m\Lambda(y_n)) = 0$ we have

$$\lim_{n\to\infty} D_m - (G(y_n) - m\Lambda(y_n)) + C_m - (F(x_n) - m\Lambda(x_n)) \leq 0.$$

Both $D_m - (G(y_n) - m\Lambda(y_n))$ and $C_m - (F(x_n) - m\Lambda(x_n))$ are non-negative so we have $\lim_{n\to\infty} G(y_n) - m\Lambda(y_n) = D_m$ as required.

The extension of the map $f \to f^o$ from bounded functions to arbitrary functions in $L^{p\downarrow}_\lambda$ is not done by continuity but by monotonicity. We need the following lemma to show that the properties of the $p$-level function carry over as well.

**Lemma 5.3.** *Let $1 \leq p \leq \infty$. Suppose that*

(1) *$\{f_n\}$ increases to $f$ pointwise $\lambda$-almost everywhere and $0 \leq f_n \in L^{p\downarrow}_\lambda$ for each $n$.*
(2) *$\{h_n\}$ increases to $h$ pointwise $\lambda$-almost everywhere and $h_n \in L^p_\lambda$ for each $n$.*
(3) *$h_n$ is a $p$-level function of $f_n$ for each $n$.*

*Then $h$ is a $p$-level function of $f$ whenever $f \in L_\lambda^{p\downarrow}$. Also, $f \in L_\lambda^{p\downarrow}$ if and only if $h \in L_\lambda^p$.*

*Proof.* Fix $g \in L_\lambda^{p'}$ with $g$ non-negative and $\lambda$-non-increasing. Since $h_n$ is a $p$-level function of $f_n$ we have $\int_{\mathbf{R}} f_n g\, d\lambda \le \int_{\mathbf{R}} h_n g\, d\lambda$ and $\|h_n\|_{p,\lambda} = \|f_n\|_{p\downarrow\lambda}$. By the Monotone Convergence Theorem (used twice),

$$\int_{\mathbf{R}} fg\, d\lambda = \lim_{n\to\infty} \int_{\mathbf{R}} f_n g\, d\lambda \le \lim_{n\to\infty} \int_{\mathbf{R}} h_n g\, d\lambda = \int_{\mathbf{R}} hg\, d\lambda. \tag{5.5}$$

Also, by the Monotone Convergence Theorem for $p < \infty$ and trivially for $p = \infty$,

$$\|h\|_{p,\lambda} = \lim_{n\to\infty} \|h_n\|_{p,\lambda} = \lim_{n\to\infty} \|f_n\|_{p\downarrow\lambda} \le \|f\|_{p\downarrow\lambda}.$$

To complete the proof we need $\|f\|_{p\downarrow\lambda} \le \|h\|_{p,\lambda}$. This follows from (5.5) (allowing $g$ to vary) and Hölders inequality.

$$\|f\|_{p\downarrow\lambda} = \sup \int_{\mathbf{R}} fg\, d\lambda \le \sup \int_{\mathbf{R}} hg\, d\lambda \le \|h\|_{p,\lambda}$$

where the suprema are taken over all non-negative, $\lambda$-non-increasing functions $g$ with $\|g\|_{p',\lambda} \le 1$.

We are now ready to prove the existence of $p$-level functions.

**Theorem 5.4.** *Suppose that $1 \le p \le \infty$. If $f \in L_\lambda^{p\downarrow}$ then $f$ has a unique $p$-level function $f^o$. Moreover, $f^o$ is independent of $p$ in the sense that if $f \in L_\lambda^p \cap L_\lambda^q$ then $f^o$ is both the $p$-level and the $q$-level function of $f$.*

*Proof.* For $n = 1, 2, \dots$ set $f_n(x) = \min(n, |f(x)|)$ when $x \le n$ and $f_n(x) = 0$ when $x > n$. Note that $f_n \in L_\lambda^1 \cap L_\lambda^\infty$. Clearly, $\{f_n\}$ increases to $|f|$ pointwise. Also, by Theorem 5.2, the level functions $f_n^o$ form an increasing sequence. Finally, Theorems 4.11 and 4.12 show that $f_n^o$ is a p-level function of $f_n$ for each $n$. The hypotheses of Lemma 5.3 are satisfied and we have $f \in L_\lambda^{p\downarrow}$ so we conclude that $f^o$, the pointwise limit of $\{f_n^o\}$, is a $p$-level function of $f$. Uniqueness was already proved in Lemma 3.5. The $p$-independence of $f^o$ is clear from the construction.

With this result the concept of a $p$-level function becomes superfluous. The level function construction of Section 4 extends unambiguously to every function in $\cup_{1\le p\le\infty} L_\lambda^{p\downarrow}$.

**Definition 5.5.** *Suppose $f \in L_\lambda^{p\downarrow}$. The level function $f^o$ of $f$ is the $p$-level function whose existence is asserted above.*

The next two results extend Theorems 5.2 and 4.10 to apply to this larger collection of level functions.

**Corollary 5.6.** *Suppose that $f$ and $g$ are in $\cup_{1\le p\le\infty} L_\lambda^{p\downarrow}$ and that $|f| \le |g|$. Then $f^o \le g^o$.*

*Proof.* For $n = 1, 2, \dots$ set $f_n(x) = \min(n, |f(x)|)$ when $x \le n$, $f_n(x) = 0$ when $x > n$, $g_n(x) = \min(n, |g(x)|)$ when $x \le n$ and $g_n(x) = 0$ when $x > n$. Note that $f_n, g_n \in L_\lambda^1 \cap L_\lambda^\infty$. Clearly $0 \le f_n \le g_n$ so, by Theorem 5.2, $f_n^o \le g_n^o$. The proof of Theorem 5.4 shows that $f^o$ and $g^o$ are the pointwise limits of $\{f_n^o\}$ and $\{g_n^o\}$ respectively. It follows that $f^o \le g^o$ as required.

