FOUR QUESTIONS RELATED TO HARDY'S INEQUALITY

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Abstract. New results are interspersed with questions and suggestions for further research. The topics considered revolve around the weighted Hardy inequality. In this paper it is restricted to hyperplanes, considered on a kind of Lebegue space with non-constant index, modified to include operators of the form $\int_0^{b(x)} f(t)\,dt$ for arbitrary $b(x) \geq 0$, and looked at as defining a partially ordered class of measures.

0. Introduction, The Weighted Hardy Inequality

All of the questions in this paper depend on or are inspired by the weight characterization for the Hardy inequality. It is appropriate to begin with that well-known result.

Proposition 0.1. *Suppose that $1 < p < \infty$, $0 < q < \infty$ and μ and ν are non-negative, regular measures on the interval (a, b) with $-\infty \leq a < b \leq \infty$. Then there exists a constant C such that*

$$\left(\int_a^b \left| \int_a^x f(t)\,d\nu(t) \right|^q d\mu(x) \right)^{1/q} \leq C \left(\int_a^b |f(t)|^p \,d\nu(t) \right)^{1/p}$$

holds for all $f \in L^p_\nu[a, b]$ if and only if either
$p \leq q$ and

$$\sup_{a \leq y \leq b} \left(\int_a^y d\nu \right)^{1/p'} \left(\int_y^b d\mu \right)^{1/q} < \infty, \text{ or}$$

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$q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_a^b \left(\int_a^y d\nu\right)^{r/p'} \left(\int_y^b d\mu\right)^{r/p} d\mu(y)\right)^{1/r} < \infty.$$

Various proofs of Proposition 0.1 in the case that μ and ν are absolutely continuous with respect to Lebesgue measure (weight functions) may be found in [5] and the references therein. The extension to measures may be found in [7] and [8]. It is important to point out that the usual form of the weighted Hardy inequality,

$$\left(\int_a^b \left|\int_a^x f(t)\,dt\right|^q u(x)dx\right)^{1/q} \le C\left(\int_a^b |f(t)|^p v(t)\,dt\right)^{1/p} \tag{0.1}$$

for non-negative weight functions u and v, can be cast in the form of Proposition 0.1 by replacing f by $fv^{1-p'}$ in (0.1) and taking $d\mu(x) = u(x)\,dx$ and $d\nu(t) = v(t)^{1-p'}\,dt$.

Although we have introduced the Hardy inequality on the interval $[a, b]$ we will work on $[0, 1]$ for simplicity except in Section 3 where it is simpler to work on $[0, \infty)$. Generally speaking, results on one interval translate readily to any other. As usual we denote the harmonic conjugate of p by p' so that $1/p + 1/p' = 1$. Integrals are taken to include their endpoints so $\int_a^b d\mu = \int_{[a,b]} d\mu$ and χ_S denotes the function with value 1 on S and 0 otherwise.

1. Hardy's Inequality on Hyperplanes

Suppose that $1 < p < \infty$ and $0 < q < \infty$, let u and v be weights, and set $w = v^{1-p'}$. Fix a function $m \in L_w^{p'}(0, 1) \equiv L_w^{p'}$ and set

$$H_m = \{h \in L_v^p : \int_0^1 hm = 0\}.$$

Question 1.1. *What conditions on p, q, u, and v are necessary and sufficient for there to exist a constant C such that*

$$\left(\int_0^1 \left|\int_0^x f(t)\,dt\right|^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^1 |f(t)|^p v(t)\,dt\right)^{1/p} \tag{1.1}$$

for all $f \in H_m$?

The case $m = 0$ is the weighted Hardy inequality of Proposition 0.1 because $H_0 = L_v^p$. If m is not trivial then H_m is genuinely a hyperplane in L_v^p. Note that multiplying m by a non-zero constant has no effect on H_m. The case $m \equiv 1$ (or any non-zero constant) was solved by P. Gurka (see [5, Chap. 1, Sect. 8]) for $1 < p \le q < \infty$ and in [4] for all p and q. With the aid of the following lemma we will be able to answer Question 1.1 in the case that both $\{x : m(x) > 0\}$ and $\{x : m(x) < 0\}$ are of positive measure.

