SCHUR'S LEMMA AND BEST CONSTANTS IN WEIGHTED NORM INEQUALITIES

Gord Sinnamon

The University of Western Ontario

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Abstract. Strong forms of Schur's Lemma and its converse are proved for maps taking non-negative functions to non-negative functions and having formal adjoints. These results are applied to give best constants in a large class of weighted Lebesgue norm inequalities for non-negative integral operators. Since general measures are used, norms of non-negative matrix operators may be calculated by the same method.

1. Introduction

Schur's Lemma is primarily a way of establishing the boundedness of integral operators with non-negative kernels, or of matrix operators with non-negative entries, between Lebesgue spaces. As a sufficient condition [1, 3, 4, 7, 11] it has been proved in many different forms and applied in a great many situations. In a typical application, the lemma deduces the boundedness of the operator in question from the hypothesis that there exist a positive function, or sequence, satisfying a certain inequality. The clever choice of such a function, or sequence, then finishes the proof. After a great many clever choices have been made, one begins to suspect that there is always some choice that will serve. The converses to Schur's Lemma [2, 3, 5, 7, 8, 9, 10, 13] assert just that.

Naturally enough, Schur's Lemma provides not only boundedness but also an estimate of the norm of the integral or matrix operator. The various converses generally show that the actual norm can be approximated arbitrarily closely by such estimates. The question of whether or not the operator norm can be reached has been addressed for sequences in [2, 9] and for certain integral operators in [3, 4, 7].

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In this paper we prove strong forms of Schur's Lemma and its converse for maps between non-negative functions on general measure spaces. The maps are only required to have formal adjoints so they include integral operators with non-negative kernels, non-negative matrix operators, composition operators, and multiplication operators. These results are given in Section 2 but some of the proofs have been deferred to Section 4 where we also give an iterative procedure for approximating the norm of such an operator. For matrix operators, the procedure is easily implemented on a computer and convergence is rapid. The delicate iteration introduced in the proofs of Lemma 4.3 and Theorem 2.5 may be of independent interest as a fixed point result.

In Section 3, Schur's Lemma and its converse are applied to establish best constants for a large class of weighted Lebesgue norm inequalities, including essentially all such inequalities for non-negative integral operators when the Lebesgue index in the domain space is larger than the index in the codomain. The method of generating inequalities with best constants is quite simple and the calculations can be readily carried out by hand, with a computer algebra system, or numerically. Several examples are given. Although these specific examples may be of direct interest to the specialist, they are included here simply as applications of Theorem 3.1 and illustrations of the method. The reader is encouraged to start with a favorite positive operator, select a domain space weight, fix a positive function, and see how simple it is to generate a weighted norm inequality with best constant.

A great deal of progress has been made recently in the understanding of weighted norm inequalities but the focus has been on establishing the finiteness of the constant rather than its best (smallest) possible value. Boyd's work [3, 4] as well as that of Howard and Schep [7] are exceptions to this and do find best constants. There is some overlap between Boyd's approach and this one but the methods and objectives are quite different. Some of the standard proofs of Schur and Gagliardo appear again here in no greater generality than in Howard and Schep's work but in a somewhat different context.

Throughout the paper we work with the extended real numbers under the convention that $0 \cdot \infty = 0$. The dual of the Lebesgue index p is denoted p' so that $1/p + 1/p' = 1$.

Schur's Lemma and its converse

Let L^+_μ and L^+_ν denote the non-negative, extended real valued functions on the measure spaces $(X, d\mu)$ and $(Y, d\nu)$ respectively. We say that a map $T : L^+_\nu \to L^+_\mu$ has a *formal adjoint* $T^* : L^+_\mu \to L^+_\nu$ provided

$$\int (Tf)g \, d\mu = \int f(T^*g) \, d\nu$$

for all $f \in L^+_\nu$ and $g \in L^+_\mu$. For fixed indices p and q with $1 < q \le p < \infty$ we define $\|T\|$ by

$$\|T\| = \sup\{\|Tf\|_{L^q_\mu} : f \in L^+_\nu, \|f\|_{L^p_\nu} \le 1\}$$

and the map $S : L_\nu^+ \to L_\nu^+$ by

$$S\varphi = (T^*((T\varphi)^{q-1}))^{p'-1}.$$

An *extremal* function for T from L_ν^p to L_μ^q is a non-zero function $f \in L_\nu^+$ satisfying $\|Tf\|_{L_\mu^q} = \|T\|\|f\|_{L_\nu^p} < \infty$.

Although the results of this section apply to any map T having a formal adjoint we are most interested in the class of non-negative integral operators. If $k(x,t)$ is a non-negative, extended real valued, $\mu \times \nu$-measurable function then the maps $T_k : L_\nu^+ \to L_\mu^+$ and $T_k^* : L_\mu^+ \to L_\nu^+$ defined by

$$T_k f(x) = \int_Y k(x,y) f(y)\, d\nu(y) \quad \text{ and } \quad T_k^* g(y) = \int_X k(x,y) g(x)\, d\mu(x)$$

are easily seen to be formal adjoints. Moreover, $\|T\|$ is just the usual norm of T_k considered as a linear transformation from L_ν^p to L_μ^q. The (non-linear) operator S_k corresponding to S is

$$S_k\varphi(y) = \left(\int_Y k(x,y) \left(\int_X k(x,z)\varphi(z)\, d\nu(z) \right)^{q-1} d\mu(x) \right)^{p'-1}.$$

Maps having formal adjoints inherit positive homogeneity, additivity, monotonicity, and Hölder's inequality from integration.