**Corollary 5.7.** *Suppose that* $1 \le p \le \infty$, $f \in L^{p\downarrow}_\lambda$ *and* $\alpha \ge 0$. *Then*

$$\int_R |f|(f^o)^\alpha \, d\lambda = \int_R (f^o)^{\alpha+1} \, d\lambda. \tag{5.6}$$

*Proof.* As usual set $f_n(x) = \min(n, |f(x)|)$ when $x \le n$ and $f_n(x) = 0$ when $x > n$. Note that $f_n \in L^1_\lambda \cap L^\infty_\lambda$ for each $n$. We can apply Theorem 4.10 to get

$$\int_R f_n(f_n^o)^\alpha \, d\lambda = \int_R (f_n^o)^{\alpha+1} \, d\lambda$$

for each $n$. Since $\{f_n\}$ and $\{f_n^o\}$ increase to $|f|$ and $f^o$ respectively, we may apply the Monotone Convergence Theorem to both sides of the integral above. This yields (5.6).

In order to prove that $L^{p\downarrow}_\lambda$ is a Banach space it remains to show that it is complete. The proof will follow along the same lines as the proof (in Royden [7]) that the $L^p$-spaces are complete. Indeed we will use the usual characterisation of completeness—that every absolutely summable series is summable. To proceed, we need analogues of some $L^p$ convergence results: The Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

Our substitute for the Monotone Convergence Theorem is

**Theorem 5.8.** *Suppose* $1 \le p \le \infty$ *and* $0 \le f_1 \le f_2 \le \dots$, $f_n \in L^{p\downarrow}_\lambda$. *Then the pointwise limit,* $f$, *of* $\{f_n\}$ *satisfies*

$$\|f\|_{p\downarrow\lambda} = \lim_{n\to\infty} \|f_n\|_{p\downarrow\lambda}. \tag{5.7}$$

*In particular* $f \in L^{p\downarrow}_\lambda$ *whenever the limit in* (5.7) *is finite.*

*Proof.* If the limit in (5.7) is infinite then the statement is trivial so suppose that the limit is finite. We will apply Lemma 5.3. We have $\{f_n\}$ increasing pointwise to $f$ and, by Corollary 5.6, $\{f_n^o\}$ increasing to some function, say g. Since each $f_n^o \in L^p_\lambda$, the usual Monotone Convergence Theorem shows that $\|g\|_{p,\lambda} = \lim_{n\to\infty} \|f_n^o\|_{p,\lambda}$ and hence $g \in L^p_\lambda$. (Note that this statement is valid for $p = \infty$ as well although not by the Monotone Convergence Theorem.) By Lemma 5.3 $f \in L^{p\downarrow}_\lambda$ and by Definition 5.5 $g = f^o$. Thus

$$\|f\|_{p\downarrow\lambda} = \|g\|_{p,\lambda} = \lim_{n\to\infty} \|f_n^o\|_{p,\lambda} = \lim_{n\to\infty} \|f_n\|_{p\downarrow\lambda}$$

as required.

Just as the Monotone Convergence Theorem leads to Fatou's Lemma we are led to

**Corollary 5.9.** *Suppose that* $1 \le p \le \infty$ *and that* $\{f_n\}$ *is a sequence of non-negative functions in* $L^{p\downarrow}_\lambda$. *Let* $f(x) = \liminf_{n\to\infty} f_n(x)$. *Then*

$$\|f\|_{p\downarrow\lambda} \le \liminf_{n\to\infty} \|f_n\|_{p\downarrow\lambda}. \tag{5.8}$$

*In particular* $f \in L^{p\downarrow}_\lambda$ *whenever the left hand side of* (5.8) *is finite.*

*Proof.* Define $h_n$ by $h_n(x) = \inf_{k\geq n} f_k(x)$. $\{h_n\}$ increases to $f$ and since $h_n \leq f_n$ we have $h_n \in L^{p\downarrow}_\lambda$ for each $n$. By Lemma 5.8 we have

$$\|f\|_{p\downarrow\lambda} = \lim_{n\to\infty} \|h_n\|_{p\downarrow\lambda} = \liminf_{n\to\infty} \|h_n\|_{p\downarrow\lambda} \leq \liminf_{n\to\infty} \|f_n\|_{p\downarrow\lambda}.$$

Passing from Fatou's Lemma to the Dominated Convergence Theorem usually involves the linearity of the integral. The $L^{p\downarrow}_\lambda$ norms are not given as integrals and the map taking $f$ to $f^o$ is not linear (it is not even sublinear!) The proof below uses Corollary 5.7 as a substitute for this lack.

**Lemma 5.10.** *Let* $1 \leq p < \infty$. *Suppose* $\{f_n\}$ *converges pointwise* $\lambda$*-almost everywhere to zero,* $0 \leq f_n \leq g$, *and* $g \in L^{p\downarrow}_\lambda$. *Then* $\{f_n\}$ *converges to zero in* $L^{p\downarrow}_\lambda$.

