HARDY-TYPE INEQUALITIES FOR A NEW CLASS OF INTEGRAL OPERATORS

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June 10, 1997

Abstract. Mapping properties between weighted Lebesgue spaces of the operator that integrates a function over a difference of two dilations of a starshaped set in $\mathbf{R}^n$ are completely characterized. This includes integrals over annuli whose inner and outer radii are arbitrary increasing functions. The general result is applied to give sufficient conditions for boundedness between weighted Lebegue spaces for operators with a large class of non-negative kernels.

1.1 Introduction

The weight conditions which characterize the weighted Hardy inequality

$$\left(\int_0^\infty \left(\int_0^s f\right)^q v(s)\,ds\right)^{1/q} \le C\left(\int_0^\infty f^p u\right)^{1/p}$$

have set a standard for weight conditions because they are easy to estimate and to verify in particular cases, they relate well to the size of the constant C in the inequality, and they are themselves essentially just weighted norms. See [2], [5] and [7].

The problem of finding comparably simple weight conditions for weighted norm inequalities involving other operators, for example, replacing $\int_0^s f$ in the above inequality by $\int_0^\infty k(s,t)f(t)\,dt$ for some kernel $k(s,t)$, is a difficult one even for non-negative k. Some results for particular kernels are available and certain classes of kernels have been investigated with success. One of the general results in this direction, begun by Martin-Reyes and Sawyer [4] and by Bloom and Kerman [1] and improved by Stepanov [8], solves the weight characterization problem for Generalized Hardy Operators: The operator $f \to \int_0^s k(s,t)f(t)\,dt$ is a GHO provided

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A33.

Key words and phrases. Hardy inequality, Weight, Averaging operator.

Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

that $k(s,t)$ is non-negative, non-decreasing in s or non-increasing in t, and for some $D>0$ satisfies the growth condition

$$D^{-1}(k(s,\xi)+k(\xi,t)\le k(s,t)\le D(k(s,\xi)+k(\xi,t))$$

whenever $0<t<\xi<s$. These are strong assumptions which, in particular, exclude kernels with sharp jumps or with zeros. The results are general enough, however, to include weighted norm inequalities for the Riemann-Liouville fractional integral operators.

Recently, in [3], weight characterizations were given for operators of the form $f\to\int_{a(s)}^{b(s)}f$ where a and b are increasing functions. In these results the kernel is the characteristic function of a set and does have sharp jumps and zeros. In this paper we extend the results of [3] to higher dimensions using arguments introduced in [6]. Necessary and sufficient weight conditions which live up to the standard of simplicity set for the Hardy operator are given for operators which integrate over differences of general starshaped regions in $\mathbf{R}^n$. The weight conditions simplify futher in the case of integration over annuli.

The higher dimensional results are then applied to give sufficient conditions for one-dimensional inequalities for operator with a large class of kernels, namely those that can be expressed in the form

$$k(s,t)=\varphi(t/b(s))-\varphi(t/a(s))$$

for some non-increasing function φ and some increasing functions a and b with $a<b$. No growth condition is assumed.

1.2 Starshaped regions

We will call a region $S\in\mathbf{R}^n$ *smoothly starshaped* provided there exists a non-negative, piecewise-C^1 function ψ defined on the unit sphere in $\mathbf{R}^n$ with $S=\{x\in\mathbf{R}^n\setminus\{0\}:|x|\le\psi(x/|x|)\}$.

If S is smoothly starshaped, let $B=\{x\in\mathbf{R}^n\setminus\{0\}:|x|=\psi(x/|x|)\}$ and note that B is contained in the boundary of S. Since ψ is not assumed to be continuous, B may not be the whole boundary of S. The family of regions we integrate over is the collection of dilations of S.

Let E be the union of all dilations of S, $E=\cup_{\alpha>0}\alpha S$, and note that $E=\mathbf{R}^n$ whenever 0 is in the interior of S. For non-zero $x\in E$, since S is starshaped, there is a least positive dilation $\alpha_x S$ which contains x. We write $S_x=\alpha_x S$ and note that $x/\alpha_x\in B$ so that x is on the boundary of S.

Throughout, a and b are taken to be increasing differentiable functions on $\mathbf{R}^+$, satisfying $a(0)=b(0)=0$, $a(x)<b(x)$ for $x>0$, and $a(\infty)=b(\infty)=\infty$.

The n-dimensional weighted inequality that we characterise in this section involves integrals over differences of the form $b(\alpha_x)S\setminus a(\alpha_x)S$.

Our first result reduces the problem to a one-dimensional one of the type studied in [3].

