TRANSFERRING MONOTONICITY IN WEIGHTED NORM INEQUALITIES

Gord Sinnamon

The University of Western Ontario

November 11, 2002

Abstract. Certain weighted norm inequalities for integral operators with non-negative, monotone kernels are shown to remain valid when the weight is replaced by a monotone majorant or minorant of the original weight. A similar result holds for operators with quasi-concave kernels. To prove these results a careful investigation of the functional properties of the monotone envelopes of a non-negative function is carried out. Applications are made to function space embeddings of the cones of monotone functions and quasi-concave functions.

Under weaker partial orders on non-negative functions, monotone envelopes are re-examined and the level function is recognized as a monotone envelope in two ways. Using the level function, monotonicity can be transferred from the kernel to the weight in inequalities restricted to a cone of monotone functions.

1. Introduction

This paper is a contribution to the theory of weighted norm inequalities for positive integral operators but will also be of interest to those studying function spaces. We show that the monotonicity of the kernel of an integral operator can be transferred to the weight in a norm inequality. The result is applied to embeddings of the cone of non-increasing functions between general rearrangement-invariant spaces and to embeddings of the cone of quasi-concave functions between weighted Lebesgue spaces.

The usual partial order on non-negative functions is pointwise, that is, $u \le v$ provided $u(x) \le v(x)$ for (almost) all x. For non-negative, λ-measurable functions

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 46E30.

Key words and phrases. Inequalities, Weights, Monotonicity, Quasi-concave functions.

Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged

on $\mathbf{R}$ we can look at the weaker order relation $u \le_\downarrow v$ given by

$$\int_{(-\infty,x]} u\,d\lambda \le \int_{(-\infty,x]} v\,d\lambda, \quad x \in \mathbf{R}.$$

This relation is important in the study of monotone functions because $u \le_\downarrow v$ if and only if

$$\int_{\mathbf{R}} fu\,d\lambda \le \int_{\mathbf{R}} fv\,d\lambda$$

for all non-negative, non-increasing functions f. (See Corollary 1.3 below.) Naturally, there is a corresponding relation $u \le_\uparrow v$ defined by

$$\int_{[x,\infty)} u\,d\lambda \le \int_{[x,\infty)} v\,d\lambda, \quad x \in \mathbf{R}$$

and satisfying $u \le_\uparrow v$ if and only if

$$\int_{\mathbf{R}} fu\,d\lambda \le \int_{\mathbf{R}} fv\,d\lambda$$

for all non-negative, non-decreasing functions f. Although the relations $\le_\downarrow$ and $\le_\uparrow$ are reflexive and transitive on the set of all non-negative functions they are not partial orders on this large domain because antisymmetry may fail when the integrals used to define them are infinite.

It is essential to understand the interplay between these order relations when working with monotone functions. To further this understanding we examine, in Section 2, the four monotone envelopes of a non-negative function u: The least non-increasing majorant, the greatest non-increasing minorant, the least non-decreasing majorant, and the greatest non-decreasing minorant. There are not just four monotone envelopes, however, as we can change order relations and thereby change our notions of least, greatest, majorant and minorant. Looking at monotone envelopes with respect to the order relations $\le_\downarrow$ and $\le_\uparrow$ leads to some surprising and useful results. For instance, the level function of u makes its appearance as both a least non-increasing majorant of u with respect to $\le_\downarrow$ and a greastest non-increasing minorant of u with respect to $\le_\uparrow$.

Studying the various monotone envelopes in Section 2 leads to our main results, in Section 3, for transferring monotonicity from the kernel to the weight in certain norm inequalities. Theorems 4.1 and 4.2 are concerned with transferring quasi-concavity. Applications extending embedding theorems for monotone functions are given in Theorem 3.7 and Corollary 3.8 and for quasi-concave functions in Theorem 4.3 and Corollary 4.4. Section 5 is devoted to proving Theorem 2.1 by a series of lemmas that set out the principle of "pushing mass." In Section 6, we expose the simple structure of the level function, a result that was previously known only for bounded functions.

In the remainder of this section we introduce notation, prove some basic results in their natural generality, and recall those properties of the level function that we will require in the sequel. For notation and background in Banach Function Spaces we refer to [1].

Let λ be a σ-finite measure on the real line. In order for monotone functions to be λ-measurable we assume that all Borel sets are λ-measurable. Let L^+_λ be the collection of all non-negative, λ-measurable functions on $\mathbf{R}$ and let $L^+_\lambda(S)$ denote those functions in L^+_λ which vanish off $S \subset \mathbf{R}$. We denote the collections of monotone functions on $\mathbf{R}$ by

$$L^\downarrow_\lambda = \{f \in L^+_\lambda : f \text{ is non-increasing}\} \text{ and } L^\uparrow_\lambda = \{f \in L^+_\lambda : f \text{ is non-decreasing}\}.$$

The two operators of integration we will need are I and I^* defined by

$$If(x) = \int_{(-\infty,x]} f\,d\lambda \quad \text{and} \quad I^*f(x) = \int_{[x,\infty)} f\,d\lambda.$$

Note that for all $u, v \in L^+_\lambda$ we have

$$\int_{\mathbf{R}} (Iu)v\,d\lambda = \int_{\mathbf{R}} u(I^*v)\,d\lambda.$$

Now that we have defined the operators I and I^* we prefer to write $Iu \le Iv$ rather than the equivalent $u \le_\downarrow v$ and to write $I^*u \le I^*v$ rather than the equivalent $u \le_\uparrow v$.

It is clear that the operator I takes non-negative functions to non-decreasing functions so $I(L^+_\lambda) \subset L^\uparrow_\lambda$. In Lemma 1.2 we show that the subset is quite a large one. To begin we show that $I(L^+_\lambda)$ has a useful lattice property.

Lemma 1.1. *If $u, v \in L^+_\lambda$ then there exists $w \in L^+_\lambda$ such that $Iw = \max(Iv, Iv)$.*

Proof. Set $W = \max(Iu, Iv)$ and let $M = \sup\{x \in \mathbf{R} : W(x) < \infty\}$. If $M = -\infty$ the result is trivial, otherwise W is non-decreasing and right continuous on $(-\infty, M)$ and $W(-\infty) = 0$. By [8, Theorem 12, p301] there exists a Borel measure μ such that

$$W(x) = \int_{(-\infty,x]} d\mu$$

for all $x < M$. We show that μ is absolutely continuous with respect to λ on $(-\infty, M)$. Note that both $u\lambda$ and $v\lambda$ are finite on compact subsets of $(-\infty, M)$ and hence are Baire measures on $(-\infty, M)$. By [8, Corollary 12, p340] both $u\lambda$ and $v\lambda$ are regular measures on $(-\infty, M)$. Thus, if $E \subset (-\infty, M)$ with $\lambda(E) = 0$ and $\varepsilon > 0$ then $u\lambda(E) = v\lambda(E) = 0$ as well so we can find an open set O with $E \subset O \subset (-\infty, M)$ such that

$$\int_O u\,d\lambda < \varepsilon/2 \quad \text{and} \quad \int_O v\,d\lambda < \varepsilon/2.$$

Now write $O = \cup_i (a_i, b_i)$, a union of its connected components, to get

$$\mu(E) = \int_E d\mu \le \int_O d\mu = \sum_i W(b_i-) - W(a_i)$$
$$\le \sum_i Iu(b_i-) - Iu(a_i) + Iv(b_i-) - Iv(a_i) = \int_O u\,d\lambda + \int_O v\,d\lambda < \varepsilon.$$

(Here we have used the observation that if $A \ge C$ and $B \ge D$ then $\max(A, B) - \max(C, D) \le A - C + B - D$.) Since ε was arbitrary, $\mu(E) = 0$ so μ is absolutely continuous with respect to λ. By the Radon-Nikodym Theorem there is a $w \in L^+_\lambda$ such that $\mu = w\lambda$ and we have

$$W(x) = \int_{(-\infty,x]} w\,d\lambda$$

for $x < M$. If $M = \infty$ we are done. If M is an atom for λ then it is a simple matter to choose a value for $w(M)$ so that

$$W(M) = \int_{(-\infty,M]} w\,d\lambda.$$

For $x > M$, $W(x) = \infty$ so we may set $w = \max(u, v)$ on (M, ∞) to complete the proof.

Lemma 1.2. *If $f \in L^\uparrow_\lambda$ then there exist $u_n \in L^+_\lambda$ such that the sequence Iu_n is non-decreasing and converges to f pointwise, λ-almost everywhere.*

Proof. We begin by replacing $f(x)$ by

$$\operatorname*{ess\,sup_\lambda}_{t \le x} f(t).$$

Since f is non-decreasing, the two functions agree λ-almost everywhere and therefore this new f satisfies

$$f(x) = \operatorname*{ess\,sup_\lambda}_{t \le x} f(t) = \sup\{y : \lambda\{t \le x : f(t) > y\} > 0\}.$$

Let $f_j = \min(f - 1/j, j)$ and note that f_j increases to f pointwise as $j \to \infty$. Since for all a,

$$f_j(a) < f(a) = \sup\{y : \lambda\{t \le a : f(t) > y\} > 0\}$$

the set

$$\{t \le a : f(t) > f_j(a)\}$$

has positive λ-measure and since λ is σ-finite we can choose a subset $E_{a,j}$ of finite, positive λ-measure. Let

$$v_{a,j} = f_j(a)\lambda(E_{a,j})^{-1}\chi_{E_{a,j}}.$$

It is easy to check that $Iv_{a,j} \le f(x)$ for $x \in \mathbf{R}$ and that $Iv_{a,j}(a) = f_j(a)$.

Since λ is σ-finite, it has at most countably many atoms. Hence we can choose a countable dense subset $\{a_i\}$ of $\mathbf{R}$ which contains all the atoms of λ. Induction applied to Lemma 1.1 shows that for each positive integer n there exists a $u_n \in L^+_\lambda$ such that

$$Iu_n = \max_{i=1,\dots,n;j=1,\dots,n}\{Iv_{a_i,j}\}.$$

It is evident that Iu_n is a non-decreasing sequence and that $Iu_n \le f$ for each n. It remains to show that

$$\lim_{n\to\infty} Iu_n(x) = f(x) \tag{1.1}$$

for λ-almost every x. For each a_i and each $n \ge i$ we have

$$f(a_i) \ge Iu_n(a_i) \ge Iv_{a_i,n}(a_i) = f_n(a_i)$$

so (1.1) holds for $x = a_i$. If x is not one of the a_i then for each $a_i < x$ we have

$$f(a_i) = \lim_{n\to\infty} Iu_n(a_i) \le \lim_{n\to\infty} Iu_n(x) \le f(x).$$

Since the a_i are dense this implies that

$$f(x-) \le \lim_{n\to\infty} Iu_n(x) \le f(x).$$

In particular, if x is a point of continuity of f then (1.1) holds. Since f is non-decreasing it has at most countably many points of discontinuity so the set

$$\{x \in \mathbf{R} \setminus \{a_i\} : f(x-) \ne f(x)\}$$

is countable and contains no atoms of λ. Therefore it has zero λ-measure. This completes the proof.

Corollary 1.3. *Suppose $v, w \in L^+_\lambda$. If $f \in L^\uparrow_\lambda$ and $I^*v \le I^*w$ or if $f \in L^\downarrow_\lambda$ and $Iv \le Iw$ then*

$$\int_{\mathbf{R}} fv\,d\lambda \le \int_{\mathbf{R}} fw\,d\lambda.$$

Proof. Suppose $f \in L^\uparrow_\lambda$ and $I^*v \le I^*w$. By Lemma 1.2 we have

$$\begin{aligned}\int_{\mathbf{R}} fv\,d\lambda &= \sup_{Iu\le f}\int_{\mathbf{R}}(Iu)v\,d\lambda = \sup_{Iu\le f}\int_{\mathbf{R}} u(I^*v)\,d\lambda \\ &\le \sup_{Iu\le f}\int_{\mathbf{R}} u(I^*w)\,d\lambda = \sup_{Iu\le f}\int_{\mathbf{R}}(Iu)w\,d\lambda \le \int_{\mathbf{R}} fw\,d\lambda.\end{aligned}$$

The second part follows from the first by the change of variable $x \to -x$.

To work with the level function we introduce a class of averaging operators. Suppose $\{J\}$ is a countable (or finite) collection of disjoint intervals each of finite, positive λ-measure and define the operator A by

$$Af(x) = \begin{cases} \frac{1}{\lambda(J)} \int_J f\,d\lambda, & x \in J \in \{J\} \\ f(x), & x \notin \cup\{J\} \end{cases}.$$

We denote the collection of all such operators A by $\mathcal{A}$.

Proposition 1.4. *Suppose that $A \in \mathcal{A}$. Then*
i) A is formally self-adjoint, that is, for all $f, g \in L^+_\lambda$,

$$\int_{\mathbf{R}} (Af)g\,d\lambda = \int_{\mathbf{R}} f(Ag)\,d\lambda.$$

ii) If $f \in L^\downarrow_\lambda$ then $Af \in L^\downarrow_\lambda$ and $IAf \le If$.
iii) If $f \in L^\uparrow_\lambda$ then $Af \in L^\uparrow_\lambda$ and $If \le IAf$.

Proof. Suppose $f, g \in L^+_\lambda$. Then

$$\begin{aligned} \int_{\mathbf{R}} (Af)g\,d\lambda &= \int_{\mathbf{R}\setminus\cup J} fg\,d\lambda + \sum_J \int_J \left(\frac{1}{\lambda(J)} \int_J f\,d\lambda\right) g\,d\lambda \\ &= \int_{\mathbf{R}\setminus\cup J} fg\,d\lambda + \sum_J \int_J f \left(\frac{1}{\lambda(J)} \int_J g\,d\lambda\right) d\lambda = \int_{\mathbf{R}} f(Ag)\,d\lambda. \end{aligned}$$

This is (i).