*Proof.* Consider the sequence $\{2g - f_n\}$ of non-negative functions in $L^{p\downarrow}_\lambda$. $\liminf_{n\to\infty} 2g - f_n = \lim_{n\to\infty} 2g - f_n = 2g$ so using our analogue of Fatou's Lemma (Lemma 5.9) we have

$$\|2g\|_{p\downarrow\lambda} \leq \liminf_{n\to\infty} \|2g - f_n\|_{p\downarrow\lambda} \leq \limsup_{n\to\infty} \|2g - f_n\|_{p\downarrow\lambda} \leq \|2g\|_{p\downarrow\lambda}$$

where the last inequality is from the hypothesis $0 \leq f_n \leq g$. Thus $\lim_{n\to\infty} \|2g - f_n\|_{p\downarrow\lambda} = \|2g\|_{p\downarrow\lambda}$. To complete the proof we will show that $\|f_n\|^p_{p\downarrow\lambda} \leq \|2g\|^p_{p\downarrow\lambda} - \|2g - f_n\|^p_{p\downarrow\lambda}$ from which it follows that $\lim_{n\to\infty} \|f_n\|_{p\downarrow\lambda} = 0$ as required.

By Corollary 5.7, with $\alpha = p - 1$, we have

$$\begin{aligned} \|2g - f_n\|^p_{p\downarrow\lambda} &= \int_R ((2g - f_n)^o)^p \, d\lambda = \int_R (2g - f_n)((2g - f_n)^o)^{p-1} \, d\lambda \\ &= \int_R 2g((2g - f_n)^o)^{p-1} \, d\lambda - \int_R f_n((2g - f_n)^o)^{p-1} \, d\lambda. \end{aligned} \tag{5.9}$$

To estimate the first integral in (5.9) we use the definition of the $L^{p\downarrow}_\lambda$ norm. (Note that $((2g - f_n)^o)^{p-1}$ is non-negative and $\lambda$-non-increasing.)

$$\begin{aligned} \int_R 2g((2g - f_n)^o)^{p-1} \, d\lambda &\leq \|2g\|_{p\downarrow\lambda} \|((2g - f_n)^o)^{p-1}\|_{p',\lambda} \\ &= \|2g\|_{p\downarrow\lambda} \|(2g - f_n)^o\|^{p-1}_{p,\lambda} = \|2g\|_{p\downarrow\lambda} \|2g - f_n\|^{p-1}_{p\downarrow\lambda} \leq \|2g\|^p_{p\downarrow\lambda}. \end{aligned}$$

To estimate the second integral in (5.9) we note that $2g - f_n \geq 2g - g = g \geq f_n$ so $(2g - f_n)^o \geq f_n^o$ by Theorem 5.6. This observation, together with another application of Corollary 5.7 yields

$$\int_R f_n((2g - f_n)^o)^{p-1} \, d\lambda \geq \int_R f_n(f_n^o)^{p-1} \, d\lambda = \int_R (f_n^o)^p \, d\lambda = \|f_n\|^p_{p\downarrow\lambda}.$$

Combining the above estimates we have

$$\|2g - f_n\|^p_{p\downarrow\lambda} \leq \|2g\|^p_{p\downarrow\lambda} - \|f_n\|^p_{p\downarrow\lambda}$$

which completes the proof.

Note that the last result does not include the case $p = \infty$. It is easy to see that the result fails in that case.

Our analogue of the Dominated Convergence Theorem now follows easily.

**Corollary 5.11.** *Let $1 \le p < \infty$ and suppose that $\{f_n\}$ converges pointwise $\lambda$-almost everywhere to $f$. If there exists a $g \in L^{p\downarrow}_\lambda$ such that $|f_n| \le g$ $\lambda$-almost everywhere then $\{f_n\}$ converges to $f$ in $L^{p\downarrow}_\lambda$.*

*Proof.* Let $h_n = |f - f_n|/2$. Clearly $|f| \le g$ $\lambda$-almost everywhere so $0 \le h_n \le g$ and $\{h_n\}$ converges pointwise to 0. By Lemma 5.10 $\{h_n\}$ converges to 0 in $L^{p\downarrow}_\lambda$. It follows that $\{f_n\}$ converges to $f$ in $L^{p\downarrow}_\lambda$.

Having established some control over convergence in $L^{p\downarrow}_\lambda$ we are ready to prove

**Theorem 5.12.** *$L^{p\downarrow}_\lambda$ is complete for $1 \le p < \infty$.*

*Proof.* It is enough to prove that every absolutely summable series in $L^{p\downarrow}_\lambda$ is summable. Suppose $f_1, f_2, \dots$ are functions in $L^{p\downarrow}_\lambda$ such that $\sum_{k=1}^\infty \|f_k\|_{p\downarrow\lambda} = M < \infty$. Set $g_n = \sum_{k=1}^n |f_k|$. Clearly $\{g_n\}$ is an increasing sequence of non-negative functions and $\|g_n\|_{p\downarrow\lambda} \le \sum_{k=1}^n \|f_k\|_{p\downarrow\lambda} \le M$. Theorem 5.8 shows that the pointwise limit $g$ of the sequence $\{g_n\}$ is in $L^{p\downarrow}_\lambda$. In particular, $g$ is finite $\lambda$-almost everywhere. Thus for $\lambda$-almost every $x \in \mathbf{R}$ the series $\sum_{k=1}^\infty f_k(x)$ is absolutely convergent and hence convergent. Set $s(x) = \sum_{k=1}^\infty f_k(x)$. Now $s_n = \sum_{k=1}^n f_k$ is a sequence of $L^{p\downarrow}_\lambda$ functions converging pointwise to $s$ $\lambda$-almost everywhere and satisfying $|s_n| \le g$. Corollary 5.12 completes the proof, showing that $\{s_n\}$ converges to $s$ in $L^{p\downarrow}_\lambda$.