Lemma 1.2. *If T is a non-negative, linear operator that satisfies*
$$\|Th\|_{L^q_u} \le C_0 \|h\|_{L^p_v}, \quad h \in H,$$
for some C_0 then
$$\|Tg\|_{L^q_u} \le C_1 \|g\|_{L^p_v}, \quad g \in L^p_v,$$
where
$$C_1 = C_0(1 + \|m\|_{L^{p'}_w} / \min(\|m\chi_{m>0}\|_{L^{p'}_w}, \|m\chi_{m<0}\|_{L^{p'}_w})).$$
(Recall that $w = v^{1-p'}$.)

Proof. If $\|m\chi_{m>0}\|_{p',w} = 0$ or $\|m\chi_{m<0}\|_{p',w} = 0$ then $C_1 = \infty$ and the conclusion holds trivially. Otherwise, fix $g \in L^p_v$ and define h by
$$h = |g| + \left(\int_0^1 |g|m\right) \chi_{m<0}|m|^{p'-1}w \Big/ \int_{m<0} |m|^{p'}w \quad \text{if } \int_0^1 |g|m \ge 0, \text{ and}$$
$$h = |g| - \left(\int_0^1 |g|m\right) \chi_{m>0}|m|^{p'-1}w \Big/ \int_{m>0} |m|^{p'}w \quad \text{if } \int_0^1 |g|m < 0.$$
In either case we clearly have $h \ge |g| \ge g$.

If $\int_0^1 |g|m \ge 0$ then
$$\int_0^1 hm = \int_0^1 |g|m + \left(\int_0^1 |g|m\right) \int_{m<0} |m|^{p'-1}mw \Big/ \int_{m<0} |m|^{p'}w = 0$$
and if $\int_0^1 |g|m < 0$ then
$$\int_0^1 hm = \int_0^1 |g|m - \left(\int_0^1 |g|m\right) \int_{m>0} |m|^{p'-1}mw \Big/ \int_{m>0} |m|^{p'}w = 0$$
so in either case $h \in H$.

Since T is a non-negative operator and $g \le h$ we have
$$\|Tg\|_{L^q_u} \le \|Th\|_{L^q_u} \le C_0\|h\|_{L^p_v}$$
so we may complete the proof by estimating $\|h\|_{L^p_v}$. If $\int_0^1 |g|m \ge 0$ then
$$\begin{aligned}\|h\|_{L^p_v} &\le \|g\|_{L^p_v} + \left(\int_0^1 |g|m\right) \|m^{p'-1}\chi_{m<0}\|_{L^p_v} \Big/ \int_{m<0} |m|^{p'}w \\ &= \|g\|_{L^p_v} + \left(\int_0^1 |g|m\right) / \|m\chi_{m<0}\|_{L^{p'}_w} \\ &\le \|g\|_{L^p_v} + \|g\|_{L^p_v}\|m\|_{L^{p'}_w} / \|m\chi_{m<0}\|_{L^{p'}_w}.\end{aligned}$$
Similarly, if $\int_0^1 |g|m < 0$ then
$$\|h\|_{L^p_v} \le \|g\|_{L^p_v} + \|g\|_{L^p_v}\|m\|_{L^{p'}_w} / \|m\chi_{m>0}\|_{L^{p'}_w}.$$
The conclusion follows.

Corollary 1.3. *Suppose that both $\{x : m(x) > 0\}$ and $\{x : m(x) < 0\}$ have positive w-measure. If there exists a finite constant C such that* (1.1) *holds for all $f \in H_m$ then there exists a (different) finite constant C such that* (1.1) *holds for all $f \in L^p_v$. In particular, Question 1.1 reduces to the usual Hardy inequality* (0.1).

2. Non-Constant Indices

Let p, q, u, and v be non-negative, measurable functions and consider the inequality

$$\int_0^1 \left| \int_0^x f(t)\, dt \right|^{q(x)} u(x)\, dx \le C \int_0^1 |f(t)|^{p(t)} v(t)\, dt. \tag{2.1}$$

If p and q are constant functions and take the same value then (2.1) reduces to the familiar weighted Hardy inequality. The theorem which follows shows that (2.1) never holds otherwise.

Definition 2.1. *Suppose that (X, μ) and (T, ν) are σ-finite measure spaces. A $\mu \times \nu$-measurable function $k(x,t)$ is called a* proper *kernel on $X \times T$ provided that if X_0 and X_1 are disjoint μ-measurable subsets of X and T_0 and T_1 are disjoint ν-measurable subsets of T such that*

$$k(x,t) = k(x,t)\,(\chi_{X_0 \times T_0}(x,t) + \chi_{X_1 \times T_1}(x,t))$$

then either $\mu(X_0) = \nu(T_0) = 0$ or $\mu(X_1) = \nu(T_1) = 0$.