Replacing a function by its average on an interval preserves monotonicity so the first statements of both (ii) and (iii) are clear.

For the second statements of (ii) and (iii), fix $y \in \mathbf{R}$ and set $\chi = \chi_{(-\infty,y]}$. Replacing χ by its average on an interval has no effect when χ is constant on the interval so applying A to χ is easy: If $y \in J$ for some J then

$$A\chi(t) = \begin{cases} 1, & t \le y, t \notin J \\ \frac{\lambda((-\infty,y]\cap J)}{\lambda(J)}, & t \in J \\ 0, & \text{otherwise.} \end{cases}$$

If $y \notin \cup J$ then $A\chi = \chi$. In the latter case we certainly have $IA\chi \le I\chi$ and $I^*\chi \le I^*A\chi$. These hold in the former case as well: Since

$$IA\chi(x) = \begin{cases} \lambda((-\infty, x]), & x \le y, x \notin J \\ \lambda((-\infty, x] \cap J^c) + \lambda((-\infty, x] \cap J)\frac{\lambda((-\infty,y]\cap J)}{\lambda(J)}, & x \in J \\ \lambda((-\infty, y]), & \text{otherwise} \end{cases}$$

we easily see that $IA\chi(x)$ is no greater than $\lambda((-\infty,\min(x,y)]) = I\chi(x)$. Since

$$I^*A\chi(x) = \begin{cases} 0, & x \geq y, x \notin J \\ \lambda([x,\infty)\cap J)\frac{\lambda((-\infty,y]\cap J)}{\lambda(J)}, & x \in J \\ \lambda([x,y]), & \text{otherwise} \end{cases}$$

we easily see that $I^*A\chi(x)$ is no less than $\lambda([x,y])\chi_{(-\infty,y]}(x) = I^*\chi(x)$. Now that we have $IA\chi \leq I\chi$ and $I^*\chi \leq I^*A\chi$ we can apply Corollary 1.3 to see that for any $f \in L^{\downarrow}_{\lambda}$,

$$IAf(y) = \int_{\mathbf{R}} (Af)\chi\, d\lambda = \int_{\mathbf{R}} f(A\chi)\, d\lambda \leq \int_{\mathbf{R}} f\chi\, d\lambda = If(y)$$

and for any $f \in L^{\uparrow}_{\lambda}$,

$$IAf(y) = \int_{\mathbf{R}} (Af)\chi\, d\lambda = \int_{\mathbf{R}} f(A\chi)\, d\lambda \geq \int_{\mathbf{R}} f\chi\, d\lambda = If(y)$$

Since y was arbitrary these complete the proof.

The level function of u depends on the underlying measure λ. To avoid technical difficulties we require that λ satisfy

$$\lambda(-\infty,x] < \infty, \quad x \in \mathbf{R}, \tag{1.2}$$

when working with the level function.

Proposition 1.5. *Suppose that λ satisfies (1.2). To each $u \in L^+_\lambda$ there corresponds a function $u^o \in L^{\downarrow}_\lambda$, called the level function of u with repect to λ such that*

i) $Iu \leq Iu^o$.

ii) If $u_n \uparrow u$ then $u^o_n \uparrow u^o$. That is, if an increasing sequence of functions converges pointwise λ-almost everywhere to u then the sequence of their level functions is increasing and converges pointwise λ-almost everywhere to the level function of u.

iii) If $u \in L^+_\lambda$ is bounded and vanishes on $[M,\infty)$ for some M then there exists an $A_u \in \mathcal{A}$ such that $u^o = A_u u$.

iv) $I^*u^o \leq I^*u$.

Proof. The structure of the level function of u is given in [10, Theorem 4.4, Definition 4.6, Corollary 4.8 and Theorem 4.9]. There it is shown that u^o is non-negative and non-increasing and that (i) holds. Regarding part (iii), we take the intervals of A_u to be the intervals I_i of [10, Definition 4.6 and Corollary 4.8]. It is not assumed in [10] that u is supported on $(-\infty,M]$ so the possibility of an interval of infinite λ-measure is considered there. An easy argument shows that if u is supported on

$(-\infty, M]$ then all the intervals are contained in $(-\infty, M]$ and hence are of finite λ-measure. Clearly we may discard those of zero λ-measure.

As a consequence of [10, Theorem 5.2] the property in part (ii) can be used to extend the level function construction by monotonicity. In [11, Theorem 5.2] it is shown that the extended construction retains the property.

In view of (ii) above and the Monotone Convergence Theorem we observe that it is enough to work with bounded u vanishing on $[M, \infty)$ for some M when proving part (iv). Since in this case we have $u^o = A_u u$ it follows that

$$\int_{\mathbf{R}} u^o \, d\lambda = \int_{\mathbf{R}} u \, d\lambda < \infty.$$

Now part (iv) follows from part (i): For each $x \in \mathbf{R}$,

$$I^* u^o(x) = \int_{\mathbf{R}} u^o \, d\lambda - I u^o(x-) \leq \int_{\mathbf{R}} u \, d\lambda - I u(x-) = I^* u(x).$$

At one point in the sequel we require an extension of the Proposition 1.5(iii) in which the restriction to bounded functions vanishing on $[M, \infty)$ is removed. Because of the technical nature of this result and the delicate argument it requires we defer its statement and proof to Section 6.

2. Monotone Envelopes

For $u \in L^+_\lambda$ we define the monotone envelopes of u as follows: The least non-increasing majorant of u is

$$u^{\downarrow}(x) = \operatorname*{ess\,sup_\lambda}_{t \geq x} u(t)$$

and the greatest non-increasing minorant of u is

$$u_{\downarrow}(x) = \operatorname*{ess\,inf_\lambda}_{t \leq x} u(t).$$

The two non-decreasing envelopes of u are defined analogously,

$$u^{\uparrow}(x) = \operatorname*{ess\,sup_\lambda}_{t \leq x} u(t) \quad \text{and} \quad u_{\uparrow}(x) = \operatorname*{ess\,inf_\lambda}_{t \geq x} u(t).$$

A routine measure theory exercise shows that for λ-almost every $x \in \mathbf{R}$

$$u_{\downarrow}(x) \leq u(x) \leq u^{\downarrow}(x) \quad \text{and} \quad u_{\uparrow}(x) \leq u(x) \leq u^{\uparrow}(x).$$

For f and g in L^+_λ the condition $Ig \leq If$ is a weaker order relation than $g \leq f$. Consequently, the supremum in the obvious identity

$$\sup_{g \leq f} \int_{\mathbf{R}} g u \, d\lambda = \int_{\mathbf{R}} f u \, d\lambda$$

may become larger when the condition $g \leq f$ is weakened to $Ig \leq If$. Our first theorem makes this observation precise.

Theorem 2.1. *Suppose $f, u \in L^+_\lambda$. Then*

$$\sup_{Ig \le If} \int_{\mathbf{R}} gu \, d\lambda = \int_{\mathbf{R}} f u^{\downarrow} \, d\lambda \tag{2.1}$$

and

$$\inf_{Ig \ge If} \int_{\mathbf{R}} gu \, d\lambda = \int_{\mathbf{R}} f u_{\downarrow} \, d\lambda. \tag{2.2}$$

Also

$$\sup_{I^*g \le I^*f} \int_{\mathbf{R}} gu \, d\lambda = \int_{\mathbf{R}} f u^{\uparrow} \, d\lambda \tag{2.3}$$

and

$$\inf_{I^*g \ge I^*f} \int_{\mathbf{R}} gu \, d\lambda = \int_{\mathbf{R}} f u_{\uparrow} \, d\lambda. \tag{2.4}$$

A careful proof is given in Section 5. Here we sketch the essential idea of the proof of (2.1): The function $u^{\downarrow}$ majorizes u and is non-increasing so by Corollary 1.3 we have

$$\int_{\mathbf{R}} gu \, d\lambda \le \int_{\mathbf{R}} gu^{\downarrow} \, d\lambda \le \int f u^{\downarrow} \, d\lambda$$

for each g with $Ig \le If$. Thus

$$\sup_{Ig \le If} \int_{\mathbf{R}} gu \, d\lambda \le \int_{\mathbf{R}} f u^{\downarrow} \, d\lambda.$$

For the other inequality in (2.1) we fix f and construct g_f as follows: A well-behaved function u agrees with $u^{\downarrow}$ except on a collection of intervals where $u^{\downarrow}$ is constant. On each such interval we push the mass of f over until it sits on the right endpoint. The result is a "function" g_f that is zero inside the intervals where $u \ne u^{\downarrow}$. Thus

$$\int_{\mathbf{R}} g_f u \, d\lambda = \int_{\mathbf{R}} g_f u^{\downarrow} \, d\lambda.$$

Also, the mass of f has been pushed to the right to form g_f so for each x

$$I g_f(x) \le If(x).$$

Now we have

$$\sup_{Ig \le If} \int_{\mathbf{R}} gu \, d\lambda \ge \int_{\mathbf{R}} g_f u \, d\lambda = \int_{\mathbf{R}} g_f u^{\downarrow} \, d\lambda = \int_{\mathbf{R}} f u^{\downarrow} \, d\lambda.$$

The last equality holds because the mass of f has been shifted only on intervals where $u^{\downarrow}$ is constant.

Despite the vagueness of "pushing mass" and the strong simplifying assumption on u the basic idea in this sketch survives in the proofs of Section 5.

For the next result we look at another kind of monotone envelope—the level function. We show that the level function u^o of u with respect to λ is a least non-increasing majorant of u when the order $u \le v$ is replaced by $Iu \le Iv$. Surprisingly, when the order $u \le v$ is replaced by $I^*u \le I^*v$ the same level function becomes a greatest non-increasing minorant of u. As suggested by the use of the indefinite article above, it may happen that a function u has more than one least non-increasing majorant when the order is $\le_{\downarrow}$. This is because the order relation lacks antisymmetry for very large functions. If $Iu^o(x) < \infty$ for $x \in \mathbf{R}$ then the level function u^o is the unique least non-increasing majorant of u with order relation $\le_{\downarrow}$. A similar comment applies to the greatest non-increasing minorant of u with order relation $\le_{\uparrow}$.

As we see in Examples 2.4 and 2.6 there need be no greatest non-increasing minorant with respect to the order $\le_{\downarrow}$ nor least non-increasing majorant with respect to the order $\le_{\uparrow}$.

Lemma 2.2. *The level function of u is a least non-increasing majorant of u with respect to the order relation $\le_{\downarrow}$. That is, $Iu \le Iu^o$ and if $v \in L^{\downarrow}_{\lambda}$ with $Iu \le Iv$ then $Iu^o \le Iv$. Also, the level function of u is a greatest non-increasing minorant of u with respect to the order relation $\le_{\uparrow}$. That is, $I^*u^o \le I^*u$ and if $v \in L^{\downarrow}_{\lambda}$ with $I^*v \le I^*u$ then $I^*v \le I^*u^o$.*

Proof. In view of Proposition 1.5(ii) and the Monotone Convergence Theorem it is enough to prove the first part of the lemma for u bounded and vanishing on $[M, \infty)$ for some M. Suppose we have such a u and a v with $v \in L^{\downarrow}_{\lambda}$ and $Iu \le Iv$. By Proposition 1.5(i), $Iu \le Iu^o$ and by Proposition 1.5(iii) we can choose $A_u \in \mathcal{A}$ so that $u^o = A_u u$. Fix $x \in \mathbf{R}$ and let $\chi = \chi_{(-\infty,x]}$. Then

$$Iu^o(x) = \int_{\mathbf{R}} \chi u^o \, d\lambda = \int_{\mathbf{R}} \chi(A_u u) \, d\lambda = \int_{\mathbf{R}} (A_u \chi) u \, d\lambda.$$

Since χ is non-increasing, so is $A_u\chi$. Therefore the hypothesis $Iu \le Iv$ and Corollary 1.3 show that

$$\int_{\mathbf{R}} (A_u \chi) u \, d\lambda \le \int_{\mathbf{R}} (A_u \chi) v \, d\lambda = \int_{\mathbf{R}} \chi(A_u v) \, d\lambda.$$

Proposition 1.4(ii) shows that $IA_u v \le Iv$ and since $\chi \in L^{\downarrow}_{\lambda}$, Corollary 1.3 implies that

$$\int_{\mathbf{R}} \chi(A_u v) \, d\lambda \le \int_{\mathbf{R}} \chi v \, d\lambda = Iv(x).$$

These together yield $Iu^o \le Iv$ as required.

For the second half of the lemma we apply Proposition 1.5(iv) to see that $I^*u^o \le I^*u$. Now suppose that $v \in L^\downarrow_\lambda$ with $I^*v \le I^*u$. Our object is to show that $I^*v \le I^*u^o$. We do this in two cases depending on whether or not u is integrable.