Lemma 2.1. *Suppose that $T : L_\nu^+ \to L_\mu^+$ has a formal adjoint, that $a \geq 0$, and that $f_1, f_2 \in L_\nu^+$. Then $T(af_1) = aTf_1$, $T(f_1 + f_2) = Tf_1 + Tf_2$, $Tf_1 \leq Tf_2$ whenever $f_1 \leq f_2$, and if $1 < q < \infty$ then $T(f_1 f_2) \leq [T(f_1^q)]^{1/q}[T(f_2^{q'})]^{1/q'}$. Also, the formal adjoint of T is unique.*

Proof. Standard arguments show that if $g_1, g_2 \in L_\mu^+$ and $\int g_1 g\, d\mu \leq \int g_2 g\, d\mu$ for all $g \in L_\mu^+$ then $g_1 \leq g_2$ μ-almost everywhere. Consequently, if $g_1, g_2 \in L_\mu^+$ and $\int g_1 g\, d\mu = \int g_2 g\, d\mu$ for all $g \in L_\mu^+$ then $g_1 = g_2$ μ-almost everywhere.

Let T^* be a formal adjoint of T. For all $g \in L_\mu^+$,

$$\int T(af_1) g\, d\mu = \int af_1 T^* g\, d\nu = a \int f_1 T^* g\, d\nu = \int aTf_1 g\, d\mu$$

so we have $T(af_1) = aTf_1$. In just the same way we show that $T(f_1 + f_2) = Tf_1 + Tf_2$. If $f_1 \leq f_2$ and $g \in L_\mu^+$ then

$$\int Tf_1 g\, d\mu = \int f_1 T^* g\, d\nu \leq \int f_2 T^* g\, d\nu = \int Tf_2 g\, d\mu$$

so $Tf_1 \leq Tf_2$.

The analogue of Hölder's inequality is proved using the positive homogeneity and additivity of T and the well-known inequality $AB \leq (1/q)A^q + (1/q')B^{q'}$ for

$A, B \geq 0$ but first we must dispense with the case where the right hand side is zero. Let $g \in L^+_\mu$ be supported on the set where $[T(f_1^q)]^{1/q}[T(f_2^{q'})]^{1/q'}$ vanishes and write $g = g_1 + g_2$ where $T(f_1^q)g_1 = 0$ and $T(f_2^{q'})g_2 = 0$. We have

$$0 = \int T(f_1^q) g_1 \, d\mu = \int f_1^q T^* g_1 \, d\nu$$

and hence $\int f_1 f_2 T^* g_1 \, d\nu = 0$. Similarly, $\int f_1 f_2 T^* g_2 \, d\nu = 0$. Putting these together yields

$$\begin{aligned}\int T(f_1 f_2) g \, d\mu &= \int T(f_1 f_2) g_1 \, d\mu + \int T(f_1 f_2) g_2 \, d\mu \\ &= \int f_1 f_2 T^* g_1 \, d\nu + \int f_1 f_2 T^* g_2 \, d\nu = 0.\end{aligned}$$

Since the only restriction on g was its support, we see that $T(f_1 f_2)$ vanishes whenever $[T(f_1^q)]^{1/q}[T(f_2^{q'})]^{1/q'}$ does.

It remains to establish the analogue of Hölder's inequality on the set where both $T(f_1^q)$ and $T(f_2^{q'})$ are positive. If either is infinite then there is nothing to prove so

$$\alpha_1 \equiv [T(f_1^q)]^{-1/q} \in (0, \infty) \quad \text{and} \quad \alpha_2 \equiv [T(f_2^{q'})]^{-1/q'} \in (0, \infty).$$

Now,

$$\begin{aligned}\alpha_1 \alpha_2 T(f_1 f_2) = T(\alpha_1 f_1 \alpha_2 f_2) &\leq T((\alpha_1 f_1)^q/q + (\alpha_2 f_2)^{q'}/q') \\ &= (\alpha_1^q/q) T(f_1^q) + (\alpha_2^{q'}/q') T(f_2^{q'}) = 1/q + 1/q' = 1\end{aligned}$$

and we have $T(f_1 f_2) \leq [T(f_1^q)]^{1/q}[T(f_2^{q'})]^{1/q'}$ as desired.

If $T^\#$ is also a formal adjoint of T and $g \in L^+_\mu$ then for all $f \in L^+_\nu$,

$$\int f T^\# g \, d\nu = \int T f g \, d\mu = \int f T^* g \, d\nu.$$

Thus $T^\# g = T^* g$ and so the formal adjoint is unique.

Our first version of Schur's Lemma is in the next theorem. Although it applies to any map having a formal adjoint it is essentially the standard result: If the positive function φ satisfies the appropriate inequality then the map T is bounded and an estimate of the norm of T is given.

Theorem 2.2. *Let $1 < q \leq p < \infty$. Suppose that $T : L^+_\nu \to L^+_\mu$ has a formal adjoint $T^* : L^+_\mu \to L^+_\nu$ and that there exist an $A > 0$ and a positive, ν-measurable function φ which is finite ν-almost everywhere and satisfies $S\varphi \leq A\varphi$ ν-almost everywhere in Y. If $p = q$ then*

$$\|T\| \leq A^{(p-1)/p}.$$

If $q < p$ then

$$\|T\| \leq A^{(p-1)/q} \|\varphi\|_{L^p_\nu}^{(p-q)/q}.$$

Proof. Let $f \in L^+_\nu$. The hypotheses on φ ensure that $\varphi^{-1/q'}\varphi^{1/q'} = 1$ ν-almost everywhere. Using the analogue of Hölder's inequality given in Lemma 2.1, we have

$$\|Tf\|^q_{L^p_\nu} = \int [T(f\varphi^{-1/q'}\varphi^{1/q'})]^q \, d\mu \leq \int T(f^q\varphi^{1-q})(T\varphi)^{q-1} \, d\mu.$$

Introducing the formal adjoint, T^*, the last expression becomes

$$\int f^q \varphi^{1-q} T^*((T\varphi)^{q-1}) \, d\nu = \int f^q\varphi^{1-q}(S\varphi)^{p-1} \, d\nu \leq A^{p-1} \int f^q \varphi^{p-q} \, d\nu.$$

Here we have used the definition of S and the hypothesis $S\varphi \leq A\varphi$.