Theorem 2.2. *Suppose that (X, μ) and (T, ν) are σ-finite measure spaces and $k(x,t)$ is a proper kernel on $X \times T$. Let $p(t)$ and $q(x)$ be non-negative, measurable functions on T and X respectively. If there exists a constant C such that*

$$\int_X \left(\int_T k(x,t) f(t)\, d\nu(t) \right)^{q(x)} d\mu(x) \le C \int_T f(t)^{p(t)}\, d\nu(t) \tag{2.2}$$

holds for all non-negative ν-measurable functions f then $p(t)$ is constant ν-almost everywhere, $q(x)$ is constant μ-almost everywhere, and the two functions take the same value.

Proof. Since (T, ν) is a σ-finite measure space there exists a positive function φ such that $\int_T \varphi(t)\, d\nu(t) < \infty$. Fix such a function φ. For each $\lambda > 0$ set

$$T_0(\lambda) = \{t : p(t) < \lambda\}, \ X_1(\lambda) = \{x : q(x) \ge \lambda\}, \text{ and } f_\lambda(t) = \varphi(t)^{1/p(t)} \chi_{T_0(\lambda)}(t).$$

For any $m \ge 1$ we have

$$\begin{aligned} &\int_{X_1(\lambda)} \left(\int_{T_0(\lambda)} k(x,t) f_\lambda(t)\, d\nu(t) \right)^{q(x)} d\mu(x) \\ \le & m^{-\lambda} \int_X \left(\int_T k(x,t) m f_\lambda(t)\, d\nu(t) \right)^{q(x)} d\mu(x) \\ \le & m^{-\lambda} C \int_T (m f_\lambda(t))^{p(t)}\, d\nu(t) \\ = & C \int_{T_0(\lambda)} m^{p(t)-\lambda} \varphi(t)\, d\nu(t). \end{aligned} \tag{2.3}$$

For $t \in T_0(\lambda)$, $m^{p(t)-\lambda} \to 0$ as $m \to \infty$ so, by the Dominated Convergence Theorem, the last integral tends to zero as $m \to \infty$. It follows that the integral (2.3) is zero. Since $f_\lambda(t) > 0$ for $t \in T_0(\lambda)$ we see that $k(x,t) = 0$ $\mu \times \nu$-almost everywhere on $X_1(\lambda) \times T_0(\lambda)$.

This time we set

$$T_1(\lambda) = \{t : p(t) > \lambda\},\ X_0(\lambda) = \{x : q(x) < \lambda\},\ \text{and } g_\lambda(t) = \varphi(t)^{1/p(t)}\chi_{T_1(\lambda)}(t).$$

For any $m \le 1$ we have

$$\begin{aligned}
(2.4)\qquad & \int_{X_0(\lambda)} \left(\int_{T_1(\lambda)} k(x,t) g_\lambda(t)\, d\nu(t) \right)^{q(x)} d\mu(x) \\
\le & m^{-\lambda} \int_X \left(\int_T k(x,t) m g_\lambda(t)\, d\nu(t) \right)^{q(x)} d\mu(x) \\
\le & m^{-\lambda} C \int_T (m g_\lambda(t))^{p(t)}\, d\nu(t) \\
= & C \int_{T_1(\lambda)} m^{p(t)-\lambda} \varphi(t)\, d\nu(t).
\end{aligned}$$

For $t \in T_1(\lambda)$, $m^{p(t)-\lambda} \to 0$ as $m \to 0$ so, by the Dominated Convergence Theorem, the last integral tends to zero as $m \to 0$. It follows that the integral (2.4) is zero. Since $g_\lambda(t) > 0$ for $t \in T_1(\lambda)$ we see that $k(x,t) = 0$ $\mu \times \nu$-almost everywhere on $X_0(\lambda) \times T_1(\lambda)$.