If $\int_{\mathbf{R}} u\,d\lambda < \infty$ then for each positive integer n set $u_n = \min(u,n)\chi_{(-\infty,n]}$ and choose $A_n \in \mathcal{A}$ such that $A_n u_n = u_n^o$. Fix $x \in \mathbf{R}$ and let $\chi = \chi_{[x,\infty)}$. Since $\chi \in L^\uparrow_\lambda$, Proposition 1.4 yields $I\chi \le IA_n\chi$ for all $n > 0$ so Corollary 1.3 shows

$$I^*v(x) = \int_{\mathbf{R}} \chi v\,d\lambda \le \int_{\mathbf{R}} (A_n\chi)v\,d\lambda$$

because v is non-increasing. Now $A_n\chi$ is non-decreasing and $I^*v \le I^*u$ so Corollary 1.3 shows that

$$\int_{\mathbf{R}} (A_n\chi)v\,d\lambda \le \int_{\mathbf{R}} (A_n\chi)u\,d\lambda = \int_{\mathbf{R}} (A_n\chi)(u-u_n)\,d\lambda + \int_{\mathbf{R}} (A_n\chi)u_n\,d\lambda.$$

Notice that for any n, $A_n\chi \le 1$. Since $\int_{\mathbf{R}} u\,d\lambda < \infty$ and $u - u_n$ tends to zero pointwise, the Dominated Convergence Theorem shows that

$$\int_{\mathbf{R}} (A_n\chi)(u-u_n)\,d\lambda \to 0 \text{ as } n \to \infty.$$

Proposition 1.5(ii) and the Monotone Convergence Theorem show that as $n \to \infty$,

$$\int_{\mathbf{R}} (A_n\chi)u_n\,d\lambda = \int_{\mathbf{R}} \chi(A_n u_n)\,d\lambda = \int_{\mathbf{R}} \chi u_n^o\,d\lambda \to \int_{\mathbf{R}} \chi u^o\,d\lambda = I^*u^o(x).$$

Putting these together yields $I^*v(x) \le I^*u^o(x)$ and since x was arbitrary this completes the proof of the case $\int_{\mathbf{R}} u\,d\lambda < \infty$.

To handle the case $\int_{\mathbf{R}} u\,d\lambda = \infty$ we first observe that by the Monotone Convergence Theorem it is enough to prove the result for v satisfying $\int_{\mathbf{R}} v\,d\lambda < \infty$. Next we need the following fact which depends on the σ-finiteness of λ: If $\int_{\mathbf{R}} v\,d\lambda < \infty$ and $\int_{\mathbf{R}} u\,d\lambda = \infty$ with $I^*v \le I^*u$ then there exists a $w \in L^+_\lambda$ with $w \le u$ such that $\int_{\mathbf{R}} w\,d\lambda < \infty$ and $I^*v \le I^*w$. The construction of such a w, an easy exercise in measure theory, is left to the reader. Having w we use the previous case to get

$$I^*v \le I^*w^o \le I^*u^o.$$

Note that since $w \le u$ it is a consequence of Proposition 1.5(ii) that $w^o \le u^o$.

Now we present an analogue of (2.1) with f and g restricted to be non-increasing. The level function appears here in its role of least non-increasing majorant of u with respect to the order $\le_\downarrow$.

Theorem 2.3. *Suppose λ satisfies (1.2) and $u \in L^+_\lambda$. If $f \in L^\downarrow_\lambda$ then*

$$\sup_{\substack{g\in L^\downarrow_\lambda\\ Ig\le If}} \int_{\mathbf{R}} gu\,d\lambda = \int_{\mathbf{R}} fu^o\,d\lambda. \tag{2.5}$$

Proof. By Proposition 1.5(ii) and the Monotone Convergence Theorem it is enough to prove the theorem assuming that u is bounded and vanishes on $[M,\infty)$ for some M. If $g \in L^\downarrow_\lambda$ and $Ig \le If$ then by Proposition 1.5(i) and Corollary 1.3 applied twice we have

$$\int_{\mathbf{R}} gu\,d\lambda \le \int_{\mathbf{R}} gu^o\,d\lambda \le \int_{\mathbf{R}} fu^o\,d\lambda$$

since u^o is non-increasing. This proves the inequality "$\le$" of (2.5).

For the reverse inequality apply Proposition 1.5(iii) to choose $A_u \in \mathcal{A}$ such that $A_u u = u^o$. Then Proposition 1.4(ii) shows that $A_u f \in L^\downarrow_\lambda$ and $IA_u f \le If$. Thus

$$\sup_{\substack{g\in L^\downarrow_\lambda\\ Ig\le If}} \int_{\mathbf{R}} gu\,d\lambda \ge \int_{\mathbf{R}} (A_u f)u\,d\lambda = \int_{\mathbf{R}} f(A_u u)\,d\lambda = \int_{\mathbf{R}} fu^o\,d\lambda.$$

This completes the proof.

Now that we have the analogue (2.5) of (2.1) it is natural to ask if there is an analogue of (2.2) with f and g restricted to be non-increasing. Suprisingly, the answer is no. The following example shows that no direct analogue is possible.

Example 2.4. *Let λ be Lebesgue measure on $(0,3)$, that is, $d\lambda(x) = \chi_{(0,3)}(x)\,dx$, and set $u = 3\chi_{(0,1)} + \chi_{(2,3)}$. Then there is no function u_o which satisfies*

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\ge If}} \int_{\mathbf{R}} gu\,d\lambda = \int_{\mathbf{R}} fu_o\,d\lambda \tag{2.6}$$

for all $f \in L^\downarrow_\lambda$.

Proof. For each $s \in (0,4)$ set

$$f_s = 4\chi_{(0,1]} + s\chi_{(1,3)}.$$

Observe that

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\ge If_s}} \int_{\mathbf{R}} gu\,d\lambda = \inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\ge If_s}} 3\int_0^1 g + \int_2^3 g \ge \inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\ge If_s}} 3\int_0^1 g \ge 3\int_0^1 f_s = 12.$$

Also, since $\int_0^1 g - \int_1^2 g \geq 0$ for $g \in L^\downarrow_\lambda$, we have

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\geq If_s}} \int_{\mathbf{R}} gu\,d\lambda = \inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\geq If_s}} \int_0^1 g + \left(\int_0^1 g - \int_1^2 g\right) + \int_0^3 g \geq \int_0^1 f_s + \int_0^3 f_s = 8+2s$$

These two observations yield

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\geq If_s}} \int_{\mathbf{R}} gu\,d\lambda \geq \max(12, 8+2s). \tag{2.7}$$

In fact this inequality is equality, as we show in two cases. If $0 < s \leq 2$ then we set $g_s = 4\chi_{(0,1]} + 2s\chi_{(1,2)}$, note that $g_s \in L^\downarrow_\lambda$ and $Ig_s \geq If_s$, and conclude that

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\geq If_s}} \int_{\mathbf{R}} gu\,d\lambda \leq \int_{\mathbf{R}} g_s u\,d\lambda = 12.$$

If $2 \leq s \leq 4$ then we set $g_s = 4\chi_{(0,2]} + (2s-4)\chi_{(2,3)}$, again note that $g_s \in L^\downarrow_\lambda$ and $Ig_s \geq If_s$, and conclude that

$$\inf_{\substack{g\in L^\downarrow_\lambda\\ Ig\geq If_s}} \int_{\mathbf{R}} gu\,d\lambda \leq \int_{\mathbf{R}} g_s u\,d\lambda = 8+2s.$$

Equality in (2.7) turns (2.6), with f replaced by f_s, into

$$\max(12, 8+2s) = \int_{\mathbf{R}} f_s u_o\,d\lambda = 4\int_0^1 u_o + s\int_1^3 u_o.$$

It is clear that this cannot hold for all $s \in (0,4)$ no matter what the function u_o may be as the right hand side has constant slope while the left hand side does not.

A similar situation occurs when we consider restricting (2.1) and (2.2) to non-decreasing functions f and g. The level function provides an analogue of (2.2) but there is no analogue of (2.1).

Theorem 2.5. *Suppose λ satisfies* (1.2) *and $u \in L^+_\lambda$. If $f \in L^\uparrow_\lambda$ then*

$$\inf_{\substack{g\in L^\uparrow_\lambda\\ Ig\geq If}} \int_{\mathbf{R}} gu\,d\lambda = \int_{\mathbf{R}} fu^o\,d\lambda. \tag{2.8}$$

Proof. If $g \in L^\uparrow_\lambda$ and $Ig \geq If$ then by Proposition 1.5(iv) and Corollary 1.3 applied twice we have

$$\int_{\mathbf{R}} gu\,d\lambda \geq \int_{\mathbf{R}} gu^o\,d\lambda \geq \int_{\mathbf{R}} fu^o\,d\lambda$$

because u^o is non-increasing. This proves the inequality "$\geq$" of (2.8).

For the reverse inequality we apply Theorem 6.1 to u to obtain intervals J_{left} and J_{right} and an operator $A \in \mathcal{A}$ such that $u^o = Au$ off $J_{\text{left}} \cup J_{\text{right}}$. Note that all the intervals of A are contained in $\mathbf{R} \setminus (J_{\text{left}} \cup J_{\text{right}})$. Define g by $g = Af$ on $\mathbf{R} \setminus J_{\text{right}}$ and

$$g = \lim_{x\to\infty} \frac{1}{\lambda((-\infty,x] \cap J_{\text{right}})} \int_{(-\infty,x]\cap J_{\text{right}}} f\,d\lambda$$

on J_{right}. Because f is non-decreasing the limit is non-decreasing and therefore exists. It is easy to check that g is non-decreasing as well. By Proposition 1.4(iii) we see that $Ig \geq If$ on $\mathbf{R} \setminus J_{\text{right}}$ and for $x \in J_{\text{right}}$

$$\begin{aligned} Ig(x) = \int_{(-\infty,x]\setminus J_{\text{right}}} g\,d\lambda + \int_{(-\infty,x]\cap J_{\text{right}}} g\,d\lambda \\ \geq \int_{(-\infty,x]\setminus J_{\text{right}}} f\,d\lambda + \int_{(-\infty,x]\cap J_{\text{right}}} f\,d\lambda = If(x) \end{aligned}$$

as well. To complete the proof we show that

$$\int_{\mathbf{R}} gu\,d\lambda = \int_{\mathbf{R}} fu^o\,d\lambda.$$

Proposition 1.4(i) implies

$$\int_{\mathbf{R}\setminus J_{\text{right}}} gu\,d\lambda = \int_{\mathbf{R}\setminus J_{\text{right}}} fu^o\,d\lambda$$

so we need only show that

$$\int_{J_{\text{right}}} gu\,d\lambda = \int_{J_{\text{right}}} fu^o\,d\lambda.$$

That is, by Theorem 6.1(iii), that

$$\begin{aligned} \left(\lim_{x\to\infty} \frac{1}{\lambda((-\infty,x] \cap J_{\text{right}})} \int_{(-\infty,x]\cap J_{\text{right}}} f\,d\lambda\right) \int_{J_{\text{right}}} u\,d\lambda \\ = \int_{J_{\text{right}}} f\,d\lambda \left(\limsup_{x\to\infty} \frac{1}{\lambda((-\infty,x] \cap J_{\text{right}})} \int_{(-\infty,x]\cap J_{\text{right}}} u\,d\lambda\right). \end{aligned}$$

It is not difficult to recognise both sides as

$$\limsup_{x\to\infty} \frac{1}{\lambda((-\infty,x] \cap J_{\text{right}})} \int_{(-\infty,x]\cap J_{\text{right}}} u\,d\lambda \int_{(-\infty,x]\cap J_{\text{right}}} f\,d\lambda$$

to complete the proof.

Example 2.6. *Let λ be Lebesgue measure on $(0,3)$, that is, $d\lambda(x) = \chi_{(0,3)}(x)\,dx$, and set $u = 3\chi_{(1,2)} + \chi_{(2,3)}$. Then there is no function u_o which satisfies*

$$\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If}} \int_{\mathbf{R}} gu\,d\lambda = \int_{\mathbf{R}} fu_o\,d\lambda \tag{2.9}$$

for all $f \in L^{\uparrow}_{\lambda}$.

Proof. For each $s \in (0,4)$ set

$$f_s = s\chi_{(0,2]} + 4\chi_{(2,3)}.$$

Then

$$\begin{aligned}\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} \int_{\mathbf{R}} gu\,d\lambda &= \sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} 3\int_1^2 g + \int_2^3 g\\ &\le \sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} 2\int_0^2 g + \int_0^3 g \le 2\int_0^2 f_s + \int_0^3 f_s = 6s+4.\end{aligned}$$

Also, since $\int_1^2 g \le \int_2^3 g$ for $g \in L^{\uparrow}_{\lambda}$, we have

$$\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} \int_{\mathbf{R}} gu\,d\lambda \le \sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} 2\int_1^2 g + 2\int_2^3 g \le 2\int_0^3 g \le 2\int_0^3 f_s = 4s+8$$

These two observations yield

$$\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} \int_{\mathbf{R}} gu\,d\lambda \le \min(6s+4, 4s+8). \tag{2.10}$$

We demonstrate that this inequality is equality. If $0 < s \le 2$ then $g_s = 2s\chi_{(1,2]} + 4\chi_{(2,3)}$ is in $L^{\uparrow}_{\lambda}$ and it is easy to check that $Ig_s \le If_s$. Therefore,

$$\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} \int_{\mathbf{R}} gu\,d\lambda \ge \int_{\mathbf{R}} g_s u\,d\lambda = 6s+4.$$

If $2 \le s \le 4$ then we set $g_s = (s+2)\chi_{(1,3]}$. Again $g_s \in L^{\uparrow}_{\lambda}$ and $Ig_s \le If_s$. We have

$$\sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\le If_s}} \int_{\mathbf{R}} gu\,d\lambda \ge \int_{\mathbf{R}} g_s u\,d\lambda = 4s+8.$$

Equality in (2.10) turns (2.9), with f replaced by f_s, into

$$\min(6s+4, 4s+8) = \int_{\mathbf{R}} f_s u_o \, d\lambda = s\int_0^2 u_o + 4\int_2^3 u_o.$$

It is clear that this cannot hold for all $s \in (0,4)$ no matter what the function u_o may be as the right hand side has constant slope while the left hand side does not.