If $q = p$ this simplifies to $\|Tf\|^p_{L^p_\mu} \leq A^{p-1}\|f\|^p_{L^p_\nu}$ and, since f was arbitrary, $\|T\| \leq A^{(p-1)/p}$ as required. If $q < p$ we apply Hölder's inequality with indices p/q and $p/(p-q)$ to get

$$\|Tf\|^q_{L^q_\mu} \leq A^{p-1} \left(\int f^p \, d\nu\right)^{q/p} \left(\int \varphi^p \, d\nu\right)^{(p-q)/p} = A^{p-1}\|\varphi\|^{p-q}_{L^p_\nu}\|f\|^q_{L^p_\nu}$$

and, since f was arbitrary, we conclude that $\|T\| \leq A^{(p-1)/q}\|\varphi\|^{(p-q)/q}_{L^p_\nu}$ to complete the proof.

In our second version we make stronger hypotheses and get a stronger conclusion.

Theorem 2.3. *Let $1 < q \leq p < \infty$. Suppose that $T : L^+_\nu \to L^+_\mu$ has a formal adjoint $T^* : L^+_\mu \to L^+_\nu$ and that there exist an $A > 0$ and a positive, ν-measurable function $\varphi \in L^p_\nu$ which satisfies $S\varphi = A\varphi$ ν-almost everywhere in Y. Then*

$$\|T\| = A^{(p-1)/q}\|\varphi\|^{(p-q)/q}_{L^p_\nu}, \tag{2.1}$$

the constant multiples of φ are extremal functions for T from L^p_ν to L^q_μ and if $q < p$ they are the only ones.

Proof. Since $\varphi \in L^p_\nu$ it is necessarily finite ν-almost everywhere. Thus Theorem 2.2 applies and we have $\|T\| \leq A^{(p-1)/q}\|\varphi\|^{(p-q)/q}_{L^p_\nu}$ for $1 < q \leq p < \infty$. (Since $\varphi \in L^p_\nu$ the second factor vanishes in the case $q = p$.) To prove the reverse inequality we use the definitions of S and $\|T\|$.

$$\begin{aligned} A^{p-1}\|\varphi\|^p_{L^p_\nu} &= \int \varphi(A\varphi)^{p-1} \, d\nu = \int \varphi(S\varphi)^{p-1} \, d\nu = \int \varphi T^*((T\varphi)^{q-1}) \, d\nu \\ &= \int T\varphi(T\varphi)^{q-1} \, d\mu = \int (T\varphi)^q \, d\mu \leq \|T\|^q\|\varphi\|^q_{L^p_\nu}. \end{aligned}$$

This may be rearranged to yield $\|T\| \geq A^{(p-1)/q}\|\varphi\|_{L^p_\nu}^{(p-q)/q}$. Not only do we have (2.1) but we also see that the inequality in the above calculation is an equality. That is, $\|T\varphi\|_{L^q_\mu} = \|T\|\|\varphi\|_{L^p_\nu}$, and therefore φ (and its constant multiples) are extremal functions for T from L^p_ν to L^q_μ.

If $q < p$ and f is an extremal function for T from L^p_ν to L^q_μ then we consider the argument of Theorem 2.2 applied to f. Since f is extremal, all inequalities necesarily become equalities. In particular, the application of Hölder's inequality that gave

$$\int f^q \varphi^{p-q}\, d\nu \leq \left(\int f^p\, d\nu\right)^{q/p} \left(\int \varphi^p\, d\nu\right)^{(p-q)/p}$$

is an equality. This occurs only when f is a constant multiple of φ.

The next two theorems explore the extent to which these versions of Schur's Lemma are reversible. First we note that if (Y, ν) is not σ-finite then there is no positive function in L^p_ν so, with the possible exception of the case $q = p$ in Theorem 2.2, the above results hold vacuously and no converse is to be expected. It may be possible to give a partial converse to Theorem 2.2 in the case $q = p$ with (Y, ν) non-σ-finite but at very least it would have to be assumed that Y contain no infinite atoms. We will not investigate this possibility further here.

The iteration used to prove Theorem 2.4 is essentially given in [5, 13]. A refinement of this iteration scheme is used to prove Theorem 2.5. Both proofs will be deferred to Section 4.

Theorem 2.4. *Suppose that (Y, ν) is σ-finite, $1 < q = p < \infty$, $T : L^+_\nu \to L^+_\mu$ has a formal adjoint $T^* : L^+_\mu \to L^+_\nu$ and $\|T\| < \infty$. Then for every $\varepsilon > 0$ there exists an $A > 0$ and a positive $\varphi \in L^p_\nu$ such that $S\varphi \leq A\varphi$ and*

$$-\varepsilon + A^{(p-1)/p} \leq \|T\| \leq A^{(p-1)/p}.$$

The next theorem gives the converse of Theorem 2.2 in the case $q < p$ and shows that A may be taken to be 1. It also gives the converse to Theorem 2.3 in the case $q < p$. In Example 2.7 it is shown that a converse to Theorem 2.3 is not possible when $q = p$.

In addition to the σ-finiteness of (Y, ν) there is another mild but necessary assumption for a converse to Theorem 2.3:

$$T^* g > 0 \text{ whenever } g > 0. \tag{2.2}$$

For the integral operator T_k, this asserts that $k(x, y)$ does not vanish identically on any tube $X \times Y_0$ with $\nu(Y_0) > 0$ so that the domain of T_k is not artificially too large. Stated in terms of the map T, (2.2) becomes:

There is no set of positive ν measure such
that $Tf = 0$ for all $f \in L^+_\nu$ supported there.

The assumption is necessary because if there were such a set then for all f supported there and all $g \in L^+_\mu$ we would have

$$\int fT^*g\, d\nu = \int Tfg\, d\mu = 0$$

so $T^*g = 0$ on the set. If $S\varphi = A\varphi$ for some positive real number A then $\varphi = A^{-1}(T^*((T\varphi)^{q-1}))^{p'-1}$ would be zero on the set and hence φ could not be positive and there could be no converse to Theorem 2.3. We view the condition (2.2) as a non-degeneracy condition on the map T.