## 6. The Dual of $L^{p\downarrow}_\lambda$.

In this section we identify the dual space of $L^{p\downarrow}_\lambda$. At the centre of our construction is the following definition.

**Definition 6.1.** *Suppose that $g$ is a $\lambda$-measurable function. Define $\bar g$ by*

$$\bar g(x) = \operatorname{ess\,sup}_\lambda(|g|, [x, \infty)).$$

Clearly $\bar g$ is non-negative, non-increasing and $\lambda$-measurable. Lemma 2.3 shows that $\bar g \ge g$ $\lambda$-almost everywhere. Indeed $\bar g$ is the smallest non-increasing majorant of $g$.

**Lemma 6.2.** *If $0 \le g_1 \le g_2 \le \dots$ and $\lim_{n\to\infty} g_n(x) = g(x)$ for $\lambda$-almost every $x$ then $0 \le \bar g_1 \le \bar g_2 \le \dots$ and $\lim_{n\to\infty} \bar g_n(x) = \bar g(x)$ for $\lambda$-almost every $x$.*

*Proof.* It is immediate from the definition that $0 \le \bar g_n \le \bar g_{n+1} \le \bar g$ for all $n$. Thus $\tilde g(x) = \lim_{n\to\infty} \bar g_n(x)$ exists and $\tilde g \le \bar g$. Moreover $\tilde g$ is a non-increasing function. Since $\tilde g \ge \bar g_n \ge g_n$ $\lambda$-almost everywhere for each $n$ we have $\tilde g \ge g$ $\lambda$-almost everywhere. Thus

$$\bar g(x) = \operatorname{ess\,sup}_\lambda(g, [x, \infty)) \le \operatorname{ess\,sup}_\lambda(\tilde g, [x, \infty)) = \tilde g(x)$$

which completes the proof.

Now we define our candidate for the dual space of $L^{p\downarrow}_\lambda$.

**Definition 6.3.** *$\|g\|_{p'*\lambda} = \|\bar g\|_{p',\lambda}$ and $L^{p'*}_\lambda$ is the collection of functions $g$ for which $\|g\|_{p'*\lambda} < \infty$.*

$L^{p'*}_\lambda$ is a subspace of $L^{p'}_\lambda$ since we have $\|g\|_{p',\lambda} \le \|g\|_{p'*\lambda}$. It is easy to see that $\|\cdot\|_{p'*\lambda}$ is a norm.

**Theorem 6.4.** *Suppose $f \in L^{p\downarrow}_\lambda$, $1 \le p \le \infty$. Then*

$$\sup\left\{\left|\int_{\mathbf{R}} fg\,d\lambda\right| : \|g\|_{p'*\lambda} \le 1\right\} = \|f\|_{p\downarrow\lambda}. \tag{6.1}$$

*Proof.* Since $\bar{g} \ge |g|$ $\lambda$-almost everywhere we have.

$$\left|\int_{\mathbf{R}} fg\,d\lambda\right| \le \int_{\mathbf{R}} |f||g|\,d\lambda \le \int_{\mathbf{R}} |f|\bar{g}\,d\lambda.$$

Since $\bar{g}$ is non-increasing, Definition 5.5 and Hölder's inequality show that

$$\int_{\mathbf{R}} |f|\bar{g}\,d\lambda \le \int_{\mathbf{R}} f^o\bar{g}\,d\lambda \le \|f^o\|_{p,\lambda}\|\bar{g}\|_{p',\lambda}.$$

By Definition 6.3 this is

$$\left|\int_{\mathbf{R}} fg\,d\lambda\right| \le \|f\|_{p\downarrow\lambda}\|g\|_{p'*\lambda}. \tag{6.2}$$

Thus we have "$\le$"in (6.1). To prove "$\ge$"note that for any non-negative, $\lambda$-non-increasing function $g$ with $\|g\|_{p',\lambda} \le 1$ we have $\|\operatorname{sgn}(f)g\|_{p'*\lambda} \le 1$ and $\left|\int_{\mathbf{R}} f\operatorname{sgn}(f)g\,d\lambda\right| = \int_{\mathbf{R}} |f|g\,d\lambda$. The definition of $\|\cdot\|_{p\downarrow\lambda}$ completes the proof.