The non-decreasing analogue of the level function is obtained by simply flipping the real line end for end. Thus if $\lambda[x,\infty) < \infty$ for $x \in \mathbf{R}$ and $u \in L^+_\lambda$ then we set $\lambda_1(x) = \lambda(-x)$, $u_1(x) = u(-x)$, and let u_1^o be the level function of u_1 with respect to λ_1. The non-decreasing level function of u with respect to λ is then $u_1^o(-x)$. This construction simultaneously yields the least non-decreasing majorant of u with respect to the order $\leq_\uparrow$ and the greatest non-decreasing minorant of u with respect to the order $\leq_\downarrow$. Lemma 2.2 and Theorems 2.3 and 2.5 have obvious counterparts for the non-decreasing level function that we leave to the reader.

3. Transferring Monotonicity

In [13] the notion of transferring monotonicity from the kernel of an operator to the weight was introduced to study a special case of the weighted Hardy inequality. The results of the previous section allow us to better express the ideas behind that notion and place them in a more general setting. A result related to Theorem 3.5 may be found in [3, Proposition 2.12].

As before, the measure λ is a σ-finite measure on $(-\infty,\infty)$ for which non-increasing functions are λ-measurable. Let μ be any measure on any set and let X be a Banach Function Space of μ-measurable functions. Define the linear operator K by

$$Kf(x) = \int_{\mathbf{R}} k(x,t) f(t)\, d\lambda(t)$$

where the kernel $k(x,t)$ is a non-negative $(\mu\times\lambda)$-measurable function.

Theorem 3.1. *Suppose $k(x,t)$ is non-increasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\|Kf\|_X \leq C\int_{\mathbf{R}} fu\, d\lambda, \quad f\in L^+_\lambda,$$

holds is unchanged when u is replaced by $u_\downarrow$. That is,

$$\sup_{f\geq 0}\frac{\|Kf\|_X}{\int_{\mathbf{R}} fu\, d\lambda} = \sup_{f\geq 0}\frac{\|Kf\|_X}{\int_{\mathbf{R}} fu_\downarrow\, d\lambda}. \tag{3.1}$$

Proof. Since $u_\downarrow \leq u$ λ-almost everywhere the inequality "$\leq$" in (3.1) is immediate. To establish the reverse inequality we apply (2.2) of Theorem 2.1 to get

$$\sup_{f\geq 0}\frac{\|Kf\|_X}{\int_{\mathbf{R}} fu_\downarrow\, d\lambda} = \sup_{f\geq 0}\frac{\|Kf\|_X}{\inf_{Ig\geq If}\int_{\mathbf{R}} gu\, d\lambda} = \sup_{f\geq 0}\sup_{Ig\geq If}\frac{\|Kf\|_X}{\int_{\mathbf{R}} gu\, d\lambda}.$$

Now if $If \le Ig$ then the monotonicity of k and Corollary 1.3 shows that $Kf \le Kg$ and since X is a Banach Function Space we have $\|Kf\|_X \le \|Kg\|_X$. Thus

$$\sup_{f\ge 0} \frac{\|Kf\|_X}{\int_{\mathbf{R}} f u_\downarrow \, d\lambda} \le \sup_{f\ge 0} \sup_{Ig \ge If} \frac{\|Kg\|_X}{\int_{\mathbf{R}} g u \, d\lambda} \le \sup_{g\ge 0} \frac{\|Kg\|_X}{\int_{\mathbf{R}} g u \, d\lambda}.$$

This completes the proof.

The substitution $x \to -x$ gives the corresponding result for non-decreasing kernels.

Corollary 3.2. *Suppose $k(x,t)$ is non-decreasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\|Kf\|_X \le C \int_{\mathbf{R}} f u \, d\lambda, \quad f \in L^+_\lambda,$$

holds is unchanged when u is replaced by $u_\uparrow$. That is,

$$\sup_{f\ge 0} \frac{\|Kf\|_X}{\int_{\mathbf{R}} f u \, d\lambda} = \sup_{f\ge 0} \frac{\|Kf\|_X}{\int_{\mathbf{R}} f u_\uparrow \, d\lambda}.$$

Next we look at the reversed inequality.

Theorem 3.3. *Suppose $k(x,t)$ is non-increasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\int_{\mathbf{R}} f u \, d\lambda \le C \|Kf\|_X, \quad f \in L^+_\lambda,$$

holds is unchanged when u is replaced by $u^\downarrow$. That is,

$$\sup_{f\ge 0} \frac{\int_{\mathbf{R}} f u \, d\lambda}{\|Kf\|_X} = \sup_{f\ge 0} \frac{\int_{\mathbf{R}} f u^\downarrow \, d\lambda}{\|Kf\|_X}. \tag{3.2}$$

Proof. Since $u \le u^\downarrow$ λ-almost everywhere the inequality "$\le$" in (3.2) is clear. For the reverse inequality we apply (2.1) of Theorem 2.1 to get

$$\sup_{f\ge 0} \frac{\int_{\mathbf{R}} f u^\downarrow \, d\lambda}{\|Kf\|_X} = \sup_{f\ge 0} \frac{\sup_{Ig\le If} \int_{\mathbf{R}} g u \, d\lambda}{\|Kf\|_X} = \sup_{f\ge 0} \sup_{Ig \le If} \frac{\int_{\mathbf{R}} g u \, d\lambda}{\|Kf\|_X}.$$

Now if $Ig \le If$ then the monotonicity of k and Corollary 1.3 shows that $Kg \le Kf$ and since X is a Banach Function Space we have $\|Kg\|_X \le \|Kf\|_X$. Thus

$$\sup_{f\ge 0} \frac{\int_{\mathbf{R}} f u^\downarrow \, d\lambda}{\|Kf\|_X} \le \sup_{f\ge 0} \sup_{Ig \le If} \frac{\int_{\mathbf{R}} g u \, d\lambda}{\|Kg\|_X} \le \sup_{g\ge 0} \frac{\int_{\mathbf{R}} g u \, d\lambda}{\|Kg\|_X}.$$

This completes the proof.

Corollary 3.4. *Suppose $k(x,t)$ is non-decreasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\int_{\mathbf{R}} fu\,d\lambda \le C\|Kf\|_X, \quad f \in L^+_\lambda,$$

holds is unchanged when u is replaced by $u^\uparrow$. That is,

$$\sup_{f\ge 0} \frac{\int_{\mathbf{R}} fu\,d\lambda}{\|Kf\|_X} = \sup_{f\ge 0} \frac{\int_{\mathbf{R}} fu^\uparrow\,d\lambda}{\|Kf\|_X}.$$

We can also transfer monotonicity in weighted norm inequalities restricted to monotone functions.

Theorem 3.5. *Suppose $k(x,t)$ is non-increasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\int_{\mathbf{R}} fu\,d\lambda \le C\|Kf\|_X, \quad f \in L^\downarrow_\lambda,$$

holds is unchanged when u is replaced by u^o. That is,

$$\sup_{f\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} fu\,d\lambda}{\|Kf\|_X} = \sup_{f\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} fu^o\,d\lambda}{\|Kf\|_X}. \tag{3.3}$$

Proof. Since $Iu \le Iu^o$, Corollary 1.3 yields the inequality "$\le$" in (3.3). To establish the reverse inequality we apply (2.5) of Theorem 2.3 to get

$$\sup_{f\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} fu^o\,d\lambda}{\|Kf\|_X} = \sup_{f\in L^\downarrow_\lambda} \frac{\sup_{\substack{g\in L^\downarrow_\lambda\\ Ig\le If}} \int_{\mathbf{R}} gu\,d\lambda}{\|Kf\|_X} = \sup_{f\in L^\downarrow_\lambda} \sup_{\substack{g\in L^\downarrow_\lambda\\ Ig\le If}} \frac{\int_{\mathbf{R}} gu\,d\lambda}{\|Kf\|_X}.$$

Now if $Ig \le If$ then the monotonicity of k and Corollary 1.3 shows that $Kg \le Kf$ and since X is a Banach Function Space we have $\|Kg\|_X \le \|Kf\|_X$. Thus

$$\sup_{f\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} fu^o\,d\lambda}{\|Kf\|_X} \le \sup_{f\in L^\downarrow_\lambda} \sup_{\substack{g\in L^\downarrow_\lambda\\ Ig\le If}} \frac{\int_{\mathbf{R}} gu\,d\lambda}{\|Kg\|_X} \le \sup_{g\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} gu\,d\lambda}{\|Kg\|_X}.$$

This completes the proof.

Theorem 3.6. *Suppose $k(x,t)$ is non-increasing in t for each x. Then the least constant C, finite or infinite, for which*

$$\|Kf\|_X \le C \int_{\mathbf{R}} fu\,d\lambda, \quad f \in L^{\uparrow}_{\lambda},$$

holds is unchanged when u is replaced by u^o. That is,

$$\sup_{f\in L^{\uparrow}_{\lambda}} \frac{\|Kf\|_X}{\int_{\mathbf{R}} fu\,d\lambda} = \sup_{f\in L^{\uparrow}_{\lambda}} \frac{\|Kf\|_X}{\int_{\mathbf{R}} fu^o\,d\lambda}. \tag{3.4}$$

Proof. Since $I^*u \ge I^*u^o$, Corollary 1.3 yields the inequality "$\le$" in (3.4). To establish the reverse inequality we apply (2.7) of Theorem 2.5 to get

$$\sup_{f\in L^{\uparrow}_{\lambda}} \frac{\|Kf\|_X}{\int_{\mathbf{R}} fu^o\,d\lambda} = \sup_{f\in L^{\uparrow}_{\lambda}} \frac{\|Kf\|_X}{\inf_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\ge If}} \int_{\mathbf{R}} gu\,d\lambda} = \sup_{f\in L^{\uparrow}_{\lambda}} \sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\ge If}} \frac{\|Kf\|_X}{\int_{\mathbf{R}} gu\,d\lambda}.$$

Now if $If \le Ig$ then the monotonicity of k and Corollary 1.3 shows that $Kf \le Kg$ and since X is a Banach Function Space we have $\|Kf\|_X \le \|Kg\|_X$. Thus

$$\sup_{f\in L^{\uparrow}_{\lambda}} \frac{\|Kf\|_X}{\int_{\mathbf{R}} fu^o\,d\lambda} \le \sup_{f\in L^{\uparrow}_{\lambda}} \sup_{\substack{g\in L^{\uparrow}_{\lambda}\\ Ig\ge If}} \frac{\|Kg\|_X}{\int_{\mathbf{R}} gu\,d\lambda} \le \sup_{g\in L^{\uparrow}_{\lambda}} \frac{\|Kg\|_X}{\int_{\mathbf{R}} gu\,d\lambda}.$$

This completes the proof.

The results for non-decreasing kernels corresponding to Theorems 3.5 and 3.6 involve the non-decreasing level function. (See the remark at the end of Section 2.) We leave their formulation to the reader.

As an application of these results we present a companion result to [11, Theorem 4.4] and then a special case related to [9, Theorem 1]. Let $\Lambda = I1$ so that $\Lambda(x) = \lambda((-\infty, x])$. Define the operator P by

$$Pf = (If)/\Lambda + If(\infty)/\Lambda(\infty).$$

Note that the second term in the definition of P is absent if λ is an infinite measure. When working in rearrangement invariant spaces it is natural to assume that the underlying measure is either non-atomic or purely atomic with all atoms having equal measure. This ensures, among other things, that the associate space of a rearrangement invariant space is again rearrangement invariant. Under this assumption [11, Theorem 4.4] shows that

$$\sup_{g\in L^{\downarrow}_{\lambda}} \frac{\int_{\mathbf{R}} fg\,d\lambda}{\|g\|_{X'}} \approx \|Pf\|_X$$

provided $P : X \to X$ is bounded. Lemma 1.2 shows that we can take the supremum over $g = I^*G \in I^*L^+_\lambda$ rather than $g \in L^\downarrow_\lambda$ so we can evaluate

$$\sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} uG\,d\lambda}{\|I^*G\|_{X'}}$$

where $u = If$. Here u is non-decreasing but by transferring monotonicity we can evaluate the above supremum for arbitrary $u \in L^+_\lambda$.

The boundedness of $P : X \to X$ is equivalent to the upper Boyd index of X being less than 1. For more information about Boyd indices see [2, 7].

Theorem 3.7. *Let λ be a σ-finite measure on $\mathbf{R}$ that is either non-atomic or purely atomic with all atoms having equal measure. Suppose that X is a rearrangement invariant Banach function space of λ-measurable functions. If $P : X \to X$ is bounded then*

$$\sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} uG\,d\lambda}{\|I^*G\|_{X'}} \approx \|u^\uparrow/\Lambda\|_X + u^\uparrow(\infty)\|1\|_X/\Lambda(\infty).$$

Here 1 *represents the constant function with value* 1*.*

Note that if $\lambda(\mathbf{R}) = \infty$ the second term on the right hand side is absent.