Theorem 2.5. *Suppose that (Y, ν) is σ-finite, $1 < q < p < \infty$, $T : L^+_\nu \to L^+_\mu$ has a formal adjoint $T^* : L^+_\mu \to L^+_\nu$ and $\|T\| < \infty$. Then for every $\varepsilon > 0$ there exists a positive φ, finite ν-almost everywhere, such that $S\varphi \le \varphi$ and*

$$-\varepsilon + \|\varphi\|^{(p-q)/q}_{L^p_\nu} \le \|T\| \le \|\varphi\|^{(p-q)/q}_{L^p_\nu}.$$

If (2.2) *is satisfied then there exists a positive function φ such that $S\varphi = \varphi$ and*

$$\|T\| = \|\varphi\|^{(p-q)/q}_{L^p_\nu}.$$

Combining the last statement of Theorem 2.5 with the last statement of Theorem 2.3 gives the following interesting result.

Corollary 2.6. *Suppose $1 < q < p < \infty$ and (Y, ν) is σ-finite. Every non-negative integral operator satisfying* (2.2) *that is bounded from L^p_ν to L^q_μ has a unique extremal function up to constant multiples. This function is never zero nor does it change sign.*

Example 2.7. *Hardy's inequality,* [6, Theorem 327]*, is a strict inequality. If $p > 1$ and $f \ge 0$ then*

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\, dy\right)^p dx\right)^{1/p} < p' \left(\int_0^\infty f(y)^p\, dy\right)^{1/p}$$

unless $f \equiv 0$. The constant is best possible.

This inequality has no positive extremal function so Theorem 2.5 and Corollary 2.6 do not extend to the case $q = p$.

Remark. The case $q = p$ in Theorem 2.3 deserves further study. The hypothesis that $\varphi \in L^p_\nu$ is too restrictive. It is not required to get an upper bound on the norm of the operator as we saw in Theorem 2.2. If this assumption can be weakened (at least finiteness ν-almost everywhere is required) then there may be a converse of a sort. The results of [2, 9] show that for matrix operators there is always a positive extremal sequence but that equality in $S\varphi \le A\varphi$ is not necessarily achieved.

The difficulty arises with operators having a direct sum decomposition. Perron showed that a (finite) matrix with positive entries has a unique positive eigenvalue but to extend the result to matrices with non-negative entries Frobenius had to impose the condition that the matrix be indecomposable. This observation is at the root of the difficulty with the case $q = p$.

We seem to be faced with two choices. Either restrict our attention to operators with no direct sum decomposition, or give up the uniqueness of the extremal and the equality in $S\varphi \le A\varphi$. We hope to return to this dilemma in future work.

3. Weighted Norm Inequalities

A weighted norm inequality for the map $K : L^+_\eta \to L^+_\xi$ is an inequality of the form

$$\left(\int (KF)^q u\, d\xi\right)^{1/q} \le C\left(\int F^p v\, d\eta\right)^{1/p}. \tag{3.1}$$

Here (X, ξ) and (Y, η) are measure spaces, $F \in L^+_\eta$, and the weights u and v are fixed functions in L^+_ξ and L^+_η repectively. The best constant in (3.1) is the smallest constant C, finite or infinite, such that the inequality holds for all $F \in L^+_\eta$.

It may seem redundant to include the weight functions u and v as well as the measures ξ and η in inequality (3.1). After all, multiplying a measure by a weight function just gives another measure. However, the measures ξ and η provide the inner products necessary for the definition of the formal adjoint and the weights allow us to consider the action of the operator on Lebesgue spaces with different measures than those used for the inner products. In order to preserve the integrability of the functions in L^+_ξ and L^+_η with respect to different measures we naturally expect the new measures to be absolutely continuous with respect to the old ones. The Radon-Nikodym Theorem leads us to the weighted measures $u\xi$ and $v\eta$.

To avoid certain exceptional cases we assume throughout this section that the weight v is positive. If this is not the case we simply replace the space Y underlying η by the support of v.

Taken together, the first two theorems of this section give a formula for the best constant in all weighted norm inequalities for non-degenerate maps with formal adjoints in the case $1 < q < p < \infty$. This includes weighted norm inequalities for integral operators with non-negative kernels. In the case $q = p$, best constants are given for a large class of weighted norm inequalities but Example 2.7 has shown that not all weighted norm inequalities are included.

So that we may easily transfer the results of the previous section to the investigation here, we make the following identifications: If u, v, ξ, η, K, and F are as above and K^* is a formal adjoint of K then

$$F = fv^{1-p'},\ \mu = u\xi,\ \nu = v^{1-p'}\eta,\ Tf = K(fv^{1-p'}),\ \text{and } T^*g = K^*(gu). \tag{3.2}$$

Note that $T : L^+_\nu \to L^+_\mu$ and $T^* : L^+_\mu \to L^+_\nu$ are formal adjoints because

$$\int (Tf)g\, d\mu = \int K(fv^{1-p'})gu\, d\xi = \int fv^{1-p'}K^*(gu)\, d\eta = \int fT^*(g)\, d\nu,$$

whenever $f \in L^+_\nu$ and $g \in L^+_\mu$.

The assumption that v does not vanish enables us to show that for each $C > 0$, (3.1) holds for all $F \in L^+_\eta$ if and only if

$$\left(\int (Tf)^q \, d\mu\right)^{1/q} \le C \left(\int f^p \, d\nu\right)^{1/p} \tag{3.3}$$

holds for all $f \in L^+_\nu$. We conclude that the best constant in (3.1) and the best constant in (3.3) coincide.