**Lemma 6.5.** *Suppose $1 < p \le \infty$. If $f \in L^{p\downarrow}_\lambda$ and $g \in L^{p'*}_\lambda$ then for each $\alpha \in (0,1)$ there exists a non-negative function $h \in L^{p\downarrow}_\lambda$ such that $\|h\|_{p\downarrow\lambda} \le \|f\|_{p\downarrow\lambda}$ and $\int_{\mathbf{R}} h|g|\,d\lambda \ge \alpha^2\int_{\mathbf{R}} |f|\bar{g}\,d\lambda$.*

*Proof.* Without loss of generality we may suppose that $f$ and $g$ are non-negative. Fix $\alpha \in (0,1)$, and for each $n \in \mathbf{Z}$ define $B_n = \{x \in \mathbf{R} : \alpha^{n+1} < \bar{g}(x) \le \alpha^n\}$. Note that since $\bar{g}$ is non-increasing the sets $B_n$ are intervals such that if $n < m$ then $B_n$ lies entirely to the left of $B_m$. (It may happen that $B_n$ is empty for some $n$ in which case this statement is vacuously true.) Moreover, if we include the sets $B_{-\infty} = \{x \in \mathbf{R} : \bar{g}(x) = \infty\}$ and $B_\infty = \{x \in \mathbf{R} : \bar{g}(x) = 0\}$ then we have $\mathbf{R} = \cup_{\mathbf{Z}\cup\{\pm\infty\}}B_n$, a union of disjoint sets. We will prove the lemma on each set $B_n$, $n \in \mathbf{Z}\cup\{\pm\infty\}$. That is, for each $n \in \mathbf{Z}\cup\{\pm\infty\}$ we will construct a function $h_n : B_n \to [0,\infty)$ such that

$$\int_{B_n} h_n g\,d\lambda \ge \alpha^2\int_{B_n} f\bar{g}\,d\lambda, \quad \text{and} \tag{6.3}$$

$$\int_{B_n} h_n\phi\,d\lambda \le \int_{B_n} f\phi\,d\lambda \tag{6.4}$$

for all non-negative, non-increasing functions $\phi$. Once each $h_n$ is constructed, the lemma will follow on setting $h(x) = h_n(x)$ for $x \in B_n$, $n \in \mathbf{Z}\cup\{\pm\infty\}$.

It remains to construct the functions $h_n$. First set $h_\infty = 0$ on $B_\infty$ and note that (6.3) and (6.4) hold. Next, if $\lambda B_n = 0$ then any definition of $h_n$ will satisfy (6.3) and (6.4) trivially. In particular, this includes the case $n = \infty$ since $g \in L_\lambda^{p'*}$ implies $\lambda B_{-\infty} = 0$. We are left with $n \in \mathbf{Z}$ and $\lambda B_n > 0$. Note that $\lambda B_n \neq \infty$ lest we violate the assumption $g \in L_\lambda^{p'*}$. We let $y_n$ be the right endpoint of $B_n$ and distinguish two cases, $y_n \in B_n$ and $y_n \notin B_n$.

Case 1. $y_n \in B_n$. The definition of the sets $B_k$ implies that

$$\operatorname{ess\,sup}_\lambda(g, (y_n, \infty)) = \operatorname{ess\,sup}_\lambda(g, \cup_{k>n} B_k) \leq \alpha^{n+1} < \bar{g}(y_n) = \operatorname{ess\,sup}_\lambda(g, [y_n, \infty)).$$

Hence $\lambda\{y_n\} > 0$ and $\bar{g}(y_n) = g(y_n)$. Set $h_n(y_n) = (1/\lambda\{y_n\}) \int_{B_n} f \, d\lambda$ and $h_n(x) = 0$ for $x \in B_n \setminus \{y_n\}$. To verify (6.3) we estimate as follows.

$$\begin{aligned}\int_{B_n} h_n g \, d\lambda = \left( (1/\lambda\{y_n\}) \int_{B_n} f \, d\lambda \right) g(y_n) \lambda\{y_n\} = \left( \int_{B_n} f \, d\lambda \right) \bar{g}(y_n) \\ \geq \alpha \int_{B_n} f \alpha^n \, d\lambda \geq \alpha \int_{B_n} f \bar{g} \, d\lambda \geq \alpha^2 \int_{B_n} f \bar{g} \, d\lambda.\end{aligned}$$

To prove (6.4) fix a non-negative, $\lambda$-non-increasing function $\phi$ and note that since $\lambda\{y_n\} > 0$, Theorem 2.4 part (2) implies that $\phi(x) \geq \phi(y_n)$ for $\lambda$-almost every $x \in B_n$. Thus

$$\int_{B_n} h\phi \, d\lambda = \left( \int_{B_n} f \, d\lambda \right) \phi(y_n) \leq \int_{B_n} f\phi \, d\lambda.$$

Case 2. $y_n \notin B_n$. Let $x \in B_n$. $y_n \in \cup_{k>n} B_k$ so

$$\operatorname{ess\,sup}_\lambda(g, [y_n, \infty)) = \bar{g}(y_n) \leq \alpha^{n+1} < \bar{g}(x) = \operatorname{ess\,sup}_\lambda(g, [x, \infty)).$$

Thus $\lambda[x, y_n) > 0$ and $\bar{g}(x) = \operatorname{ess\,sup}_\lambda(g, [x, y_n))$. If we define the set $A$ by $A = \{x \in \mathbf{R} : \alpha\bar{g}(x) < g(x)\}$, we have

$$\begin{aligned}\operatorname{ess\,sup}_\lambda(g, [x, y_n) \setminus A) &\leq \operatorname{ess\,sup}_\lambda(\alpha\bar{g}, [x, y_n) \setminus A) \\ &\leq \operatorname{ess\,sup}_\lambda(\alpha\bar{g}, [x, y_n)) = \alpha\bar{g}(x) < \bar{g}(x) = \operatorname{ess\,sup}_\lambda(g, [x, y_n))\end{aligned}$$

so $\lambda([x, y_n) \cap A) > 0$ and $\bar{g}(x) = \operatorname{ess\,sup}_\lambda(g, [x, y_n) \cap A)$. Now choose $x_0, x_1, x_2, \ldots$ in $B_n$ converging to $y_n$ such that $\lambda([x_k, x_{k+1}) \cap A) > 0$ for $k = 0, 1, 2, \ldots$. Define $h_n$ by

$$h_n(x) = \begin{cases} 0, & x \in B_n, x < x_0 \\ (1/\lambda([x_0, x_1) \cap A)) \int_{(-\infty, x_0) \cap B_n} f \, d\lambda, & x_0 \leq x < x_1, x \in A \\ (1/\lambda([x_k, x_{k+1}) \cap A)) \int_{[x_{k-1}, x_k)} f \, d\lambda, & x_k \leq x < x_{k+1}, x \in A \\ 0, & x \in B_n \setminus A. \end{cases}$$