Proof. The kernel of I^* is $\chi_{[x,\infty)}(t)$ which is non-decreasing in t for each x. We may apply Corollary 3.4 to get

$$\sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} uG\,d\lambda}{\|I^*G\|_{X'}} = \sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} u^\uparrow G\,d\lambda}{\|I^*G\|_{X'}}.$$

Since $u^\uparrow$ is non-decreasing it can be approximated from below by integrals. Thus

$$\sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} u^\uparrow G\,d\lambda}{\|I^*G\|_{X'}} = \sup_{G\in L^+_\lambda} \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \frac{\int_{\mathbf{R}} (If)G\,d\lambda}{\|I^*G\|_{X'}} = \sup_{G\in L^+_\lambda} \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \frac{\int_{\mathbf{R}} f(I^*G)\,d\lambda}{\|I^*G\|_{X'}}.$$

Using the fact that a non-increasing function can be approximated from below by integrals we have

$$\sup_{G\in L^+_\lambda} \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \frac{\int_{\mathbf{R}} f(I^*G)\,d\lambda}{\|I^*G\|_{X'}} = \sup_{g\in L^\downarrow_\lambda} \sup_{\substack{G\in L^+_\lambda\\ I^*G\le g}} \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \frac{\int_{\mathbf{R}} f(I^*G)\,d\lambda}{\|I^*G\|_{X'}} = \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \sup_{g\in L^\downarrow_\lambda} \frac{\int_{\mathbf{R}} fg\,d\lambda}{\|g\|_{X'}}.$$

The inner supremum above is equivalent to $\|Pf\|_X$ by [11, Theorem 4.4] so we have

$$\begin{aligned}\sup_{G\in L^+_\lambda} \frac{\int_{\mathbf{R}} u^\uparrow G\,d\lambda}{\|I^*G\|_{X'}} \approx \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \|Pf\|_X &\approx \sup_{\substack{f\in L^+_\lambda\\ If\le u^\uparrow}} \|If/\Lambda\|_X + If(\infty)\|1\|_X/\Lambda(\infty)\\ &= \|u^\uparrow/\Lambda\|_X + u^\uparrow(\infty)\|1\|_X/\Lambda(\infty).\end{aligned}$$

This completes the proof.

Corollary 3.8. *Suppose that $1 < p < \infty$, $1/p + 1/p' = 1$, and v is a non-negative, Lebesgue measurable function defined on $(0, \infty)$ which is finite almost everywhere. Then*

$$\sup_{g \in L_\lambda^+} \frac{\int_0^\infty ug}{\left(\int_0^\infty \left(\int_x^\infty g\right)^{p'} v(x)\,dx\right)^{1/p'}} \tag{3.5}$$
$$\approx \left(\int_0^\infty u^\uparrow(x)^p \left(\int_0^x v\right)^{-p} v(x)\,dx\right)^{1/p} + u^\uparrow(\infty)\left(\int_0^\infty v\right)^{-1/p'}.$$

Proof. Since v is finite almost everywhere, the measure λ defined by

$$d\lambda(x) = v(x)\chi_{(0,\infty)}(x)\,dx$$

is σ-finite and non-atomic. With respect to this underlying measure the weighted Lebesgue space L_v^p having norm

$$\|f\|_{L_v^p} = \left(\int_0^\infty |f|^p v\right)^{1/p}$$

is rearrangement invariant and its associate space is $L_v^{p'}$. Moreover, since $1 < p < \infty$, the upper Boyd index of L_v^p is 1/p which is less than 1. Therefore the conclusion of Theorem 3.5 holds. We have

$$\sup_{G \in L_\lambda^+} \frac{\int_0^\infty uGv}{\left(\int_0^\infty \left(\int_x^\infty Gv\right)^{p'} v(x)\,dx\right)^{1/p'}}$$
$$\approx \left(\int_0^\infty u^\uparrow(x)^p \left(\int_0^x v\right)^{-p} v(x)\,dx\right)^{1/p} + u^\uparrow(\infty)\left(\int_0^\infty v\right)^{1/p}\left(\int_0^\infty v\right)^{-1}.$$

With $g = Gv$ this reduces to (3.5) to complete the proof.

4. Quasi-Concave Functions

We call a function $h : (0, \infty) \to [0, \infty)$ *quasi-concave* and write $h \in \Omega_{0,1}$, provided $h(t)$ is non-decreasing and $t^{-1}h(t)$ is non-increasing. The two quasi-concave envelopes of a Lebesgue measurable function $u : (0, \infty) \to [0, \infty)$ are the least quasi-concave majorant of u, given by

$$\bar{u}(x) = x \operatorname*{ess\,sup}_{t \geq x} t^{-1} \operatorname*{ess\,sup}_{0 \leq s \leq t} u(s),$$

and the greatest quasi-concave minorant of u, given by

$$\underline{u}(x) = x \operatorname*{ess\,inf}_{0 \leq t \leq x} t^{-1} \operatorname*{ess\,inf}_{s \geq t} u(s).$$

It is easy to check that $\underline{u} \leq u \leq \bar{u}$, that $\bar{u}$ and $\underline{u}$ are both quasi-concave, and that they are envelopes in the following sense: If $u \leq h$ and h is quasi-concave then $\bar{u} \leq h$. Also, if $h \leq u$ and h is quasi-concave then $h \leq \underline{u}$.

In this section we take the measure λ to be Lebesgue measure on $(0, \infty)$ and recall the definitions of $u^\downarrow$, $u_\downarrow$, $u^\uparrow$, and $u_\uparrow$ given in Section 2.

Theorem 4.1. *Suppose that for each x, $k(x,t)$ is a quasi-concave function of t. Then the least constant C, finite or infinite, for which*

$$\int_0^\infty fu \le C\|Kf\|_X, \quad f \ge 0,$$

is unchanged when u is replaced by $\bar{u}$. That is,

$$\sup_{f\ge 0} \frac{\int_0^\infty fu}{\|Kf\|_X} = \sup_{f\ge 0} \frac{\int_0^\infty f\bar{u}}{\|Kf\|_X}.$$

Proof. Since $k(x,t)$ is non-decreasing in t we may apply Corollary 3.4 to get

$$\sup_{f\ge 0} \frac{\int_0^\infty fu}{\|Kf\|_X} = \sup_{f\ge 0} \frac{\int_0^\infty fu^\uparrow}{\|Kf\|_X}.$$

Let $l(x,t) = t^{-1}k(x,t)$ and define the operator L by

$$Lg(x) = \int_0^\infty l(x,t)g(t)\,dt.$$

Note that if we set $g(t) = tf(t)$ then $Kf = Lg$. Now $l(x,t)$ is non-increasing in t so by Theorem 3.3 we have

$$\sup_{f\ge 0} \frac{\int_0^\infty fu^\uparrow}{\|Kf\|_X} = \sup_{g\ge 0} \frac{\int_0^\infty gw}{\|Lg\|_X} = \sup_{g\ge 0} \frac{\int_0^\infty gw^\downarrow}{\|Lg\|_X}$$

where $w(t) = t^{-1}u^\uparrow(t)$. The definition of $\bar{u}$ shows that $xw^\downarrow(x) = \bar{u}(x)$. Therefore

$$\sup_{g\ge 0} \frac{\int_0^\infty gw^\downarrow}{\|Lg\|_X} = \sup_{f\ge 0} \frac{\int_0^\infty f\bar{u}}{\|Kf\|_X}$$

which completes the proof.

In just the same way the next theorem follows from Corollary 3.2 and Theorem 3.1. We omit the details.

Theorem 4.2. *Suppose that for each x, $k(x,t)$ is a quasi-concave function of t. Then the least constant C, finite or infinite, for which*

$$\|Kf\|_X \le C\int_0^\infty fu, \quad f \ge 0,$$

is unchanged when u is replaced by $\underline{u}$. That is,

$$\sup_{f\ge 0} \frac{\|Kf\|_X}{\int_0^\infty fu} = \sup_{f\ge 0} \frac{\|Kf\|_X}{\int_0^\infty f\underline{u}}.$$

With our choice of λ the operators I and I^* become

$$If(x) = \int_0^x f \quad \text{and} \quad I^*f(x) = \int_x^\infty f.$$

The composition II^* maps the cone of non-negative functions to the cone of quasi-concave functions. It is well known, see for example [12, Lemma 2.3], that every quasi-concave function is equivalent to an increasing limit of functions in the image $II^*(L^+_\lambda)$. Work of [5, 6, 12] has characterized weighted Lebesgue space imbeddings of the cone of quasi-concave functions. In the next theorem we apply a special case of [5, Theorem 5.1(ii)].

Theorem 4.3. *Suppose* $1 < p < \infty$, $1/p + 1/p' = 1$, *and* $f \in L^+_\lambda$. *Then*

$$\sup_{f\ge 0} \frac{\int_0^\infty fg}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}} \approx \|\bar{g}\|_{L^{p'}_\sigma}$$

where

$$\sigma(x) = \frac{1}{x}\left(\int_0^x t^p v(t)\,dt\right)\left(x^p\int_x^\infty v(t)\,dt\right)\left(\int_0^x t^p v(t)\,dt + x^p\int_x^\infty v(t)\,dt\right)^{-p'-1}.$$

Proof. Interchanging the order of integration in the composition II^* yields

$$II^*f(x) = \int_0^x \int_s^\infty f(t)\,dt\,ds = \int_0^\infty \min(x,t) f(t)\,dt.$$

The kernel, $\min(x,t)$, is a quasi-concave function of t for each x so by Theorem 4.1 we have

$$\sup_{f\ge 0} \frac{\int_0^\infty fg}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}} = \sup_{f\ge 0} \frac{\int_0^\infty f\bar{g}}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}}.$$

The quasi-concave function $\bar{g}$ is equivalent to an increasing limit of functions in $II^*(L^+_\lambda)$ so

$$\sup_{f\ge 0} \frac{\int_0^\infty f\bar{g}}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}} \approx \sup_{f\ge 0} \sup_{\substack{h\ge 0\\ II^*h\le \bar{g}}} \frac{\int_0^\infty f(II^*h)}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}}$$
$$= \sup_{\substack{h\ge 0\\ II^*h\le \bar{g}}} \sup_{f\ge 0} \frac{\int_0^\infty (II^*f)h}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}}.$$

Is is easy to check that

$$II^*(L^+_\lambda) \subset \{tf^{**}(t) : f \in L^+_\lambda\} \subset \Omega_{0,1}$$

where f^* is the non-increasing rearrangement of f and $f^{**}(t) = \frac{1}{t}\int_0^t f^*$. So for any $h \in L^+_\lambda$ we have

$$\sup_{f\ge 0} \frac{\int_0^\infty (II^*f)h}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}} = \sup_{f\ge 0} \frac{\int_0^\infty (tf^{**}(t))\, h(t)\, dt}{\left(\int_0^\infty (tf^{**}(t))^p\, v(t)\, dt\right)^{1/p}}.$$

By [5, Theorem 5.1(ii)], with $q = 1$, $u \equiv 1$, $w(s) = sh(s)$ and $v(s)$ replaced by $s^p v(s)$, the last expression is equivalent to

$$\left(\int_0^\infty \left(\int_0^\infty \min(1, x/t) th(t)\, dt\right)^{p'} \sigma(x)\, dx\right)^{1/p'}.$$

This reduces to $\|II^*h\|_{L^{p'}_\sigma}$ so we have

$$\sup_{f\ge 0} \frac{\int_0^\infty fg}{\left(\int_0^\infty (II^*f)^p v\right)^{1/p}} \approx \sup_{\substack{h\ge 0\\ II^*h\le \bar g}} \|II^*h\|_{L^{p'}_\sigma} = \|\bar g\|_{L^{p'}_\sigma}$$

as required.

As a consequence we are able to give necessary and sufficient conditions for an inequality studied in [14].

Corollary 4.4. *Suppose that* $1 < p < \infty$, $1/p + 1/p' = 1$, *and* $v, h \in L^+_\lambda$. *Then*

$$\sup_{g\in L^\downarrow_\lambda} \frac{\int_0^\infty gh}{\left(\int_0^\infty (\frac{1}{x}\int_0^x g)^p v(x)\, dx\right)^{1/p}} \approx \|H\|_{L^{p'}_\tau}$$

where $H(x)$ *is the least non-increasing majorant of* $\frac{1}{x}\int_0^x h$ *and*

$$\tau(x) = \frac{1}{x}\left(\frac{1}{x^p}\int_0^x v(t)\, dt\right)\left(\int_x^\infty v(t)\,\frac{dt}{t^p}\right)\left(\frac{1}{x^p}\int_0^x v(t)\, dt + \int_x^\infty v(t)\,\frac{dt}{t^p}\right)^{-p'-1}.$$

Proof. The supremum over $L^\downarrow_\lambda$ can be replaced by a supremum over $I^*L^+_\lambda$ so we have

$$\begin{aligned}\sup_{g\in L^\downarrow_\lambda} \frac{\int_0^\infty gh}{\left(\int_0^\infty (\frac{1}{x}\int_0^x g)^p v(x)\, dx\right)^{1/p}} &= \sup_{f\in L^+_\lambda} \frac{\int_0^\infty (I^*f)h}{\left(\int_0^\infty (II^*f)^p x^{-p} v(x)\, dx\right)^{1/p}}\\ &= \sup_{f\in L^+_\lambda} \frac{\int_0^\infty f(Ih)}{\left(\int_0^\infty (II^*f)^p x^{-p} v(x)\, dx\right)^{1/p}} \approx \|\overline{Ih}\|_{L^{p'}_{\sigma_1}}.\end{aligned}$$

The equivalence follows from Theorem 4.3 with g replaced by Ih and $v(x)$ replaced by $x^{-p}v(x)$. Here σ_1 is given by

$$\sigma_1(x) = \frac{1}{x}\left(\int_0^x v(t)\,dt\right)\left(x^p\int_x^\infty v(t)\,\frac{dt}{t^p}\right)\left(\int_0^x v(t)\,dt + x^p\int_x^\infty v(t)\,\frac{dt}{t^p}\right)^{-p'-1}.$$

Now Ih is non-decreasing so

$$\overline{Ih}(x) = x \operatorname*{ess\,sup}_{t\ge x} t^{-1} \operatorname*{ess\,sup}_{0\le s\le t} Ih(s) = x \operatorname*{ess\,sup}_{t\ge x} t^{-1} Ih(t) = xH(x).$$

Taking the factor of x into the weight we have

$$\|\overline{Ih}\|_{L^{p'}_{\sigma_1}} = \|H\|_{L^{p'}_{\tau}}$$

to complete the proof.