Theorem 3.1. *Suppose that $1 < q \le p < \infty$, $K : L^+_\eta \to L^+_\xi$ has a formal adjoint $K^* : L^+_\xi \to L^+_\eta$, and $0 < v \in L^+_\eta$. Let h be in L^+_ξ, set $\varphi = [K^*h]^{p'-1}$ and $u = [K(\varphi v^{1-p'})]^{1-q}h$. If $0 < \varphi \in L^p_{(v^{1-p'})\eta}$ then the best constant in* (3.1) *is*

$$C = \left(\int \varphi^p v^{1-p'} \, d\eta\right)^{(p-q)/(pq)}.$$

Proof. The best constant in (3.1) is also the best constant in (3.3) which was earlier denoted $\|T\|$. Therefore the desired result will follow from Theorem 2.3 once we show that $S\varphi = \varphi$:

$$S\varphi = [T^*([T\varphi]^{q-1})]^{p'-1} = [K^*([K(\varphi v^{1-p'})]^{q-1}u)]^{p'-1} = [K^*h]^{p'-1} = \varphi.$$

For the converse, in the case $q < p$, we impose non-degeneracy conditions similar to (2.2) on both K and K^*:

$$\begin{gathered} K(fv^{1-p'}) > 0 \text{ whenever } f > 0, \text{ and} \\ K^*(gu) > 0 \text{ whenever } g > 0. \end{gathered} \tag{3.4}$$

Theorem 3.2. *Suppose that $1 < q < p < \infty$, (Y, η) is σ-finite, $K : L^+_\eta \to L^+_\xi$ has a formal adjoint $K^* : L^+_\xi \to L^+_\eta$, $u \in L^+_\xi$, $0 < v \in L^+_\eta$, and* (3.4) *holds. If the best constant in* (3.1) *is finite then there exists a function $h \in L^+_\xi$ such that $0 < \varphi = (K^*h)^{p'-1} \in L^p_{(v^{1-p'})\eta}$ and $u = [K(\varphi v^{1-p'})]^{1-q}h$.*

Proof. The best constant in (3.1), and hence in (3.3), is finite which means that $\|T\| < \infty$. Note that since $v > 0$, the σ-finiteness of (Y, η) implies that (Y, ν) is also σ-finite. Also note that (3.4) implies (2.2). By Theorem 2.5 there exists a positive $\varphi \in L^p_{(v^{1-p'})\eta}$ with $S\varphi = \varphi$. Set $h = [K(\varphi v^{1-p'})]^{q-1}u$ and we immediately have

$$\varphi = S\varphi = [T^*([T\varphi]^{q-1})]^{p'-1} = [K^*([K(\varphi v^{1-p'})]^{q-1}u)]^{p'-1} = (K^*h)^{p'-1}.$$

If $0 < K(\varphi v^{1-p'}) < \infty$ then we may divide by it in the definition of h to see that $u = [K(\varphi v^{1-p'})]^{1-q}h$. Since $\varphi > 0$, (3.4) implies $K(\varphi v^{1-p'}) > 0$ and since

$\varphi \in L^p_{(v^{1-p'})\eta}$ we have $\varphi v^{1-p'} \in L^p_{v\eta}$ and the finiteness of the constant in (3.1) shows that $K(\varphi v^{1-p'}) \in L^q_{u\xi}$. Thus, on the set where $K(\varphi v^{1-p})$ is infinite u vanishes ξ-almost everywhere and so in this case, too, $u = (K(\varphi v^{1-p'}))^{1-q} h$.

Theorem 3.1 gives a simple method of generating inequalities with best constants. We give several examples to illustrate the ease and versatility of the method. The details of calculating the weight u and the norm of φ have been omitted.

We begin with a one parameter family of weighted Hardy inequalities.

Example 3.3. *Suppose* $1 < q \le p < \infty$ *and fix* $\alpha > 0$. *The inequality*

$$\left(\int_0^\infty \left(\int_0^x f(y)\,dy\right)^q [\log(1+x)]^{\alpha(1-q)} (x+1)^{-p}\,dx\right)^{1/q} \le C\left(\int_0^\infty f(y)^p [\log(1+y)]^{(\alpha-1)(1-p)}\,dy\right)^{1/p}$$

holds with best constant

$$C = (p'-1)^{1/q+\alpha(1/q-1/p)} \alpha^{1/q-1} \Gamma(\alpha)^{1/q-1/p}.$$

Proof. Take both (X,ξ) and (Y,η) to be the half line $[0,\infty)$ with Lebesgue measure, take $v(y) = [\log(1+y)]^{(\alpha-1)(1-p)}$, and let $h(x) = (x+1)^{-p}$.

Sometimes simpler is better.

Example 3.4. *Suppose* $1 < p < \infty$. *For any non-negative* f,

$$\int_0^\infty \left(\frac{1}{(1+x)\log(1+x)} \int_0^x f(y)\,dy\right)^p \log(1+x)\,dx \le \frac{1}{p-1}\int_0^\infty f(y)^p\,dy.$$

The constant is best possible.

Proof. Take $p = q$ and $\alpha = 1$ in the previous example.

Here is an inequality for a Steklov operator.

Example 3.5. *Suppose* $1 < q \le p < \infty$. *The best constant in the inequality*

$$\left(\int_{-2}^2 \left(\int_{x-1}^{x+1} f(y)\chi_{[-1,1]}(y)\,dy\right)^q (2-|x|)^{1-q}\,dx\right)^{1/q} \le C\left(\int_{-1}^1 f(y)^p\,dy\right)^{1/p}$$

is $C = 4^{1/q}/2^{1/p}$.

Proof. Take (X,ξ) to be $[-2,2]$ with Lebesgue measure, take (Y,η) to be $[-1,1]$ with Lebesgue measure, take $v = 1$ and $h = 1$.

By choosing the measures ξ and η appropriately it is simple to mix integrals and sums. The best constant is expressed in terms of the Riemann zeta function.