To prove (6.3) note that for $x \in B_n \cap A$ we have $g(x) > \alpha \bar{g}(x) > \alpha^{n+2}$.

$$
\begin{aligned}
\int_{B_n} h_n g \, d\lambda &= \int_{[x_0,x_1)\cap A} h_n g \, d\lambda + \sum_{k=1}^{\infty} \int_{[x_k,x_{k+1})} h_n g \, d\lambda \\
&= (1/\lambda([x_0,x_1)\cap A)) \int_{(-\infty,x_0)\cap B_n} f \, d\lambda \int_{[x_0,x_1)\cap A} g \, d\lambda \\
&\quad + \sum_{k=1}^{\infty} (1/\lambda([x_k,x_{k+1})\cap A)) \int_{(x_{k-1},x_k)} f \, d\lambda \int_{[x_k,x_{k+1})\cap A} g \, d\lambda \\
&\geq \alpha^{n+2} \int_{(-\infty,x_0)\cap B_n} f \, d\lambda + \sum_{k=1}^{\infty} \alpha^{n+2} \int_{(x_{k-1},x_k)\cap B_n} f \, d\lambda \\
&\geq \alpha^2 \left( \int_{(-\infty,x_0)\cap B_n} fg \, d\lambda + \sum_{k=1}^{\infty} \int_{[x_{k-1},x_k)} fg \, d\lambda \right) = \alpha^2 \int_{B_n} fg \, d\lambda.
\end{aligned}
$$

The proof of (6.4) uses Theorem 2.4 part (5). Let $\phi$ be an arbitrary, non-negative, $\lambda$-non-increasing function.

$$
\begin{aligned}
\int_{B_n} h_n \phi \, d\lambda &= (1/\lambda([x_0,x_1)\cap A)) \int_{(-\infty,x_0)\cap B_n} f \, d\lambda \int_{[x_0,x_1)\cap A} \phi \, d\lambda \\
&\quad + \sum_{k=1}^{\infty} (1/\lambda([x_k,x_{k+1})\cap A)) \int_{(x_{k-1},x_k)} f \, d\lambda \int_{[x_k,x_{k+1})\cap A} \phi \, d\lambda \\
&\leq \operatorname{ess\,sup}_\lambda(\phi, [x_0,\infty)) \int_{(-\infty,x_0)\cap B_n} f \, d\lambda + \sum_{k=1}^{\infty} \operatorname{ess\,sup}_\lambda(\phi, [x_k,\infty)) \int_{(x_{k-1},x_k)} f \, d\lambda \\
&\leq \operatorname{ess\,inf}_\lambda(\phi, (-\infty,x_0]) \int_{(-\infty,x_0)\cap B_n} f \, d\lambda + \sum_{k=1}^{\infty} \operatorname{ess\,inf}_\lambda(\phi, (\infty,x_k]) \int_{(x_{k-1},x_k)} f \, d\lambda \\
&\leq \int_{(-\infty,x_0)\cap B_n} f\phi \, d\lambda + \sum_{k=1}^{\infty} \int_{(x_{k-1},x_k)} f\phi \, d\lambda = \int_{B_n} f\phi \, d\lambda.
\end{aligned}
$$

This completes the proof.

**Theorem 6.6.** *Suppose $g \in L_\lambda^{p'*}$, $1 \leq p \leq \infty$. Then*

$$
\sup \left\{ \left| \int_{\mathbf{R}} fg \, d\lambda \right| : \|f\|_{p\downarrow\lambda} \leq 1 \right\} = \|g\|_{p'*\lambda}.
$$

*Proof.* The theorem is well known in the case $p = 1$ since $\|\cdot\|_{1\downarrow\lambda} = \|\cdot\|_{1,\lambda}$ and $\|\cdot\|_{\infty*\lambda} = \|\cdot\|_{\infty,\lambda}$. For $1 \leq p \leq \infty$ "$\leq$"follows by (6.2) above. The proof of "$\geq$"in the case $1 < p \leq \infty$ uses Lemma 6.5. Without loss of generality assume that $g$ is not $\lambda$-almost everywhere 0. Set $f = (\bar{g}/\|\bar{g}\|_{p',\lambda})^{p'-1}$ if $p \neq \infty$ and set $f \equiv 1$ if $p = \infty$. $f$ is non-increasing so by

Theorem 3.2 $\|f\|_{p\downarrow\lambda} = \|f\|_{p,\lambda} = 1$. Fix $\alpha \in (0,1)$ and let $h$ be the function given by Lemma 6.5. We have $\|h\,\mathrm{sgn}(g)\|_{p\downarrow\lambda} \leq 1$ so

$$\begin{aligned}\sup\left\{\left|\int_{\mathbf{R}} fg\,d\lambda\right| : \|f\|_{p\downarrow\lambda} \leq 1\right\} \geq \left|\int_{\mathbf{R}} h\,\mathrm{sgn}(g)g\,d\lambda\right| &= \int_{\mathbf{R}} h|g|\,d\lambda \\ &\geq \alpha^2 \int_{\mathbf{R}} f\bar{g}\,d\lambda = \alpha^2\|\bar{g}\|_{p',\lambda} = \alpha^2\|g\|_{p'*\lambda}.\end{aligned}$$

Let $\alpha \to 1$ to complete the theorem.