If $p(t)$ is not constant as a ν-measurable function then we can find a $\bar\lambda > 0$ such that $\nu(T_0(\bar\lambda)) > 0$ and $\nu(T_1(\bar\lambda)) > 0$. Moreover, using σ-finiteness again, we can choose such a $\bar\lambda$ satisfying $\nu(\{t : p(t) = \bar\lambda\}) = 0$. Since $k(x,t)$ is zero $\mu \times \nu$-almost everywhere on $X_1(\bar\lambda) \times T_0(\bar\lambda)$ and on $X_0(\bar\lambda) \times T_1(\bar\lambda)$ we have

$$k(x,t) = k(x,t) \left(\chi_{X_0(\bar\lambda)\times T_0(\bar\lambda)}(x,t) + \chi_{X_1(\bar\lambda)\times T_1(\bar\lambda)}(x,t) \right)$$

contradicting our hypothesis that $k(x,t)$ is a proper kernel on $X \times T$. Thus $p(t)$ is constant ν-almost everywhere. We denote its constant value by p.

If $\lambda > p$ then $T_0(\lambda)$ has full ν-measure in T and, since $k(x,t)$ is zero $\mu \times \nu$-almost everywhere on $X_1(\lambda) \times T_0(\lambda)$ we see that

$$k(x,t) = k(x,t) \left(\chi_{X_0(\lambda)\times T_0(\lambda)}(x,t) + \chi_{X_1(\lambda)\times T_1(\lambda)}(x,t) \right).$$

Since $k(x,t)$ is proper we have $\mu(X_1(\lambda)) = 0$ so $q(x) < \lambda$ μ-almost everywhere. As $\lambda \to p^+$ we see that $q(x) \ge p$ μ-almost everywhere.

If $\lambda < p$ then $T_1(\lambda)$ has full ν-measure in T and, since $k(x,t)$ is zero $\mu \times \nu$-almost everywhere on $X_0(\lambda) \times T_1(\lambda)$ we see that

$$k(x,t) = k(x,t) \left(\chi_{X_0(\lambda)\times T_0(\lambda)}(x,t) + \chi_{X_1(\lambda)\times T_1(\lambda)}(x,t) \right).$$

Since $k(x,t)$ is proper we have $\mu(X_0(\lambda)) = 0$ so $q(x) \le \lambda$ μ-almost everywhere. As $\lambda \to p^-$ we see that $q(x) \le p$ μ-almost everywhere. Thus $q(x) = p$ μ-almost everywhere, completing the proof.

Corollary 2.3. *If there exists a constant C such that* (2.1) *holds for all non-negative f then there exists $b \in [0,1]$ such that $v(t) > 0$ for almost every $t < b$ and $u(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant value almost everywhere for $t \in (0,b)$ and $x \in (0,b) \cap \{u \neq 0\}$.*

Proof. Define b to be the essential infimum in $[0,1]$ of the set $\{\bar{b} : u(x) = 0$ almost everywhere on $(\bar{b},1)\}$. Since (2.1) holds it is easy to see that $u = 0$ almost everywhere on the interval

$$(\operatorname{ess\,inf}\{t : v(t) = 0\}, 1)$$

so $v(t) > 0$ for almost every $t < b$. To complete the proof we observe that $\chi_{(0,x)}(t)$ is a proper kernel on $[(0,b) \cap \{x : u(x) \neq 0\}] \times (0,b)$ and apply Theorem 2.2.

The reason for the failure of (2.1) for non-constant p and q is evident from the proof of Theorem 2.1—homogeneity fails in a disastrous way. There is, however, a standard way to restore lost homogeneity. Set

$$\|f\|_{p(t),v(t)} = \inf\{\eta > 0 : \int_0^\infty |f(t)/\eta|^{p(t)} v(t)\,dt \le 1\}.$$

Question 2.4. *Set $If(x) = \int_0^x f(t)\,dt$ and consider the inequality*

$$\|If\|_{q(x),u(x)} \le C\|f\|_{p(t),v(t)}. \tag{2.5}$$

Does there exist pair of functions $p(t)$ and $q(x)$, not both constant, and a constant C such that (2.5) *holds for all non-negative f? For which $p(t)$ and $q(x)$ does such a C exist?*

3. More General Limits of Integration

In [3] and [1] the operator

$$\int_{a(x)}^{b(x)} f(t)\,dt \tag{3.1}$$

is studied, where a and b are non-decreasing functions with $a(x) \le b(x)$. The results in [3] are for increasing, differentiable a and b and include necessary and sufficient conditions for the boundedness of the operator from $L^p_v(0,\infty)$ to $L^q_u(0,\infty)$ for $1 < p < \infty$ and $0 < q < \infty$ while the results of [1] are for all a and b described above and include necessary and sufficient conditions for the operator and related operators to be bounded between pairs of Banach function spaces satisfying Berezhnoi's ℓ-condition. Such pairs include $(L^p_v(0,\infty), L^q_u(0,\infty))$ for $1 < p \le q < \infty$ but not for $0 < q < p$, $1 < p < \infty$.