Theorem 1.1. *Let S be a smoothly starshaped region in $\mathbf{R}^n$ and let B, E and α_x be defined as above. Suppose $0 < q < \infty$, $1 < p < \infty$, and u and v are non-negative weight functions on E. Then the inequality*

$$\left(\int_E \left|\int_{b(\alpha_x)S\setminus a(\alpha_x)S} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q} \le C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p} \tag{1.1}$$

holds for all locally integrable functions f on E if and only if

$$\left(\int_0^\infty \left|\int_{a(s)}^{b(s)} F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} \le C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p} \tag{1.2}$$

holds for all locally integrable functions $F : (0,\infty) \to \mathbf{R}$. Here

$$V(s) = \int_B v(s\eta)s^{n-1}\,d\eta, \quad \text{and} \quad U(t) = \left(\int_B u(t\tau)^{1-p'} t^{n-1}\,d\tau\right)^{1-p}. \tag{1.3}$$

In particular, the best constants in inequalities (1.1) *and* (1.2) *coincide.*

Proof. Suppose (1.2) holds and fix a locally integrable function $f : E \to \mathbf{R}$. Set

$$F(t) = \int_B f(t\tau)t^{n-1}\,d\tau. \tag{1.4}$$

Make the changes of variable $x = s\eta$ and $y = t\tau$ in the left hand side of (1.1) and notice that for $\eta \in B$, $\alpha_{s\eta} = s$.

$$\begin{aligned}
&\left(\int_E \left|\int_{b(\alpha_x)S\setminus a(\alpha_x)S} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q}\\
=&\left(\int_E \left|\int_{a(\alpha_x)}^{b(\alpha_x)} \int_B f(t\tau)t^{n-1}\,d\tau\,dt\right|^q v(x)\,dx\right)^{1/q}\\
=&\left(\int_0^\infty \int_B \left|\int_{a(s)}^{b(s)} \int_B f(t\tau)t^{n-1}\,d\tau\,dt\right|^q v(s\eta)s^{n-1}\,d\eta\,ds\right)^{1/q}\\
=&\left(\int_0^\infty \left|\int_{a(s)}^{b(s)} F(t)\,dt\right|^q V(s)\,ds\right)^{1/q}\\
\le& C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}.
\end{aligned}$$

The last inequality is the hypothesis (1.2). Use Hölder's inequality in the integral defining F to estimate the last line above as follows.

$$\begin{aligned}
&C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty \left|\int_B f(t\tau)t^{n-1}\,d\tau\right|^p U(t)\,dt\right)^{1/p}\\
\le&C\left(\int_0^\infty \left(\int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\right)\times\right.\\
&\qquad\left.\left(\int_B u(t\tau)^{1-p'}t^{n-1}\,d\tau\right)^{p/p'} U(t)\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty \int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}.
\end{aligned}$$

Thus (1.1) holds.

To prove the converse, suppose that (1.1) holds and fix a locally integrable function $F:(0,\infty)\to\mathbf{R}$. Define $f:E\to\mathbf{R}$ by

$$f(t\tau)=F(t)U(t)^{p'-1}u(t\tau)^{1-p'}$$

and use the definition of U to see that the relationship (1.4) is still valid. As in the first part of the proof we have

$$\left(\int_0^\infty \left|\int_{a(s)}^{b(s)} F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} = \left(\int_E \left|\int_{b(\alpha_x)S\setminus a(\alpha_x)S} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q}.$$

Now the inequality (1.1) becomes

$$\left(\int_0^\infty \left|\int_{a(s)}^{b(s)} F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} \le C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}.$$

Using the definitions of f and U we recognize the right hand side above as the right hand side of (1.2).

$$\begin{aligned}
&C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}\\
=&C\left(\int_0^\infty \int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty |F(t)|^p U(t)^{p'}\int_B u(t\tau)^{(1-p')p+1}t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}.
\end{aligned}$$

This establishes (1.2) and completes the proof.

To complete the characterization we apply the results of [3] to give the following two theorems.