Example 3.6. *Suppose* $1 < q \le p$. *The best constant in the inequality*

$$\left(\sum_{n=1}^{\infty}\left(n\int_0^\infty e^{-ny}f(y)\,dy\right)^q\right)^{1/q} \le C\left(\int_0^\infty \frac{f(y)^p}{y^{(q'-1)(p-1)}(e^y-1)}\,dy\right)^{1/p}$$

is

$$C = \Gamma(q')^{1/p'}\zeta(q')^{1/q-1/p}.$$

Proof. Take (X,ξ) to be $\{1,2,\dots\}$ with counting measure, take (Y,η) to be $[0,\infty)$ with Lebesgue measure, take, $v(y) = (y^{(q'-1)(p-1)}(e^y-1))^{-1}$ and $h = 1$.

Estimates for the norm of the Hardy operator on weighted spaces with $q < p$ have been available for some time. We can now compare them with best constants. We use estimates from [12].

Example 3.7. *Suppose* $1 < q \le p$. *The best constant in the inequality*

$$\left(\int_0^1\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^q x\,dx\right)^{1/q} \le C\left(\int_0^1 f(y)^p(1-y)\,dy\right)^{1/p}$$

is $C = 2^{1/p-1/q}$.

With $q = 2$ *and* $p = 3$ *the upper and lower estimates for* C *given in* [12] *are not very close, giving approximately*

$$0.3509 < 0.8909 < 1.4472.$$

Proof. Take (X,ξ) to be the $[0,1]$ with Lebesgue measure, take (Y,η) to be $[0,1]$ with Lebesgue measure, take $v = 1-y$ and $h = 1$.

We end this section with two inequalities for operators with formal adjoints that are not, strictly speaking, integral operators.

Example 3.8. *Suppose* $1 < q \le p$. *The inequality*

$$\left(\int_0^1\left(f(x)+\int_0^1 f(y)\,dy\right)^q dx\right)^{1/q} \le 2\left(\int_0^1 f(y)^p\,dy\right)^{1/p}$$

holds for all non-negative f. *The constant is best possible.*

Proof. Take (X,ξ) and (Y,η) to be $[0,1]$ with Lebesgue measure, take $v = 1$ and $h = 1$.

Example 3.9. *Suppose* $1 < q \le p$. *If* F *is continuously differentiable and* $F(0) = 0$ *then*

$$\left(\int_0^1 |F(x)+F'(x)|^q\,(1+x)^{1-q}\,dx\right)^{1/q}$$
$$\le (3/2)^{1/q-1/p}\left(\int_0^1 |F'(y)|^p(2-y)\,dy\right)^{1/p}.$$

The constant is best possible.

Proof. Take (X,ξ) and (Y,η) to be $[0,1]$ with Lebesgue measure, take $v(x) = 2-x$ and $h = 1$. The operator used here is $Kf(x) = f(x) + \int_0^x f(y)\,dy$ applied to F'.

Two Iterations

The proofs of a converse to Schur's Lemma given in [5, 13] use a clever iteration which is imitated here to prove Theorem 2.4. Although the same iteration can be used to get the conclusion $S\varphi \le \varphi$ in Theorem 2.5, we introduce a variation of the argument in Lemma 4.2 which is only valid in the case $q < p$. The main advantage of the modification is that it permits us to exercise greater control over the output of the process. This is important because an instance of the first iteration is used as the recursion step in the second iteration, a more delicate argument which yields a positive fixed point of the map S to complete the proof of Theorem 2.5.

There is another advantage to the iteration given in Lemma 4.2. In order to proceed, the first iteration requires that $\|T\|$ be known or at least that an upper bound for $\|T\|$ be known. As we see in Corollary 4.4, it is possible to carry out the modified iteration when no upper bound for $\|T\|$ is known, even if $\|T\| = \infty$. This result is of independent interest and may be important in computations.

Before presenting the proofs of Theorem 2.4 we need to understand the action of the map S. For convenience, the non-negative functions in the closed ball of radius σ in L^p_ν will be denoted by $B^+_\sigma(L^p_\nu)$.

Lemma 4.1. *Suppose* $1 < q \le p < \infty$, $T : L^+_\nu \to L^+_\mu$ *has a formal adjoint* $T^* : L^+_\mu \to L^+_\nu$, *and* $\|T\|$ *is finite and set* $\alpha = (q-1)/(p-1)$. *Then for any* $\lambda > 0$ *the operator* S *given by* $S\varphi = (T^*((T\varphi)^{q-1}))^{p'-1}$ *is a continuous, order preserving map from* $B^+_\lambda(L^p_\nu)$ *to* $B^+_{\lambda^\alpha \|T\|^{\alpha q'}}(L^p_\nu)$. *Also,* $S(\lambda\varphi) = \lambda^\alpha S\varphi$ *for every* $\varphi \in L^+_\nu$.

Proof. Let E_s denote the (non-linear) operator on non-negative functions defined by $E_s f(t) = f(t)^s$. It is straightforward to check that if s and σ are positive then $E_s : B^+_\sigma(L^p_\nu) \to B^+_{\sigma^s}(L^{p/s}_\nu)$ is continuous and order-preserving.

The definition of $\|T\|$ shows that $\|Tf\|_{L^q_\mu} \le \|T\|\|f\|_{L^p_\nu}$ for all $f \in L^+_\nu$ and a routine calculation shows that $\|T^*g\|_{L^{p'}_\nu} \le \|T\|\|g\|_{L^{q'}_\mu}$ for all $g \in L^+_\mu$. It follows that

$$T : B^+_\sigma(L^p_\nu) \to B^+_{\sigma\|T\|}(L^q_\mu) \quad \text{and} \quad T^* : B^+_\sigma(L^{q'}_\mu) \to B^+_{\sigma\|T\|}(L^{p'}_\nu)$$

are continuous. (A minor modification of the proof that bounded, linear operators are continuous is needed here.) Lemma 2.2 shows that maps with formal adjoints preserve order so both T and T^* are order preserving. The diagram,

$$\begin{tikzcd}
B^+_\lambda(L^p_\nu) \arrow[d, "T"] & & \\
B^+_{\lambda\|T\|}(L^q_\mu) \arrow[r, "E_{q-1}"] & B^+_{\lambda^{q-1}\|T\|^{q-1}}(L^{q'}_\mu) \arrow[d, "T^*"] & \\
 & B^+_{\lambda^{q-1}\|T\|^{q}}(L^{p'}_\nu) \arrow[r, "E_{p'-1}"] & B^+_{\lambda^\alpha\|T\|^{\alpha q'}}(L^p_\nu)
\end{tikzcd}$$

shows that the map $S = E_{p'-1}T^*E_{q-1}T$ is a continuous, order preserving map from

$B^+_\lambda(L^p_\nu)$ to $B^+_{\lambda^\alpha \|T\|^{\alpha q'}}(L^p_\nu)$. Since T and T^* are positive homogeneous by Lemma 2.1 it is easy to see that $S(\lambda\varphi) = \lambda^\alpha S\varphi$.