Question 3.1. *Can the monotonicity restriction on a and b be removed?*

This question seems to be a difficult one but we are able to characterize the boundedness from $L^p_v(0,\infty)$ to $L^q_u(0,\infty)$ for $1 < p < \infty$ and $0 < q < \infty$ of the operator (3.1) with $a = 0$ and b non-negative and measurable.

Theorem 3.2. *Suppose b is a non-negative, measurable function, $0 < q < \infty$, $1 < p < \infty$, and u and v are weights. Suppose also that either $q > 1$ or $v^{1-p'}$ is locally integrable on $[0,\infty)$. Then*

$$\left(\int_0^\infty \left(\int_0^{b(x)} f(t)\,dt\right)^q u(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{1/p} \tag{3.2}$$

holds for all non-negative functions f if and only if
either $p \le q$ and

$$\sup_{y>0}\left(\int_0^y v^{1-p'}\right)^{1/p'}\left(\int_{\{x:b(x)\ge y\}} u(x)\,dx\right)^{1/q} < \infty,$$

or $q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_0^\infty \left(\int_{\{x:b(x)\ge y\}} u(x)\,dx\right)^{r/q}\left(\int_0^y v^{1-p'}\right)^{r/q'} v(y)^{1-p'}\,dy\right)^{1/r} < \infty.$$

Before we prove Theorem 3.2 we need the following lemma.

Lemma 3.3. *Let b be a non-negative, measurable function, u be a non-negative, integrable function, and $q > 0$. Then there exists a regular Borel measure μ on $(0,\infty)$ such that*

$$\int_y^\infty d\mu = \int_{\{x:b(x)\ge y\}} u(x)\,dx \tag{3.3}$$

and

$$\int_0^\infty \left(\int_0^{b(x)} f(t)\,dt\right)^q u(x)\,dx = \int_0^\infty \left(\int_0^z f(t)\,dt\right)^q d\mu(z)$$

for all non-negative functions f.

Proof. Since u is integrable, the expression $\int_{\{x:b(x)\ge y\}} u(x)\,dx$ is a non-negative, non-increasing function of y which tends to zero as y tends to infinity. Using [6, Theorem 12, page 262] we see that there exists a finite Borel measure μ satisfying (3.3). Following the construction in the book it is an exercise to show that μ is regular. To complete the proof we calculate as follows:

$$\begin{aligned}
&\int_0^\infty \left(\int_0^{b(x)} f(t)\,dt\right)^q u(x)\,dx \\
&= \int_0^\infty \int_0^{b(x)} q\left(\int_0^t f(s)\,ds\right)^{q-1} f(t)\,dt\,u(x)\,dx \\
&= \int_0^\infty \int_0^\infty q\left(\int_0^t f(s)\,ds\right)^{q-1} f(t)u(x)\chi_{(0,b(x))}(t)\,dx\,dt \\
&= \int_0^\infty q\left(\int_0^t f(s)\,ds\right)^{q-1} f(t)\int_{\{x:t\le b(x)\}} u(x)\,dx\,dt \\
&= \int_0^\infty q\left(\int_0^t f(s)\,ds\right)^{q-1} f(t)\int_t^\infty d\mu(z)\,dt \\
&= \int_0^\infty \int_0^z q\left(\int_0^t f(s)\,ds\right)^{q-1} f(t)\,dt\,d\mu(z) \\
&= \int_0^\infty \left(\int_0^z f(t)\,dt\right)^q d\mu(z).
\end{aligned}$$

Proof of Theorem 3.2. It is enough to establish the claim in the case that u is integrable since the general result then follows using the Monotone Convergence Theorem. If μ is the regular Borel measure given by Lemma 3.3, inequality (3.2) becomes

$$\left(\int_0^\infty \left(\int_0^z f(t)\,dt\right)^q d\mu(z)\right)^{1/q} \le C\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{1/p}.$$

Using Proposition 0.1 and the remark on page 93 in [9] we see that this inequality holds if and only if

either $p \le q$ and

$$\sup_{y>0}\left(\int_0^y v^{1-p'}\right)^{1/p'}\left(\int_y^\infty d\mu\right)^{1/q} < \infty,$$

or $q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_0^\infty \left(\int_y^\infty d\mu\right)^{r/q}\left(\int_0^y v^{1-p'}\right)^{r/q'} v(y)^{1-p'}\,dy\right)^{1/r} < \infty.$$

Replacing μ by u according to (3.3) completes the proof.