Theorem 1.2. *Let S be a smoothly starshaped region in $\mathbf{R}^n$ and let B, E and α_x be defined as above. Suppose $1 < p \leq q < \infty$, and u and v are non-negative weight functions on E. Then* (1.1) *holds for all locally integrable f on E if and only if*

$$\sup_{\substack{t\leq s\\ a(s)\leq b(t)}} \left(\int_{b(t)S\setminus a(s)S} u^{1-p'}\right)^{1/p'} \left(\int_{sS\setminus tS} v\right)^{1/q} < \infty. \tag{1.5}$$

Proof. Theorem 2.2 of [3] shows that (1.2) holds for all locally integrable F if and only if

$$\sup_{\substack{t\leq s\\ a(s)\leq b(t)}} \left(\int_{a(s)}^{b(t)} U^{1-p'}\right)^{1/p'} \left(\int_t^s V\right)^{1/q} < \infty.$$

Here U and V are defined by (1.3) as before. Since Theorem 1.1 shows that (1.1) holds for all f if and only if (1.2) holds for all F, it only remains to show that the last expression is equivalent to (1.5). Using the definitions of U and V the last expression becomes

$$\sup_{\substack{t\leq s\\ a(s)\leq b(t)}} \left(\int_{a(s)}^{b(t)} \int_B u(\xi\tau)^{1-p'}\xi^{n-1}\,d\tau\,d\xi\right)^{1/p'} \left(\int_t^s \int_B v(\xi\eta)\xi^{n-1}\,d\eta\,d\xi\right)^{1/q} < \infty,$$

which reduces easily to (1.5).

To give an analogue of the above corollary in the case $q < p$ we need to define the normalizing function introduced in [3]. Since a^{-1} and b^{-1} exist and are increasing we may define the sequence $\{M_k\}_{k\in\mathbf{Z}}$ recursively as follows: Fix $M_0 = b^{-1}(1)$ and define

$$M_{k+1} = a^{-1}(b(M_k)), \text{ if } k \geq 0 \text{ and } M_k = b^{-1}(a(M_{k+1})), \text{ if } k < 0.$$

Clearly $a(M_{k+1}) = b(M_k)$ for all $k \in \mathbf{Z}$. The *normalizing function* σ is defined by

$$\sigma(t) = \sum_{k\in\mathbf{Z}} \chi_{(M_k,M_{k+1})}(t)\frac{d}{dt}(b^{-1}\circ a)^k(t)$$

where $(b^{-1}\circ a)^k$ denotes k times repeated composition.

Theorem 1.3. *Let S be a smoothly starshaped region in $\mathbf{R}^n$ and let B, E and α_x be defined as above. Suppose $0 < q < p$, $1 < p < \infty$, and u and v are non-negative*

weight functions on E. Then (1.1) *holds for all locally integrable f on E if and only if both*
(1.6)
$$\left(\int_0^\infty \int_{tS\backslash b^{-1}(a(t))S} \left(\int_{b(\alpha_x)S\backslash a(t)S} u^{1-p'}\right)^{r/p'} \left(\int_{tS\backslash \alpha_x S} v\right)^{r/p} v(x)\,dx\,\sigma(t)dt\right)^{1/r}$$

and
(1.7)
$$\left(\int_0^\infty \int_{a^{-1}(b(t))S\backslash tS} \left(\int_{b(t)S\backslash a(\alpha_x)S} u^{1-p'}\right)^{r/p'} \left(\int_{\alpha_x S\backslash tS} v\right)^{r/p} v(x)\,dx\,\sigma(t)dt\right)^{1/r}$$

are finite

Proof. Proceed as in the previous proof applying Theorem 2.5 of [3] instead of Theorem .2 of [3].

As mentioned, these results include operators which integrate over annuli. If we take the starshaped region S to be the unit ball in $\mathbf{R}^n$ we see that $E = \mathbf{R}^n$ and $\alpha_x = |x|$. The operator becomes

$$f \to \int_{b(|x|)\le|y|\le a(|x|)} f(y)\,dy,$$

an integral over annuli whose inner and outer radii are given by the increasing functions a and b. In the next two corollaries we record this result in the special case that a and b are lines through the origin.

Corollary 1.4. *Suppose $1 < p \le q < \infty$ and u and v are weights on $\mathbf{R}^n$. Fix real numbers A and B with $0 < A < B$. Then the inequality*

$$\left(\int_{\mathbf{R}^n} \left(\int_{A|x|\le|y|\le B|x|} f(y)\,dy\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbf{R}^n} f^p u\right)^{1/p} \tag{1.8}$$

holds for all non-negative f if and only if

$$\sup_{|y|\le|x|\le(B/A)|y|} \left(\int_{A|x|\le|z|\le B|y|} u(z)^{1-p'}\,dz\right)^{1/p'} \left(\int_{|y|\le|z|\le|x|} v(z)\,dz\right)^{1/q} < \infty.$$

Proof. This is Theorem 1.2, taking S to be the unit ball in $\mathbf{R}^n$, $a(t) = At$ and $b(t) = Bt$.