5. Proof of Theorem 2.1

Some preparation is required before we give our proof of Theorem 2.1 but the intermediate results are themselves worthy of note. In particular, Lemma 5.2 and Corollary 5.3 are useful tools since they make precise the notion of pushing mass mentioned in the sketch proof of (2.1). We begin by showing that even when the function u is not well behaved, each of the envelopes of u is constant except where it is close to u.

Lemma 5.1. *Suppose $u \in L^+_\lambda$, $a < b$ and $y \in \mathbf{R}$. If*

$$\{x \ge y : a < u^\downarrow(x) \le b\} \ne \emptyset \tag{5.1}$$

then

$$\lambda\{x \ge y : a < u(x) \le u^\downarrow(x) \le b\} > 0. \tag{5.2}$$

Similarly, if

$$\{x \le y : a \le u_\downarrow(x) < b\} \ne \emptyset \tag{5.3}$$

then

$$\lambda\{x \le y : a \le u_\downarrow(x) \le u(x) < b\} > 0. \tag{5.4}$$

Proof. To prove the first implication we suppose that (5.1) holds and (5.2) fails and derive a contradiction. Fix $x \ge y$ such that $a < u^\downarrow(x) \le b$. If $t \ge x$ then $u^\downarrow(t) \le u^\downarrow(x) \le b$ so either $u^\downarrow(t) \le a$ or $a < u^\downarrow(t) \le b$. For λ-almost every $t \ge x$

satisfying $u^\downarrow(t) \le a$ we have $u(t) \le u^\downarrow(t) \le a$. Since (5.2) fails and $x \ge y$ we also have $u(t) \le a$ for λ-almost every $t \ge x$ satisfying $a < u^\downarrow(t) \le b$. It follows that $u(t) \le a$ for λ-almost every $t \ge x$. Therefore

$$u^\downarrow(x) = \operatorname*{ess\,sup_\lambda}_{t \ge x} u(t) \le a$$

which contradicts the choice of x.

To prove the second implication we suppose that (5.3) holds and (5.4) fails and derive a contradiction. Fix $x \le y$ such that $a \le u_\downarrow(x) < b$. If $t \le x$ then $u_\downarrow(t) \ge u_\downarrow(x) \ge a$ so either $u_\downarrow(t) \ge b$ or $a \le u_\downarrow(t) < b$. For λ-almost every $t \le x$ satisfying $u_\downarrow(t) \ge b$ we have $u(t) \ge u_\downarrow(t) \ge b$. Since (5.4) fails and $x \le y$ we also have $u(t) \ge b$ for λ-almost every $t \le x$ satisfying $a \le u_\downarrow(t) < b$. If follows that $u(t) \ge b$ for λ-almost every $t \le x$. Therefore

$$u_\downarrow(x) = \operatorname*{ess\,inf_\lambda}_{t \le x} u(t) \ge b$$

which contradicts the choice of x.

In the next lemma we show how the mass of a function f can be "pushed" to the right and onto a small subset. Recall that $L^+_\lambda(S)$ is the collection of non-negative λ-measurable functions which vanish off S.

Lemma 5.2. *Suppose $x \in \mathbf{R}$ and E is a λ-measurable subset of $(-\infty, x]$ satisfying $\lambda(E \cap (y, \infty)) > 0$ for all $y < x$. If $f \in L^+_\lambda((-\infty, x])$ then there exists a function $g \in L^+_\lambda(E)$ such that $Ig \le If$ and $I^*g \ge I^*f$.*

Proof. We look at the simple case first. If $x \in E$ and x is an atom for λ then the σ-finiteness of λ ensures that $0 < \lambda\{x\} < \infty$. In this case we can push all the mass of f onto a single point in E. Set

$$g = \left(\int_{\mathbf{R}} f \, d\lambda \right) \frac{\chi_{\{x\}}}{\lambda\{x\}}.$$

Since f is zero on $[x, \infty)$ we have

$$Ig(y) = \left\{ \begin{array}{ll} 0 & y < x \\ \int_{\mathbf{R}} f \, d\lambda & y \ge x \end{array} \right\} \le If(y)$$

and

$$I^*g(y) = \left\{ \begin{array}{ll} \int_{\mathbf{R}} f \, d\lambda & y \le x \\ 0 & y > x \end{array} \right\} \ge I^*f(y).$$

In the remaining case, either $x \notin E$ or x is not an atom for λ. Then we have $\lambda(E \cap (y, x)) > 0$ for all $y < x$. Hence we can choose an increasing sequence $y_1 < y_2 < \dots$ converging to x such that

$$\lambda(E \cap (y_n, y_{n+1}]) > 0.$$

The σ-finiteness of λ allows us to choose subsets $E_n \subset E \cap (y_n, y_{n+1}]$ of finite, positive λ-measure. Now let $y_0 = -\infty$ and set

$$g = \sum_{n=1}^{\infty} \left(\int_{(y_{n-1}, y_n]} f \, d\lambda \right) \frac{\chi_{E_n}}{\lambda(E_n)}.$$

Here the mass of f has been cascaded to the right with each interval's mass being pushed onto a small subset of the adjacent interval.

To see that $If \le Ig$ and $I^*g \le I^*f$ we argue as follows: If $y \ge x$ then, since f and g are zero on $[x, \infty)$,

$$Ig(y) = \int_{\mathbf{R}} g \, d\lambda = \sum_{n=1}^{\infty} \int_{(y_{n-1}, y_n]} f \, d\lambda = \int_{\mathbf{R}} f \, d\lambda = If(y).$$

Also $I^*g(y) = 0 = I^*f(y)$ whenever $y \ge x$. If $y < x$ then we choose N so that $y \in (y_{N-1}, y_N]$. None of the sets $E_N, E_{N+1}, \ldots$ intersect $(-\infty, y]$ so

$$Ig(y) \le \sum_{n=1}^{N-1} \int_{(y_{n-1}, y_n]} f \, d\lambda = If(y_{N-1}) \le If(y)$$

and

$$I^*g(y) \ge \sum_{n=N}^{\infty} \int_{(y_{n-1}, y_n]} f \, d\lambda = \int_{(y_{N-1}, x)} f \, d\lambda = \int_{(y_{N-1}, \infty)} f \, d\lambda \ge I^*f(y).$$

This completes the proof.

Of course, mass can also be pushed to the left.

Corollary 5.3. *Suppose $x \in \mathbf{R}$ and E is a λ-measurable subset of $[x, \infty)$ satisfying $\lambda(E \cap (-\infty, y)) > 0$ for all $y > x$. If $f \in L^+_\lambda([x, \infty))$ then there exists a function $g \in L^+_\lambda(E)$ such that $Ig \ge If$ and $I^*g \le I^*f$.*

Lemma 5.4. *Suppose that $f, u \in L^+_\lambda$ and $\varepsilon > 0$. Then there exists a function $g \in L^+_\lambda$ such that $Ig \le If$ and*

$$\int_{\mathbf{R}} gu \, d\lambda \ge \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda - \varepsilon.$$

Proof. If $\int_{\mathbf{R}} fu^{\downarrow} \, d\lambda = 0$ then $g = f$ satisfies the conclusion of the lemma. Otherwise, choose $\alpha > 1$ so close to 1 that

$$\alpha^{-1} \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda \ge \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda - \varepsilon/2.$$

Define

$$\begin{aligned} J_\infty &= \{x \in \mathbf{R} : u^\downarrow(x) = \infty\}, \\ J_n &= \{x \in \mathbf{R} : \alpha^{n-1} < u^\downarrow(x) \le \alpha^n\}, \quad n \in \mathbf{Z}, \\ J_{-\infty} &= \{x \in \mathbf{R} : u^\downarrow(x) = 0\}. \end{aligned}$$

Since $u^\downarrow$ is non-negative and non-increasing we see that each J_n, $n \in \mathbf{Z} \cup \{\pm\infty\}$, is an interval (possibly a singleton or an empty set) and that $\mathbf{R}$ is the disjoint union of the J_n's. We construct the desired function g by defining $f_n = f\chi_{J_n}$ and constructing functions g_n to satisfy

$$\begin{aligned} &Ig_n \le If_n, \quad n \in \mathbf{Z} \cup \{\pm\infty\}, \\ &\int_{\mathbf{R}} g_n u \, d\lambda \ge \alpha^{-1} \int_{\mathbf{R}} f_n u^\downarrow \, d\lambda, \quad n \in \mathbf{Z} \cup \{-\infty\}, \text{ and} \\ &\int_{\mathbf{R}} g_\infty u \, d\lambda \ge \alpha^{-1} \int_{\mathbf{R}} f_\infty u^\downarrow - \varepsilon/2. \end{aligned} \tag{5.5}$$

Since $f = \sum_{n \in \mathbf{Z} \cup \{\pm\infty\}} f_n$, it is easy to see that the function $g = \sum_{n \in \mathbf{Z} \cup \{\pm\infty\}} g_n$ will satisfy the conclusion of the lemma.

If $J_n = \emptyset$ for some n then $g_n = 0$ satisfies (5.5). If $n = -\infty$ then $g_\infty = 0$ satisfies (5.5).

If $n \in \mathbf{Z}$ and $J_n \ne \emptyset$ we let $x_n = \sup J_n$ be the right endpoint of J_n and set

$$E_n = \{x \in \mathbf{R} : \alpha^{n-1} < u(x) \le u^\downarrow(x) \le \alpha^n\} \subset (-\infty, x_n].$$

If $y > x_n$ then $J_n \cap [y, \infty) \ne \emptyset$ so Lemma 5.1 yields

$$\lambda(E_n \cap [y, \infty)) > 0.$$

Now $f_n \in L^+_\lambda((-\infty, x_n])$ so by Lemma 5.2 there exists a $g_n \in L^+_\lambda(E_n)$ such that $Ig_n \le If_n$ and $I^*g_n \ge I^*f_n$. Since g_n is zero off E_n and f_n is zero off J_n, we have

$$\begin{aligned} \int_{\mathbf{R}} g_n u \, d\lambda \ge \alpha^{n-1} \int_{\mathbf{R}} g_n \, d\lambda &= \alpha^{n-1} I^* g_n(-\infty) \\ &\ge \alpha^{n-1} I^* f_n(-\infty) = \alpha^{n-1} \int_{\mathbf{R}} f_n \, d\lambda \ge \alpha^{-1} \int_{\mathbf{R}} f_n u^\downarrow \, d\lambda. \end{aligned}$$

The remaining case is $n = \infty$ and $J_\infty \ne \emptyset$. If $\int_{\mathbf{R}} f_\infty u^\downarrow \, d\lambda = 0$ then we can take $g_\infty = 0$. Otherwise, f_∞ does not vanish so $\int_{\mathbf{R}} f_\infty \, d\lambda > 0$. Set $x_\infty = \sup J_\infty$. If $x_\infty \in J_\infty$ then $u^\downarrow(x_\infty) = \infty$ so for $k = 1, 2, \dots$ using the definition of $u^\downarrow$ we can choose subsets

$$U_k \subset \{x \ge x_\infty : u(x) > 2^k\}$$

of finite positive λ-measure and set

$$g_\infty = \left(\int_{(-\infty, x_\infty]} f_\infty \, d\lambda \right) \sum_{k=1}^{\infty} 2^{-k} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

Otherwise, $u^\downarrow(x_\infty - 1/k) = \infty$ for each positive integer k, so we can choose subsets

$$U_k \subset \{x \geq x_{-\infty} - 1/k : u(x) > 2^k\}$$

of finite positive λ-measure and set

$$g_\infty = \sum_{k=1}^{\infty} \left(\int_{(-\infty, x_\infty - 1/k]} f_\infty \, d\lambda \right) 2^{-k} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

Either way the integral of $g_\infty u$ is infinite because

$$2^{-k} \int \frac{\chi_{U_k}}{\lambda(U_k)} u \, d\lambda \geq 1.$$

If $y \geq x_\infty$ then

$$Ig_\infty(y) \leq \int_{(-\infty, x_\infty]} f_\infty \, d\lambda \leq If_\infty(y).$$

If $y < x_\infty$ then either none of the U_k's intersects $(-\infty, y]$ or only those U_k with k satisfying $x_\infty - 1/k \leq y$ intersect $(-\infty, y]$. Therefore either $Ig_\infty(y) = 0$ or

$$Ig_\infty(y) \leq \sum_{1/k \geq x_\infty - y} \left(\int_{(-\infty, x_\infty - 1/k]} f_\infty \, d\lambda \right) 2^{-k} \leq If_\infty(y).$$

This completes the proof.

The corresponding result for the lower envelope is just different enough that a separate proof is required.