If a and b are similarly ordered in the sense of [2, page 43] the argument of Theorem 3.2 should extend to the operator (3.1). If not then Question 3.1 is quite a different sort of problem than the Hardy operator because the sections of the kernel are no longer a totally ordered set.

4. The Higher Order Hardy Inequality with One Weight Fixed

We look at the inequality

$$\left(\int_0^1 \left(\int_0^x (x-t)^k f(t)\,dt\right)^q d\mu(x)\right)^{1/q} \le C\left(\int_0^1 f(t)^p v(t)\,dt\right)^{1/p}, \tag{4.1}$$

for a fixed weight v. Here k is a non-negative integer, $1 < p < \infty$, and $0 < q < \infty$. The weights for which (4.1) holds have been characterized, see [10].

Let W_k denote the collection of those non-negative, regular measures μ for which there exists a constant C such that (4.1) holds for all non-negative f.

Definition 4.1. *If μ_1 and μ_2 are non-negative, regular measures on $[0,1]$ we say that $\mu_1 \preceq_k \mu_2$ provided*

$$\int_y^1 (y-x)^k\,d\mu_1(x) \le \int_y^1 (y-x)^k\,d\mu_2(x)$$

for all $y \in [0,1]$.

There is a natural connection between the partial order $\preceq_k$ and the class W_k.

Lemma 4.2. *Suppose $1 < p < \infty$, $0 < q < \infty$, and $0 \le k \le q$. If $\mu_1 \in W_k$ and $\mu_2 \preceq_k c\mu_1$ for some $c > 0$ then $\mu_2 \in W_k$.*

Proof. It is enough to verify (4.1), with μ replaced by μ_2, for continuous functions f since they are dense in $L^p_v(0,1)$. For a non-negative, continuous function f it is easy to check that, since $q \ge k$,

$$F(x) = \left(\int_0^x (x-t)^k f(t)\,dt\right)^q$$

satisfies

$$F^{(j)}(x) \ge 0 \text{ for } 0 \le j \le k+1 \text{ and } F^{(j)}(0) = 0 \text{ for } 0 \le j \le k.$$

Therefore

$$F(x) = \int_0^x (x-t)^k F^{(k+1)}(t)/(k!)\,dt.$$

We have

$$\begin{aligned}
\left(\int_0^1 F(x)\,d\mu_2(x)\right)^{1/q} &= \left(\int_0^1 \int_t^1 (x-t)^k\,d\mu_2(x) F^{(k+1)}(t)/(k!)\,dt\right)^{1/q} \\
&\le c^{1/q}\left(\int_0^1 \int_t^1 (x-t)^k\,d\mu_1(x) F^{(k+1)}(t)/(k!)\,dt\right)^{1/q} \\
&= c^{1/q}\left(\int_0^1 F(x)\,d\mu_1(x)\right)^{1/q} \\
&\le c^{1/q} C\left(\int_0^1 f(t)^p v(t)\,dt\right)^{1/p}.
\end{aligned}$$

which completes the proof.

In the case $p \le q$ the weight class W_0 has a largest element (up to constant multiples) with respect to the partial order $\preceq_0$. In view of Theorem 4.2 this shows that W_0 is completely determined by this maximum element. Moreover, the maximum measure can be expressed in terms of p, q, and the fixed weight v. Define the measure ω_0 by

$$d\omega_0(x) = (q/p')\left(\int_0^x v^{1-p'}\right)^{-1-q/p'} v(x)^{1-p'}\,dx + \left(\int_0^1 v^{1-p'}\right)^{-q/p'} d\delta_1(x).$$

Here δ_1 is the Dirac measure at 1.