Corollary 1.5. *Suppose $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$ and u and v are weights on $\mathbf{R}^n$. Fix real numbers A and B with $0 < A < B$. Then the inequality (1.8) holds for all non-negative f if and only if both*

$$\left(\int_0^\infty \int_{At/B\le|x|\le t}\left(\int_{At\le|y|\le B|x|} u(y)^{1-p'}\,dy\right)^{r/p'}\left(\int_{|x|\le|y|\le t} v(y)\,dy\right)^{r/p} v(x)\,dx\,\frac{dt}{t}\right)^{1/r}$$

and

$$\left(\int_0^\infty \int_{t\le|x|\le Bt/A}\left(\int_{A|x|\le|y|\le Bt} u(y)^{1-p'}\,dy\right)^{r/p'}\left(\int_{t\le|y|\le|x|} v(y)\,dy\right)^{r/p} v(x)\,dx\,\frac{dt}{t}\right)^{1/r}$$

are finite.

Proof. We apply Theorem 1.3, taking S to be the unit ball in $\mathbf{R}^n$, $a(t) = At$ and $b(t) = Bt$. It was shown in the proof of Corollary 2.6 of [3] that the normalizing function $\sigma(t)$ satisfies $1/B \le t\sigma(t) \le 1/A$ in this case. This estimate completes the proof.

1.3 From regions to kernels

In this section we define particular starshaped regions and use the results of the previous section to prove one-dimensional inequalities for operators with more general kernels.

Definition 1.1. *Suppose $\varphi : (0,\infty) \to (0,\infty)$ is a decreasing, continuously differentiable function with $\varphi(0) = 1$ and $\varphi(\infty) = 0$. Define*

$$S = \{(x_1, x_2) : 0 \le x_1, 0 \le x_2 \le x_1\varphi(x_1)\}$$

Lemma 1.6. *S is smoothly starshaped and the union of all dilations of S is*

$$E = \{(x_1, x_2) : 0 \le x_1, 0 \le x_2 < x_1\}.$$

For $x = (x_1, x_2) \in E$, S_x, the least dilation of S that contains x, is $\alpha_x S$, where α_x satisfies $x_2 = x_1\varphi(x_1/\alpha_x)$.

Proof. If $(x_1, x_2) \in S$ and $0 \le r \le 1$ then we have $0 \le rx_2 \le rx_1\varphi(x_1) \le rx_1\varphi(rx_1)$ so $(rx_1, rx_2) \in S$. Thus S is starshaped. It is clear that S is smoothly starshaped.

If $(x_1, x_2) \in \alpha S$ with $\alpha > 0$ then $0 \le x_1/\alpha$ and hence $0 \le x_1$. Also $0 \le x_2/\alpha \le (x_1/\alpha)\varphi(x_1/\alpha)$ so $0 \le x_2 \le x_1\varphi(x_1/\alpha) \le x_1\varphi(0)$. Thus $(x_1, x_2) \in E$. Conversely, if $(x_1, x_2) \in E$ then for sufficiently large α we have both $0 \le x_1/\alpha \le 1$ and $0 \le x_2/\alpha \le (x_1/\alpha)\varphi(x_1/\alpha)$ so (x_1, x_2) is in some dilation of S.

The least dilation of S that contains the point (x_1, x_2) is the unique α for which $(x_1/\alpha, x_2/\alpha)$ is on the graph of $x_1\varphi(x_1)$. Thus $x_2 = x_1\varphi(x_1/\alpha_x)$. This completes the proof.

Theorem 1.7. *Suppose $1 < p \le q < \infty$ and u and v are weights. Let a and b be as above and φ and S be as in Definition 1.1. Then the inequality*

$$(1.9)\quad \left(\int_0^\infty \left(\int_0^\infty [\varphi(t/b(s)) - \varphi(t/a(s))]g(t)\,dt\right)^q v(s)\,ds\right)^{1/q} \le C\left(\int_0^\infty g^p u\right)^{1/p}$$

holds for all non-negative g provided

$$(1.10)\qquad \sup_{\substack{t\le s\\ a(s)\le b(t)}} \left(\int_0^\infty [\varphi(\xi/b(t)) - \varphi(\xi/a(s))]u(\xi)^{1-p'}\,d\xi\right)^{1/p'} \left(\int_t^s v\right)^{1/q} < \infty.$$

Proof. We apply Theorem 1.2 with S defined in terms of φ as in Definition 1.1 and with $v(x_1,x_2)$ replaced by $\delta_1(x_1)v(1/\varphi^{-1}(x_2))\frac{d}{dx_2}(1/\varphi^{-1}(x_2))$ and $u(y_1,y_2)$ replaced by $y_1^{p-1}u(y_1)$. Here δ_1 is the measure consisting of a single atom of weight 1 at 1. It is straightforward to extend Theorem 1.2 to such measures.