**Theorem 6.7.** *Suppose $1 \leq p < \infty$. The dual space of $L^{p\downarrow}_\lambda$ is $L^{p'*}_\lambda$. More precisely, each function $g \in L^{p'*}_\lambda$ gives rise to a continuous linear functional $L_g$ on $L^{p\downarrow}_\lambda$ given by $L_g(f) = \int_{\mathbf{R}} fg\,d\lambda$. The norm of $L_g$ is $\|g\|_{p'*\lambda}$ and every continuous linear functional on $L^{p\downarrow}_\lambda$ is $L_g$ for some $g \in L^{p'*}_\lambda$.*

*Proof.* If $g \in L^{p'*}_\lambda$ then Theorem 6.6 shows that $L_g$ is defined on $L^{p\downarrow}_\lambda$ and that it is continuous, having norm $\|g\|_{p'*\lambda}$. ($L_g$ is clearly linear.) Suppose now that $L$ is a continuous, linear functional on $L^{p\downarrow}_\lambda$. We wish to show that $L = L_g$ for some $g \in L^{p'*}_\lambda$.

Since $L^p_\lambda$ is a subspace of $L^{p\downarrow}_\lambda$ (with $\|\cdot\|_{p\downarrow\lambda} \leq \|\cdot\|_{p,\lambda}$) we may consider $L$ as a continuous linear functional on $L^p_\lambda$. By the Riesz Representation Theorem there is a function $g \in L^{p'}_\lambda$ such that $L(f) = \int_{\mathbf{R}} fg\,d\lambda$ for all $f \in L^p_\lambda$. To complete the proof we show that $L(f) = \int_{\mathbf{R}} fg\,d\lambda$ for all $f \in L^{p\downarrow}_\lambda$ and that $g \in L^{p'*}_\lambda$.

To do the first we set $f_n(x) = \min(n, \max(-n, f(x)))$ for $x \leq n$ and $f_n(x) = 0$ for $x > n$. Consider the sequence $\{|f_n g|\}$. This increases pointwise to $|fg|$. The Monotone Convergence Theorem yields

$$\begin{aligned}\int_{\mathbf{R}} |fg|\,d\lambda = \lim_{n\to\infty}\int_{\mathbf{R}} |f_n g|\,d\lambda &= \lim_{n\to\infty} L(|f_n|\,\mathrm{sgn}(g)) \\ &\leq \|L\| \lim_{n\to\infty}\|f_n\|_{p\downarrow\lambda} \leq \|L\|\|f\|_{p\downarrow\lambda} < \infty.\end{aligned}$$

Thus $fg \in L^1_\lambda$. Now consider $\{f_n\}$ as a sequence in $L^{p\downarrow}_\lambda$. $|f_n| \leq |f| \in L^{p\downarrow}_\lambda$ for each $n$ so by Corollary 5.11 $\{f_n\}$ converges to $f$ in $L^{p\downarrow}_\lambda$. Since $L$ is continuous,

$$L(f) = \lim_{n\to\infty} L(f_n) = \lim_{n\to\infty}\int_{\mathbf{R}} f_n g\,d\lambda = \int_{\mathbf{R}} fg\,d\lambda$$

where the last inequality follows from the Dominated Convergence Theorem using our observation that $fg \in L^1_\lambda$.

The second task, to show that $g \in L^{p'*}_\lambda$, uses Lemma 6.2. Set $g_n(x) = \min(n, |g(x)|)$ when $x \leq n$ and set $g_n(x) = 0$ when $x > n$. Note that $g_n \in L^{p'*}_\lambda$ and $\{g_n\}$ increases pointwise to $|g|$. Thus $\{\bar{g}_n\}$ increases pointwise to $\bar{g}$. The Monotone Convergence Theorem implies that

$$\lim_{n\to\infty}\|g_n\|_{p'*\lambda} = \lim_{n\to\infty}\|\bar{g}_n\|_{p',\lambda} = \|\bar{g}\|_{p',\lambda} = \|g\|_{p'*\lambda}.$$

Also, by Theorem 6.6,

$$\|g_n\|_{p'*\lambda} = \sup \left| \int_{\mathbf{R}} f g_n \, d\lambda \right| \le \sup \int_{\mathbf{R}} |f||g| \, d\lambda = \sup L(|f| \operatorname{sgn}(g)) \le \|L\|.$$

Here the suprema are taken over all functions $f$ with $\|f\|_{p\downarrow\lambda} \le 1$. The conclusion is that $\|g\|_{p'*\lambda} \le \|L\|$ so that $g \in L_\lambda^{p'*}$ as required.

**Corollary 6.8.** *$L_\lambda^{p'*}$ is complete for $1 \le p < \infty$.*

*Proof.* The dual space of any normed linear space is complete.