Proof of Theorem 2.4. If $\|T\| = 0$ then T is the zero map so we may prove the result by choosing $A = \varepsilon^{p'}$ and taking any positive $\varphi \in L^p_\nu$.

Suppose, then, that $\|T\| > 0$ and choose $\delta \in (0,1)$ such that $A = (1-\delta)^{-1}\|T\|^{p'}$ satisfies $-\varepsilon + A^{(p-1)/p} \le \|T\| \le A^{(p-1)/p}$. Note that A is positive. Since (Y,ν) is assumed to be σ-finite we may choose a positive function ψ_0 with $\|\psi_0\|_{L^p_\nu} = \delta$. Define a sequence in L^+_ν recursively by setting $\psi_{n+1} = \psi_0 + A^{-1}S\psi_n$.

Using induction we see that the sequence is non-decreasing since $\psi_1 - \psi_0 = A^{-1}S\psi_0 \ge 0$ and if $\psi_n - \psi_{n-1} \ge 0$ then $\psi_{n+1} - \psi_n = A^{-1}(S\psi_n - S\psi_{n-1}) \ge 0$.

Moreover, $\|\psi_0\|_{L^p_\nu} = \delta \le 1$ and if $\|\psi_n\|_{L^p_\nu} \le 1$ then by Lemma 4.1 with $q = p$ and $\lambda = 1$,

$$\|\psi_{n+1}\|_{L^p_\nu} \le \delta + A^{-1}\|S\psi_n\|_{L^p_\nu} \le \delta + A^{-1}\|T\|^{p'} = 1.$$

By induction, $\|\psi_n\|_{L^p_\nu} \le 1$ for all n.

This is a non-decreasing sequence whose terms are bounded above in L^p_ν. Therefore, it converges in L^p_ν to its pointwise limit φ and $\varphi \in L^p_\nu$. The continuity of S shows that $\varphi = \psi_0 + A^{-1}S\varphi > A^{-1}S\varphi \ge 0$ so we have $\varphi > 0$ and $S\varphi \le A\varphi$.

The next lemma uses a variation of the iteration above to prepare for the recursion step in the proof of Theorem 2.5.

Lemma 4.2. *Suppose that* $1 < q < p < \infty$, $T : L^+_\nu \to L^+_\mu$ *has a formal adjoint* $T^* : L^+_\mu \to L^+_\nu$, *and* $\|T\| < \infty$. *Let* $C = \|T\|^{q/(p-q)}$ *and* $\alpha = (q-1)/(p-1)$. *If* ψ *is a positive function in* $B^+_{\lambda^2 C}(L^p_\nu)$ *for some* $\lambda > 1$ *then there exists a positive function* $\varphi \in B^+_{\lambda C}(L^p_\nu)$ *satisfying* $\varphi = ((\lambda - \lambda^\alpha)/\lambda^2)\psi + S\varphi$. *If* $S\psi \le \psi$ *then* $\varphi \le \lambda\psi$.

Proof. Set $\psi_0 = ((\lambda - \lambda^\alpha)/\lambda^2)\psi$. For $n \ge 0$ define $\psi_{n+1} = \psi_0 + S\psi_n$.

Since S is order preserving, $\psi_1 - \psi_0 = S\psi_0 \ge 0$ and if $\psi_n - \psi_{n-1} \ge 0$ then $\psi_{n+1} - \psi_n = S\psi_n - S\psi_{n-1} \ge 0$. Induction shows that $\psi_0, \psi_1, \dots$ is a non-decreasing sequence. We define φ to be the pointwise limit of this sequence.

Now $\psi \in B^+_{\lambda^2 C}(L^p_\nu)$ so $\|\psi_0\|_{L^p_\nu} \le (\lambda - \lambda^\alpha)C \le \lambda C$. If $\|\psi_n\|_{L^p_\nu} \le \lambda C$ then $\|S\psi_n\|_{L^p_\nu} \le \lambda^\alpha C$ by Lemma 4.1 so $\|\psi_{n+1}\|_{L^p_\nu} \le \|\psi_0\|_{L^p_\nu} + \|S\psi_n\|_{L^p_\nu} \le (\lambda - \lambda^\alpha)C + \lambda^\alpha C = \lambda C$. By induction $\|\psi_n\|_{L^p_\nu} \le \lambda C$ for all n and so the sequence converges in L^p_ν and $\|\varphi\|_{L^p_\nu} \le \lambda C$.

By the continuity of S and the hypothesis that ψ is positive we see that $\varphi = \psi_0 + S\varphi > S\varphi$ and that φ is positive.

To prove the last statement of the lemma we suppose that $S\psi \le \psi$. Since $\lambda > 1$, $\psi_0 = ((\lambda - \lambda^\alpha)/\lambda^2)\psi \le (\lambda - \lambda^\alpha)\psi \le \lambda\psi$. If $\psi_n \le \lambda\psi$ for some n then $\psi_{n+1} = \psi_0 + S\psi_n \le (\lambda - \lambda^\alpha)\psi + S(\lambda\psi) \le (\lambda - \lambda^\alpha)\psi + \lambda^\alpha\psi = \lambda\psi$. By induction $\psi_n \le \lambda\psi$ for all n and in the limit we have $\varphi \le \lambda\psi$. This completes the proof.