Theorem 4.3. *Suppose that $1 < p \le q < \infty$. Then $\mu \in W_0$ if and only if $\mu \preceq c\omega_0$ for some $c \ge 0$.*

Proof. The measure ω_0 was defined so that

$$\begin{aligned}\int_y^1 d\omega_0(x) &= -\left(\int_0^x v(t)^{1-p'}\,dt\right)^{-q/p'}\Bigg|_y^1 + \left(\int_0^1 v(t)^{1-p'}\,dt\right)^{-q/p'}\\ &= \left(\int_0^y v(t)^{1-p'}\,dt\right)^{-q/p'}\end{aligned}$$

which shows that

$$\left(\int_y^1 d\omega_0(x)\right)^{1/q}\left(\int_0^y v(t)^{1-p'}\,dt\right)^{1/p'} = 1.$$

By Proposition 0.1 the Hardy inequality (4.1) (with $k = 0$) holds with μ replaced by ω_0. That is, $\omega_0 \in W_0$. Theorem 4.2 shows that if $\mu \preceq_0 c\omega_0$ for some $c \ge 0$ then $\mu \in W_0$.

To prove the converse we suppose that $\mu \in W_0$, fix $y \in (0,1)$, and substitute $f(t) = v(t)^{1-p'}\chi_{(0,y)}(t)$ into (4.1) to see that

$$\left(\int_y^1 \left(\int_0^y v(t)^{1-p'}\,dt\right)^q d\mu(x)\right)^{1/q} \le C\left(\int_0^y v(t)^{1-p'}\,dt\right)^{1/p}.$$

It follows that

$$\int_y^1 d\mu(x) \le C^q\left(\int_0^y v(t)^{1-p'}\,dt\right)^{-q/p'} = C^q\int_y^1 d\omega_0(x)$$

so that $\mu \preceq_0 C^q\omega_0$ as required.

Question 4.4. *Is there a measure ω_k such that $\mu \in W_k$ if and only if $\mu \preceq_k c\omega_k$ for some constant $c \geq 0$?*

We have already answered the question in the case $k = 0$ and $1 < p \leq q < \infty$ and we can also answer it in the case $k = 0$ and $0 < q < p$, $1 < p < \infty$. The next Theorem shows that if $q < p$ then W_0 has no maximal element.

Theorem 4.5. *Suppose that $0 < q < p$, $1 < p < \infty$ and that either $q > 1$ or $v^{1-p'}$ is locally integrable. If $\mu \in W_0$ then there exists a measure $\mu^+ \in W_0$ such that $\mu^+ \not\preceq_0 c\mu$ for any constant $c \geq 0$.*

Proof. Proposition 0.1 and the remark on page 93 of [9] shows that (4.1) holds if and only if

$$\left(\int_0^1 \left(\int_y^1 d\mu\right)^{r/q} \left(\int_0^y v^{1-p'}\right)^{r/q'} v(y)^{1-p'}\,dy\right)^{1/r} < \infty,$$

where r is defined by $1/r = 1/q - 1/p$. If we set $V(y) = \left(\int_a^y v^{1-p'}\right)^{r/q'} v(y)^{1-p'}$ then we see that $\mu \in W_0$ if and only if $\int_0^1 \left(\int_y^1 d\mu\right)^{r/q} V(y)\,dy < \infty$.

Fix a measure $\mu \in W_0$. Our object is to construct a measure μ^+ such that $\int_0^1 \left(\int_y^1 d\mu^+\right)^{r/q} V(y)\,dy < \infty$ and $\int_y^1 d\mu^+ / \int_y^1 d\mu$ is an unbounded function of y.

Set $F(y) = \left(\int_y^1 d\mu\right)^{r/q}$. Since $V(y)\,dy$ is non-atomic we can choose a decreasing sequence $y_0 = 1, y_1, y_2, \ldots$, converging to 0, such that $\int_0^{y_k} FV = 2^{-k}\int_0^1 FV$. Now let h be the function whose graph is the polygonal path connecting the points (y_k, k). Clearly, h is a continuous, non-increasing function on $(0,1)$. Since $(hF)^{q/r}$ is non-decreasing there exists a Borel measure μ^+ satisfying $\int_y^1 d\mu^+ = (h(y)F(y))^{q/r}$ for almost every y. Because h is unbounded it is clear that $\mu^+ \not\preceq_0 \mu$ and for the other requirement we estimate as follows.

$$\int_0^1 \left(\int_y^1 d\mu^+\right)^{r/q} V(y)\,dy = \int_0^1 hFV \leq \sum_{k=1}^{\infty} k \int_{y_k}^{y_{k-1}} FV = \sum_{k=1}^{\infty} k2^{-k} \int_0^1 FV < \infty.$$

This completes the proof.

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