We begin by verifying the weight condition (1.5). Using the definition of S in terms of φ we have

$$\begin{aligned} b(t)S\setminus a(s)S &= \{(y_1,y_2): 0\le y_1, y_1\varphi(y_1/a(s)) \le y_2 \le y_1\varphi(y_1/b(t))\},\\ sS\setminus tS &= \{(x_1,x_2): 0\le x_1, x_1\varphi(x_1/t)\le x_2\le x_1\varphi(x_1/s)\}, \text{ and}\\ (sS\setminus tS)\cap\{(x_1,x_2): x_1=1\} &= \{(1,x_2):\varphi(1/t)\le x_2\le\varphi(1/s)\} \end{aligned}$$

The weight condition (1.5) becomes

$$\sup_{\substack{t\le s\\ a(s)\le b(t)}} \left(\int_0^\infty \int_{y_1\varphi(y_1/a(s))}^{y_1\varphi(y_1/b(t))} y_1^{-1}u(y_1)^{1-p'}\,dy_2\,dy_1\right)^{1/p'} \times\left(\int_{\varphi(1/t)}^{\varphi(1/s)} v(1/\varphi^{-1}(x_2))\,d(1/\varphi^{-1}(x_2))\right)^{1/q} < \infty$$

which, replacing the variable y_1 by ξ in the first factor and making the substitution $\xi = 1/\varphi^{-1}(x_2)$ in the second factor, reduces to the hypothesis (1.10).

We have verified the weight condition of Theorem 1.2 so we may conclude that the inequality (1.1) holds for all $f(y_1,y_2)$. In particular, it holds with $f(y_1,y_2)$ replaced by $y_1^{-1}g(y_1)$ for any non-negative function g. To simplify the left hand side of (1.1) we observe that the choice of v means that we may take $x_1=1$. Lemma 1.6 shows that $\alpha_{(1,x_2)} = 1/\varphi^{-1}(x_2)$ and applying Definition 1.1 we see that

$$\begin{aligned} &b(\alpha_{(1,x_2)})S\setminus a(\alpha_{(1,x_2)})S\\ &\qquad= \{(y_1,y_2): y_1\varphi(y_1/a(1/\varphi^{-1}(x_2)))\le y_2\le y_1\varphi(y_1/b(1/\varphi^{-1}(x_2)))\}. \end{aligned}$$

The left hand side of (1.1) becomes

$$\left(\int_0^1\left(\int_0^\infty\int_{y_1\phi(y_1/a(1/\varphi^{-1}(x_2)))}^{y_1\phi(y_1/b(1/\varphi^{-1}(x_2)))} y_1^{-1}g(y_1)\,dy_2\,dy_1\right)^q v(1/\varphi^{-1}(x_2))\,d(1/\varphi^{-1}(x_2))\right)^{1/q}$$

which, replacing y_1 by t and making the substitution $s = 1/\varphi^{-1}(x_2)$, becomes the left hand side of (1.9).

The description of E in Lemma 1.6 shows that the right hand side of (1.1) is just

$$\left(\int_0^\infty \int_0^{y_1} (y_1^{-1} g(y_1))^p y_1^{p-1} u(y_1)\, dy_2\, dy_1\right)^{1/p} = \left(\int_0^\infty g^p u\right)^{1/p}.$$

This completes the proof.

Theorem 1.8. *Suppose* $0 < q < p$, $1 < p < \infty$, $1/r = 1/q - 1/p$ *and* u *and* v *are weights. Let* a *and* b *be as above and* φ *and* S *be as in Definition 1.1. Then the inequality* (1.9) *holds for all non-negative* g *provided both*

$$\left(\int_0^\infty \int_{b^{-1}(a(t))}^t \left(\int_0^\infty [\varphi(\xi/b(s)) - \varphi(\xi/a(t))] u(\xi)^{1-p'}\, d\xi\right)^{r/p'} \times \left(\int_s^t v\right)^{r/p} v(s)\, ds\sigma(t)\, dt\right)^{1/r}$$

and

$$\left(\int_0^\infty \int_t^{a^{-1}(b(t))} \left(\int_0^\infty [\varphi(\xi/b(t)) - \varphi(\xi/a(s))] u(\xi)^{1-p'}\, d\xi\right)^{r/p'} \times \left(\int_t^s v\right)^{r/p} v(s)\, ds\sigma(t)\, dt\right)^{1/r}$$

are finite.

Proof. Proceed as in the previous proof applying Theorem 1.3 instead of Theorem 1.2.

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