**Example 6.9.** *$L_\lambda^{p\downarrow}$ is not reflexive for $1 < p < \infty$.*

Suppose $\lambda$ satisfies the following mild conditions. There exist $a, b \in \mathbf{R}$ such that $0 < \lambda(-\infty, a] < \infty$, $0 < \lambda(a, b] < \infty$, and $\lambda$ is not supported on a finite set in $(a, b]$. In this case we will show that $L_\lambda^\infty(a, b]$ may be viewed as a subspace of $L_\lambda^{p'*}$ (for any $p \in (1, \infty]$) with equivalent norms.

If $g \in L_\lambda^\infty(a, b]$ with $\|g\|_{L_\lambda^\infty(a,b]} = M$ then extend $g$ to be defined on all of $\mathbf{R}$ by setting $g(x) = 0$ for $x \notin (a, b]$. Clearly, $\bar{g}(x) = M$ for $x \in (-\infty, a]$ and $\bar{g}(x) = 0$ for $x \in (b, \infty)$. Thus

$$M^{p'} \lambda(-\infty, a] = \int_{-\infty}^a \bar{g}^{p'} \, d\lambda \le \|\bar{g}\|_{p',\lambda}^{p'} \le \int_{-\infty}^b \bar{g}^{p'} \, d\lambda \le M^{p'} \lambda(-\infty, b]$$

and so $\|g\|_{L_\lambda^\infty(a,b]} \sim \|g\|_{p'*\lambda}$. By the Hahn-Banach Theorem every linear functional on $L_\lambda^\infty(a, b]$ extends to a linear functional on $L_\lambda^{p'*}$. Since $\lambda$ is not supported on a finite set, there are linear functionals on $L_\lambda^\infty(a, b]$ which do not arise via integration against any function on $(a, b]$. It follows that the dual space of $L_\lambda^{p'*}$ is not $L_\lambda^{p\downarrow}$.

The characterisation of the dual space of $L_\lambda^{p\downarrow}$ in terms of the function $\bar{g}$ makes it possible to explicitly calculate the Peetre K-functional for the pair $(L_\lambda^1, L_\lambda^\infty)$. The interpolation results obtained in this way may be found in [10].

## 7. Hardy's Inequality

For which indices $p$ and $q$ and which non-negative weight functions $u$ and $v$ does there exist a constant $C$ for which

$$\left( \int_{-\infty}^\infty \left| \int_{-\infty}^x f(t) u(t) \, dt \right|^q v(x) \, dx \right)^{1/q} \le C \left( \int_{-\infty}^\infty |f(t)|^p u(t) \, dt \right)^{1/p} \tag{7.1}$$

holds for all $f$? The answer to this question is the Hardy inequality with weights provided by [6], [2], [5, §1.3.2], [9] and others.

For which indices $p$ and $q$ and which non-negative sequences $\{u_k\}$ and $\{v_n\}$ does there exist a constant $C$ for which

$$\left( \sum_{n=0}^\infty \left| \sum_{k=0}^n f_k u_k \right|^q v_n \right)^{1/q} \le C \left( \sum_{k=0}^\infty |f_k|^p u_k \right)^{1/p}$$

for all $\{f_k\}$? Unsurprisingly, this similar question has also been answered. (See [1].)

Hardy's inequality with measures is more general than both of these answers.

**Theorem 7.1.** *Suppose* $0 < q < \infty$ *and* $1 < p < \infty$. *Let* $\mu$ *and* $\nu$ *be regular, Borel measures on* $\mathbf{R}$ *and take* $C$ *to be the smallest positive constant (possibly infinite) such that*

$$\left(\int_{\mathbf{R}} \left| \int_{-\infty}^{x} f\, d\mu \right|^q d\nu(x)\right)^{1/q} \leq C \left(\int_{\mathbf{R}} |f|^p\, d\mu\right)^{1/p}$$

*holds for all* $\mu$*-measurable functions* $f$. *Then*

(1) *If* $p \leq q$ *then*

$$C \sim \sup_{y \in \mathbf{R}} \left(\int_y^\infty d\nu\right)^{1/q} \left(\int_{-\infty}^y d\mu\right)^{1/p'}.$$

(2) *If* $1 < q < p < \infty$ *then*

$$C \sim \left(\int_{\mathbf{R}} \left(\int_y^\infty d\nu\right)^{r/q} \left(\int_{-\infty}^y d\mu\right)^{r/q'} d\mu(y)\right)^{1/r}.$$

(3) *If* $0 < q < 1 < p < \infty$ *then*

$$C \sim \left(\int_{\mathbf{R}} \left(\int_{-\infty}^y d\mu\right)^{r/p} \left(\int_y^\infty d\nu\right)^{r/p'} d\nu(y)\right)^{1/r}.$$

*Here* $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, $1/r = 1/q - 1/p$, *and* $A \sim B$ *means that there exist positive constants* $c_1$ *and* $c_2$ *such that* $c_1 A \leq B \leq c_2 A$.

The proofs of (1), (2), and (3) all appear in [8] but parts (1) and (2) are quite similar to the analogous results for (7.1) found in [6], [2] and [5, §1.3.2]. Part (3) is proved for weights in [9] but the result relies on Halperin's level function with respect to weights. The level function with respect to measures, as constructed here, enables us to extend the Hardy inequality in [9] to the statement (3) above.

To avoid a tedious enumeration, no mention of endpoint cases ($p = 1, \infty$; $q = 0, 1, \infty$) has been made in the statement of Theorem 7.1. Results are available in the references cited which carry over to this more general setting.

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