Lemma 5.5. *Suppose that $f, u \in L^+_\lambda$ and $\varepsilon > 0$. Then there exists a function $g \in L^+_\lambda$ such that $Ig \geq If$ and*

$$\int_{\mathbf{R}} gu \, d\lambda \leq \int_{\mathbf{R}} fu_\downarrow \, d\lambda + \varepsilon.$$

Proof. If $\int_{\mathbf{R}} fu_\downarrow \, d\lambda = \infty$ then $g = f$ satisfies the conclusion of the lemma. Otherwise, choose $\alpha > 1$ so close to 1 that

$$\alpha \int_{\mathbf{R}} fu_\downarrow \, d\lambda < \int_{\mathbf{R}} fu_\downarrow \, d\lambda + \varepsilon/2.$$

Define

$$
\begin{aligned}
J_\infty &= \{x \in \mathbf{R} : u_\downarrow(x) = \infty\},\\
J_n &= \{x \in \mathbf{R} : \alpha^n \le u_\downarrow(x) < \alpha^{n+1}\}, \quad n \in \mathbf{Z},\\
J_{-\infty} &= \{x \in \mathbf{R} : u_\downarrow(x) = 0\}.
\end{aligned}
$$

Since $u_\downarrow$ is non-negative and non-increasing we see that each J_n, $n \in \mathbf{Z} \cup \{\pm\infty\}$, is an interval (possibly a singleton or an empty set) and that $\mathbf{R}$ is the disjoint union of the J_n's. We construct the desired function g by defining $f_n = f\chi_{J_n}$ and constructing functions g_n to satisfy

$$
\begin{gathered}
Ig_n \ge If_n, \quad n \in \mathbf{Z} \cup \{\pm\infty\},\\
\int_{\mathbf{R}} g_n u \, d\lambda \le \alpha \int_{\mathbf{R}} f_n u_\downarrow \, d\lambda, \quad n \in \mathbf{Z} \cup \{\infty\}, \text{ and}\\
\int_{\mathbf{R}} g_{-\infty} u \, d\lambda \le \varepsilon/2.
\end{gathered} \tag{5.6}
$$

Since $f = \sum_{n\in\mathbf{Z}\cup\{\pm\infty\}} f_n$, it is easy to see that the function $g = \sum_{n\in\mathbf{Z}\cup\{\pm\infty\}} g_n$ will satisfy the conclusion of the lemma.

If $J_n = \emptyset$ for some n then $g_n = 0$ satisfies (5.6). If $n = \infty$ then

$$
\int_{\mathbf{R}} f_\infty u_\downarrow \, d\lambda \le \int_{\mathbf{R}} f u_\downarrow \, d\lambda < \infty
$$

so f_∞ necessarily vanishes λ-almost everywhere. Thus $g_\infty = 0$ satisfies (5.6).

If $n \in \mathbf{Z}$ and $J_n \ne \emptyset$ we set $x_n = \inf J_n$ and

$$
E_n = \{x \in \mathbf{R} : \alpha^n \le u_\downarrow(x) \le u(x) < \alpha^{n+1}\} \subset [x_n, \infty).
$$

If $y < x_n$ then $J_n \cap (-\infty, y] \ne \emptyset$ so Lemma 5.1 yields

$$
\lambda(E_n \cap (-\infty, y]) > 0.
$$

Now $f_n \in L^+_\lambda([x_n,\infty))$ so by Corollary 5.3 there exists a $g \in L^+_\lambda(E_n)$ such that $Ig_n \ge If_n$ and $I^*g_n \le I^*f_n$. Since g_n is zero off E_n and f_n is zero off J_n, we have

$$
\begin{aligned}
\int_{\mathbf{R}} g_n u \, d\lambda \le \alpha^{n+1} \int_{\mathbf{R}} g_n \, d\lambda &= \alpha^{n+1} I^* g_n(-\infty)\\
&\le \alpha^{n+1} I^* f_n(-\infty) = \alpha^{n+1} \int_{\mathbf{R}} f_n \, d\lambda \le \alpha \int_{\mathbf{R}} f_n u_\downarrow \, d\lambda.
\end{aligned}
$$

The remaining case is $n = -\infty$ and $J_{-\infty} \ne \emptyset$. Set $x_{-\infty} = \inf J_{-\infty}$. If $x_{-\infty} \in J_{-\infty}$ then $u_\downarrow(x_{-\infty}) = 0$ so for $k = 1, 2, \ldots$ we can choose subsets

$$
U_k \subset \{x \le x_{-\infty} : u(x) < 2^{-k}\}
$$

of finite positive λ-measure. If $x_{-\infty} \notin J_{-\infty}$ then $u_{\downarrow}(x_{-\infty} + 1/k) = 0$ for each positive integer k, so we can choose subsets

$$U_k \subset \{x \le x_{-\infty} + 1/k : u(x) < 2^{-k}\}$$

of finite positive λ-measure. Set

$$g_{-\infty} = \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{\chi_{U_k}}{\lambda(U_k)}.$$

The choice of U_k ensures that the integral of $g_{-\infty} u$ is small. We have

$$\int_{\mathbf{R}} g_{-\infty} u \, d\lambda = \frac{\varepsilon}{2} \sum_{k=1}^{\infty} \frac{\int_{U_k} u \, d\lambda}{\lambda(U_k)} \le \frac{\varepsilon}{2} \sum_{k=1}^{\infty} 2^{-k} = \frac{\varepsilon}{2}.$$

If $x_{-\infty} \in J_{-\infty}$ then If is zero on $(-\infty, x_{-\infty})$ and, since all the U_k's are contained in $(-\infty, x_{-\infty}]$, Ig is infinite on $[x_{-\infty}, \infty)$. Hence $Ig \ge If$. If $x_{-\infty} \notin J_{-\infty}$ then If is zero on $(-\infty, x_{-\infty}]$ and, since infinitely many of the U_k's are contained in $(-\infty, x)$ for any $x > x_{-\infty}$, Ig is infinite on $(x_{-\infty}, \infty)$. Again $Ig \ge If$. This completes the proof.

Proof of Theorem 2.1. We prove only (2.1) and (2.2) since (2.3) and (2.4) follow from them by making the substitution $x \to -x$ throughout.

If $Ig \le If$ then, since $u(x) \le u^{\downarrow}(x)$ λ-almost everywhere, we may use Corollary 1.3 to get

$$\int_{\mathbf{R}} gu \, d\lambda \le \int_{\mathbf{R}} gu^{\downarrow} \, d\lambda \le \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda.$$

If $Ig \ge If$ then, since $u(x) \ge u_{\downarrow}(x)$ λ-almost everywhere, we may use Corollary 1.3 to get

$$\int_{\mathbf{R}} gu \, d\lambda \ge \int_{\mathbf{R}} gu_{\downarrow} \, d\lambda \ge \int_{\mathbf{R}} fu_{\downarrow} \, d\lambda.$$

Thus

$$\sup_{Ig \le If} \int_{\mathbf{R}} gu \, d\lambda \le \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda \quad \text{and} \quad \inf_{Ig \ge If} \int_{\mathbf{R}} gu \, d\lambda \ge \int_{\mathbf{R}} fu_{\downarrow} \, d\lambda.$$

For the reverse inequalities we use Lemmas 5.4 and 5.5. For each $\varepsilon > 0$ we apply Lemma 5.4 to get a function g_ε satisfying $Ig_\varepsilon \le If$ and

$$\int_{\mathbf{R}} fu^{\downarrow} \, d\lambda \le \int_{\mathbf{R}} g_\varepsilon u \, d\lambda + \varepsilon.$$

Now

$$\sup_{Ig \le If} \int_{\mathbf{R}} gu \, d\lambda \ge \sup_{\varepsilon > 0} \int_{\mathbf{R}} g_\varepsilon u \, d\lambda \ge \int_{\mathbf{R}} fu^{\downarrow} \, d\lambda.$$

We apply Lemma 5.5 to functions f and u to get a function g_ε satisfying $Ig_\varepsilon \geq If$ and

$$\int_{\mathbf{R}} f u_\downarrow \, d\lambda \geq \int_{\mathbf{R}} g_\varepsilon u \, d\lambda - \varepsilon.$$

Now

$$\inf_{Ig \geq If} \int_{\mathbf{R}} g u \, d\lambda \leq \inf_{\varepsilon > 0} \int_{\mathbf{R}} g_\varepsilon u \, d\lambda \leq \int_{\mathbf{R}} f u_\downarrow \, d\lambda.$$

This completes the proof.

6. The Structure of General Level Functions

In Proposition 1.5(iii) the structure of the level function of a bounded function, supported on $(-\infty, M]$ for some M is given in terms of an averaging operator $A \in \mathcal{A}$. Although one would expect that taking increasing limits of such level functions would destroy this simple structure we show here that it does not. The structure of the level function of an arbitrary function remains attractively simple.

Theorem 6.1. *Suppose u is non-negative and λ-measurable. Then there exists an $A \in \mathcal{A}$ and (possible empty) intervals J_{left} and J_{right} such that*
i) if $J_{\text{left}} \neq \emptyset$ then $\inf J_{\text{left}} = -\infty$ and $u^o \equiv \infty$ on J_{left};
ii) $u^o = Au$ λ-almost everywhere off $J_{\text{left}} \cup J_{\text{right}}$;
iii) if $J_{\text{right}} \neq \emptyset$ then $\sup J_{\text{right}} = \infty$, $\lambda(J_{\text{right}}) = \infty$, and on J_{right}

$$u^o = \limsup_{x \to \infty} \frac{1}{\lambda((-\infty, x] \cap J_{\text{right}})} \int_{(-\infty, x] \cap J_{\text{right}}} u \, d\lambda. \tag{6.1}$$

Proof. Set $u_n = \min(u, n)\chi_{(-\infty,n]}$ so that $u_n \uparrow u$ and, by Proposition 1.5(ii), $u_n^o \uparrow u^o$ pointwise on $\mathbf{R}$. Each u_n is bounded and supported on $(-\infty, n]$ so there is an operator $A_n \in \mathcal{A}$ such that $u_n^o = A_n u_n$. We call the intervals of A_n the *level intervals* of u_n and note that u_n^o is constant on each of its level intervals. Also note that level intervals have positive λ-measure by definition. If an interval J has the property that every level interval of u_n that intersects J is contained in J we say that the interval *respects* u_n. If J respects u_n then $A_n \chi_J = \chi_J$ so

$$\int_J u_n \, d\lambda = \int_{\mathbf{R}} A_n \chi_J u_n \, d\lambda = \int_{\mathbf{R}} \chi_J A_n u_n \, d\lambda = \int_J u_n^o \, d\lambda$$

and for any $x \in J$ the interval $(-\infty, x] \setminus J$ also respects u_n^o so, by Proposition 1.5(i),

$$\begin{aligned}\int_{(-\infty,x] \cap J} u_n \, d\lambda &= \int_{(-\infty,x]} u_n \, d\lambda - \int_{(-\infty,x] \setminus J} u_n \, d\lambda \\ &\leq \int_{(-\infty,x]} u_n^o \, d\lambda - \int_{(-\infty,x] \setminus J} u_n^o \, d\lambda = \int_{(-\infty,x] \cap J} u_n^o \, d\lambda\end{aligned}$$

These two together show that if J respects u_n and $x \in \mathbf{R}$ then

$$\int_J u_n \, d\lambda = \int_J u_n^o \, d\lambda \quad \text{and} \quad \int_{(-\infty,x]\cap J} u_n \, d\lambda \le \int_{(-\infty,x]\cap J} u_n^o \, d\lambda.$$

Notice that for any interval J the interval $(u_n^o)^{-1}(u_n^o(J))$ respects u_n. In particular, if $s < t$ then with $J = (u_n^o)^{-1}(u_n^o[s,\infty])$ we have

$$\int_{[s,t]} u_n \, d\lambda \le \int_{(-\infty,t]\cap J} u_n \, d\lambda \le \int_{(-\infty,t]\cap J} u_n^o \, d\lambda \le u^o(s)\lambda(-\infty,t].$$

Letting $n \to \infty$ we conclude that

$$\int_{[s,t]} u \, d\lambda \le u^o(s)\lambda(-\infty,t] \quad \text{and thus} \quad \int_{(s,t]} u \, d\lambda \le u^o(s+)\lambda(-\infty,t] \tag{6.2}$$

Set $J_{\text{left}} = (u^o)^{-1}(\infty)$. Since u^o is non-increasing, $J_{\text{left}} = \emptyset$ or else $\inf J_{\text{left}} = -\infty$ and we have proved part (i). For $0 \le y < \infty$ set

$$J_y = (u^o)^{-1}(y).$$

The J_y are (possibly empty or singleton) intervals that partition $\mathbf{R} \setminus J_{\text{left}}$. Since $\lambda(-\infty,x] < \infty$ for all $x \in \mathbf{R}$ at most one of the intervals J_y has infinite λ-measure. If no J_y has infinite λ-measure then let $J_{\text{right}} = \emptyset$. Otherwise, let J_{right} be the unique interval J_y having infinite λ-measure and note that $\sup J_{\text{right}} = \infty$. To define the averaging operator A we specify its intervals. See the definition of $\mathcal{A}$ preceding Proposition 1.4. We take the intervals of A to be all the intervals J_y having interior and satisfying $0 < \lambda(J_y) < \infty$.

To show that $u^o = Au$ off $J_{\text{left}} \cup J_{\text{right}}$ we first show that $u^o(x) = u(x)$ for λ-almost every x in some J_y with no interior or with zero λ-measure. The inequalities (6.2) show that u is locally λ-integrable on $\mathbf{R} \setminus J_{\text{left}}$ so the set of points in the λ-Lebesgue sets [4, p156] of all the functions $u, u_1, u_2, \dots$ has full λ-measure. We consider only these points. Also, since the J_y are disjoint, at most countably many have interior. The union of those J_y with interior and zero λ-measure also has zero λ-measure so we may disregard points in such intervals.

The remaining points x each lie in some interval J_y with no interior, Thus $J_y = \{x\}$. If $u_n(x) = u_n^o(x)$ for infinitely many n then we have $u(x) = u^o(x)$ as required. If not, then for some N, x is in a level interval L_n of u_n for all $n \ge N$. We have $0 < \lambda(L_n) < \infty$ and u_n^o is constant on L_n, taking the value

$$\frac{1}{\lambda(L_n)} \int_{L_n} u \, d\lambda.$$

Since $J_y = \{x\}$, for any a, d with $a < x < d$ we have $u^o(a) > u^o(x) > u^o(d)$ and hence $u_n^o(a) > u_n^o(x) > u_n^o(d)$ for sufficiently large n. It follows that $x \in L_n \subset (a,d)$

for sufficiently large n. We have shown that the diameter of L_n converges to zero with n. Since x is in the λ-Lebesgue set of u and u_m for all m we have

$$\lim_{n\to\infty} \frac{1}{\lambda(L_n)} \int_{L_n} u\, d\lambda = u(x) \text{ and } \lim_{n\to\infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_m)\, d\lambda = u(x) - u_m(x)$$

for all m. Now

$$\begin{aligned} |u(x) - u^o(x)| &= \lim_{n\to\infty} \left| \left(\frac{1}{\lambda(L_n)} \int_{L_n} u\, d\lambda \right) - u_n^o(x) \right| \\ &= \lim_{n\to\infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_n)\, d\lambda \\ &\le \lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{\lambda(L_n)} \int_{L_n} (u - u_m)\, d\lambda \\ &= \lim_{m\to\infty} u(x) - u_m(x) = 0. \end{aligned}$$

Therefore $u(x) = u^o(x)$ for λ-almost every $x \in \mathbf{R} \setminus (J_{\text{left}} \cup J_{\text{right}})$ off the intervals of A.

To complete the proof of part (ii) we must show that

$$y = \frac{1}{\lambda(J_y)} \int_{J_y} u\, d\lambda$$

for all y such that J_y has interior and $0 < \lambda(J_y) < \infty$. Since

$$\lim_{n\to\infty} \int_{J_y} u_n^o\, d\lambda = \int_{J_y} u^o\, d\lambda = y\lambda(J_y) < \infty \quad \text{and} \quad \lim_{n\to\infty} \int_{J_y} u_n\, d\lambda = \int_{J_y} u\, d\lambda$$

it is enough to show that

$$\lim_{n\to\infty} \left| \int_{J_y} (u_n^o - u_n)\, d\lambda \right| = 0. \tag{6.3}$$

To do this we let $s = \inf J_y$ and proceed in the case that $s \notin J_y$. The case $s \in J_y$ is similar. Since $s \notin J_y$ we have $u^o(s+) = y$ and (6.2) implies that for any $t > s$,

$$\int_{(s,t]} u\, d\lambda \le y\lambda(-\infty, t] < \infty.$$

Therefore, for any $\varepsilon > 0$ we may choose a, b, c, and d satisfying

$$\begin{gathered} -\infty \le a < b \le c < d \le \infty, \\ [b,c] \subset J_y \subset (a,d), \\ y\lambda((a,b) \cup (c,d)) < \varepsilon/4, \text{ and} \\ \int_{(s,b)\cup(c,d)} u\, d\lambda < \varepsilon/2. \end{gathered}$$

(Note that if $\sup J_y = \infty$ then $\lambda(J_y) < \infty$ implies $\lambda(\mathbf{R}) < \infty$ so this is possible even if d is forced to be ∞.) Since $a \notin J_y$ and $b \in J_y$ we have either $a = -\infty$ or else $u^o(a) > u^o(b)$ and hence $u_n^o(a) > u_n^o(b)$ for sufficiently large n. Also, $c \in J_y$ and $d \notin J_y$ so either $d = \infty$ or else $u^o(c) > u^o(d)$ and hence $u_n^o(c) > u_n^o(d)$ for sufficiently large n. Therefore, if we set

$$J_y(n) = (u_n^o)^{-1}(u_n^o[b, c]),$$

we have $[b, c] \subset J_y(n) \subset (a, d)$ and we can estimate the symmetric difference of J_y and $J_y(n)$ by

$$J_y \triangle J_y(n) \equiv (J_y \setminus J_y(n)) \cup (J_y(n) \setminus J_y) \subset (a, b) \cup (c, d)$$

for sufficiently large n. The choice of $J_y(n)$ ensures that $J_y(n)$ respects u_n so we have

$$\int_{J_y(n)} (u_n^o - u_n)\, d\lambda = 0 \quad \text{and} \quad \int_{(-\infty,s] \cap J_y(n)} (u_n^o - u_n)\, d\lambda \geq 0$$

and therefore

$$\begin{aligned}
&\lim_{n\to\infty} \left| \int_{J_y} (u_n^o - u_n)\, d\lambda \right| \\
&\leq \lim_{n\to\infty} \int_{J_y \triangle J_y(n)} (u_n^o + u_n)\, d\lambda \\
&\leq \lim_{n\to\infty} \int_{J_y \triangle J_y(n)} u_n^o\, d\lambda + \int_{(-\infty,s]\cap J_y(n)} u_n\, d\lambda + \int_{(s,b)\cup(c,d)} u_n\, d\lambda \\
&\leq \lim_{n\to\infty} \int_{J_y \triangle J_y(n)} u_n^o\, d\lambda + \int_{(-\infty,s]\cap J_y(n)} u_n^o\, d\lambda + \int_{(s,b)\cup(c,d)} u\, d\lambda \\
&\leq \lim_{n\to\infty} 2y\lambda((a, b) \cup (c, d)) + \int_{(s,b)\cup(c,d)} u\, d\lambda < \varepsilon.
\end{aligned}$$

Here we have used the fact that $u_n^o \leq u^o = y$ on J_y and $u_n^o \leq u_n^o(b) \leq u^o(b) = y$ on $J_y(n)$. Since ε was arbitrary, this proves (6.3) and completes part (ii).

If $J_{\text{right}} = \emptyset$ there is nothing more to prove. Otherwise $J_{\text{right}} = J_y$ for the unique y satisfying $\lambda(J_y) = \infty$. We have already argued that $\sup J_{\text{right}} = \infty$. To prove (6.1) we set $s = \sup J_y$ as before. The two cases $s \in J_y$ and $s \notin J_y$ are similar again so this time we look at the case $s \in J_y$. We have $u^o(s) = y$ and (6.2) implies that for any $t > s$,

$$\int_{[s,t]} u\, d\lambda \leq y\lambda(-\infty, t] < \infty.$$

Therefore, for any $\varepsilon > 0$ and any $x > s$ we may choose a, b, and c satisfying

$$\begin{gathered}
-\infty \le a < b < x \le c < \infty, \\
[b, c] \subset J_y \subset (a, \infty), \\
y\lambda((a, b)) < \varepsilon/4, \text{ and} \\
\int_{[s,b)} u \, d\lambda < \varepsilon/2.
\end{gathered}$$

Since $a \notin J_y$ and $b \in J_y$ we have either $a = -\infty$ or else $u^o(a) > u^o(b)$ and hence we may choose N so that $u^o_n(a) > u^o_n(b)$ for $n > N$. Therefore, if we set

$$J_y(n) = (u^o_n)^{-1}(u^o_n[b, c]),$$

we have $[b, c] \subset J_y(n) \subset (a, \infty)$ for $n > N$ and, provided $t \in J_y(n)$, we can estimate the symmetric difference of $(-\infty, t] \cap J_y$ and $(-\infty, t] \cap J_y(n)$ by

$$[(-\infty, t] \cap J_y] \triangle [(-\infty, t] \cap J_y(n)] \subset (a, b)$$

for $n > N$. Again $J_y(n)$ respects u_n so for all $t \in \mathbf{R}$ we have

$$\int_{(-\infty,t]\cap J_y(n)} (u_n - u^o_n) \, d\lambda \le 0 \quad \text{and thus} \quad \int_{(-\infty,s)\cap J_y(n)} (u_n - u^o_n) \, d\lambda \le 0.$$

For $n > N$ and any $t \in J_y(n)$,

$$\begin{aligned}
&\left| \int_{(-\infty,t]\cap J_y} (u_n - u^o_n) \, d\lambda - \int_{(-\infty,t]\cap J_y(n)} (u_n - u^o_n) \, d\lambda \right| \\
&\le \int_{[(-\infty,t]\cap J_y]\triangle[(-\infty,t]\cap J_y(n)]} (u_n + u^o_n) \, d\lambda \\
&\le \int_{[(-\infty,t]\cap J_y]\triangle[(-\infty,t]\cap J_y(n)]} u^o_n \, d\lambda + \int_{(-\infty,s)\cap J_y(n)} u_n \, d\lambda + \int_{[s,b)} u_n \, d\lambda \\
&\le \int_{[(-\infty,t]\cap J_y]\triangle[(-\infty,t]\cap J_y(n)]} u^o_n \, d\lambda + \int_{(-\infty,s)\cap J_y(n)} u^o_n \, d\lambda + \int_{[s,b)} u \, d\lambda \\
&\le 2y\lambda(a, b) + \int_{[s,b)} u \, d\lambda < \varepsilon.
\end{aligned}$$

Here we have used the fact that $u^o_n \le u^o = y$ on J_y and $u^o_n \le u^o_n(b) \le u^o(b) = y$ on $J_y(n)$.

Since $x \in [b, c] \subset J_y(n)$ for $n > N$, $\varepsilon > 0$ was arbitrary, and

$$\lim_{n\to\infty} \int_{(-\infty,x]\cap J_y(n)} (u_n - u^o_n) \, d\lambda \le 0$$

it follows that

$$\int_{(-\infty,x]\cap J_y} (u - u^o)\, d\lambda = \lim_{n\to\infty} \int_{(-\infty,x]\cap J_y} (u_n - u_n^o)\, d\lambda \le 0.$$

This proves the inequality "$\ge$" in (6.1). For the other inequality we fix $\varepsilon > 0$ and increase N so that for $n > N$, $u_n^o(c) > u^o(c) - \varepsilon$. Set $t_n = \sup J_y(n)$. If $t_n \in J_y(n)$ then

$$\left| \int_{(-\infty,t_n]\cap J_y} (u_n - u_n^o)\, d\lambda - \int_{(-\infty,t_n]\cap J_y(n)} (u_n - u_n^o)\, d\lambda \right| < \varepsilon$$

but

$$\int_{(-\infty,t_n]\cap J_y(n)} (u_n - u_n^o)\, d\lambda = \int_{J_y(n)} (u_n - u_n^o)\, d\lambda = 0$$

so

$$\begin{aligned}
\int_{(-\infty,t_n]\cap J_y} u\, d\lambda &\ge \int_{(-\infty,t_n]\cap J_y} u_n\, d\lambda \\
&\ge \int_{(-\infty,t_n]\cap J_y} u_n^o\, d\lambda - \varepsilon \\
&\ge u_n^o(c)\lambda((-\infty,t_n]\cap J_y) - \varepsilon \\
&\ge (u^o(c) - \varepsilon)\lambda((-\infty,t_n]\cap J_y) - \varepsilon.
\end{aligned}$$

Therefore

$$\frac{1}{(-\infty,t_n]\cap J_y} \int_{(-\infty,t_n]\cap J_y} u\, d\lambda \ge u^o(c) - \varepsilon - \varepsilon/\lambda((-\infty,t_n]\cap J_y).$$

If $t_n \notin J_y(n)$ then a similar agument shows that

$$\frac{1}{(-\infty,t_n)\cap J_y} \int_{(-\infty,t_n)\cap J_y} u\, d\lambda \ge u^o(c) - \varepsilon - \varepsilon/\lambda((-\infty,t_n)\cap J_y).$$

For $n > N$, $x \le t_n$ so

$$\sup_{t\ge x} \frac{1}{\lambda((-\infty,t]\cap J_y)} \int_{(-\infty,t]\cap J_y} u\, d\lambda \ge u^o(c) - \varepsilon - \varepsilon/ \limsup_{n\to\infty} \lambda((-\infty,t]\cap J_y).$$

Since ε was arbitrary this proves the inequality "$\le$" in (6.1) to complete the proof of part (iii) and the theorem.

References

1. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Orlando, Florida, 1988.
2. D. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245–1254.
3. M. Carro, L. Pick, J. Soria, and V. D. Stepanov, *On embeddings between classical Lorentz spaces*, Mathematical Inequalities and Applications **4** (2001), 397–428.
4. H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.
5. A. Gogatishvili and L. Pick, *Discretization and anti-discretization of rearrangement invariant norms* (to appear).
6. M. L. Gol'dman, H. P. Heinig and V. D. Stepanov, *On the principle of duality in Lorentz spaces*, Canad. J. Math. **48** (1996), 959–979.
7. L. Maligranda, *Indices and Interpolation*, Dissert. Math. **234** (1985), 1–49.
8. H. L. Royden, *Real Analysis, Third Edition*, Prentice Hall, Englewood Cliffs, New Jersey, 1988.
9. E. T. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 19–52.
10. G. Sinnamon, *Spaces defined by the level function and their duals*, Studia Math. **111** (1994), 19–52.
11. G. Sinnamon, *The level function in rearrangement invariant spaces*, Publ. Mat. **45** (2001 pages 175–198).
12. G. Sinnamon, *Embeddings of concave functions and duals of Lorentz space* (to appear).
13. G. Sinnamon and V. D. Stepanov, *The weighted Hardy inequality: New proofs and the case $p = 1$.*, London Math. Soc. (2) **54** (1996), 89–101.
14. V. D. Stepanov, *Integral operators on the cone of monotone functions*, London Math. Soc. (2) **48** (1993), 465–487.

Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, CANADA

E-mail address: sinnamon@uwo.ca