Proof of Theorem 2.5. Define C and α as in Lemma 4.2 and note that $0 < \alpha < 1$. Let φ_0 be a positive function with $\|\varphi_0\|_{L^p_\nu} \le 4C$. Such a function exists because (Y,ν) is σ-finite. For $n = 0, 1, \dots$ apply Lemma 4.2 with $\psi = \varphi_n$ and $\lambda = 2^{2^{-n}}$ to recursively produce φ_{n+1} satisfying

$$\|\varphi_{n+1}\|_{L^p_\nu} \le 2^{2^{-n}} C \text{ and } \varphi_{n+1} = ((2^{2^{-n}} - 2^{\alpha 2^{-n}})/2^{2^{1-n}})\varphi_n + S\varphi_{n+1}.$$

Since $S\varphi_{n+1} \le \varphi_{n+1}$, Theorem 2.2 yields $C \le \|\varphi_{n+1}\|_{L^p_\nu}$ so we have

$$\|T\| \le \|\varphi_{n+1}\|_{L^p_\nu}^{(p-q)/q} \le (2^{2^{-n}})^{(p-q)/q}\|T\|$$

and we see that $\|\varphi_n\|_{L^p_\nu}^{(p-q)/q}$ is eventually less than $\|T\| + \varepsilon$ for any $\varepsilon > 0$. This proves the first part of the theorem.

Applying the last statement of Lemma 4.2, we see that for $n \ge 1$, $S\varphi_n \le \varphi_n$ so $\varphi_{n+1} \le 2^{2^{-n}}\varphi_n$. Thus we have a decreasing sequence of positive L^p_ν functions

$$2\varphi_1 \ge \cdots \ge 2^{2^{1-n}}\varphi_n \ge 2^{2^{1-(n+1)}}\varphi_{n+1} \cdots .$$

We denote the pointwise limit of this sequence by φ and apply the Dominated Convergence Theorem and the above estimates of $\|\varphi_{n+1}\|_{L^p_\nu}$ to get

$$\|\varphi\|_{L^p_\nu}^{(p-q)/q} = \lim_{n\to\infty} \|2^{2^{-n}}\varphi_{n+1}\|_{L^p_\nu}^{(p-q)/q} = \|T\|.$$

The continuity of S yields

$$\varphi = \lim_{n\to\infty} 2^{2^{-n}}\varphi_{n+1} = \lim_{n\to\infty} 2^{-2^{-n}}(2^{2^{-n}} - 2^{\alpha 2^{-n}})\varphi_n + 2^{2^{-n}}S\varphi_{n+1} = S\varphi$$

in L^p_ν because $2^{2^{-n}} - 2^{\alpha 2^{-n}} \to 0$ as $n \to \infty$ and the φ_n are uniformly bounded in L^p_ν norm.

To complete the proof of the second part we have to show that φ is positive ν-almost everywhere. Since φ is the pointwise limit of a sequence of positive functions, it is clearly non-negative. Thus it is enough to show that $\{y \in Y : \varphi(y) = 0\}$ has ν measure zero. That is the object of the next lemma.

Lemma 4.3. *Suppose* $1 < q < p < \infty$, $S\varphi = \varphi$, *and* $\|\varphi\|_{L^p_\nu} = \|T\|^{q/(p-q)}$. *If* (2.2) *holds then* φ *is positive* ν*-almost everywhere.*

Proof. Let Y_0 be the set where φ is zero and Y_1 be its complement in Y. Let X_0 be the set where $T\varphi$ is zero and X_1 be its complement in X. If f is supported on Y_0 then

$$\int Tf(T\varphi)^{q-1}\,d\mu = \int fT^*((T\varphi)^{q-1})\,d\nu = \int f(S\varphi)^{p-1}\,d\nu = \int f\varphi^{p-1}\,d\nu$$

which is zero because f and φ have disjoint supports. Since $(T\varphi)^{q-1}$ is positive on X_1 we see that Tf is supported on X_0.

Similarly, if g is supported on X_0 then $\int \varphi T^* g\, d\nu = \int T\varphi g\, d\mu = 0$ and since φ is positive on Y_1 we see that $T^* g$ is supported on Y_0. Thus, if f is supported on Y_1 then for all $g \in L^+_\mu$,

$$\int Tf \chi_{X_0} g\, d\mu = \int f T^*(\chi_{X_0} g)\, d\nu = 0$$

which implies that

$$\int Tfg\, d\mu = \int Tf\chi_{X_0} g\, d\mu + \int Tf\chi_{X_1} g\, d\mu = \int \chi_{X_1} Tfg\, d\mu.$$

It follows that $Tf = \chi_{X_1} Tf$ so Tf is supported on X_1. These observations give the decomposition

$$Tf = \chi_{X_0} T(f\chi_{Y_0}) + \chi_{X_1} T(f\chi_{Y_1}) \tag{4.1}$$

for all $f \in L^+_\nu$.

For $i = 0, 1$; let $\mu_i = \chi_{X_i}\mu$, $\nu_i = \chi_{Y_i}\nu$, and define $T_i : L^+_{\nu_i} \to L^+_{\mu_i}$ and $T^*_i : L^+_{\mu_i} \to L^+_{\nu_i}$ by $T_i f = Tf$ and $T^*_i g = T^* g$. It is easy to check that T_i and T^*_i are formal adjoints. In view of (4.1) it is a routine calculation to show that

$$\|T\|^{pq/(p-q)} = \|T_0\|^{pq/(p-q)} + \|T_1\|^{pq/(p-q)}.$$

Now we apply Theorem 2.3 to the operator T_1. Since φ is positive ν_1-almost everywhere and

$$\varphi = (T^*((T\varphi)^{q-1}))^{p'-1} = (T^*_1((T_1\varphi)^{q-1}))^{p'-1},$$

we conclude that

$$\|T_1\| = \|\varphi\|^{(p-q)/q}_{L^p_{\nu_1}} = \|\varphi\|^{(p-q)/q}_{L^p_\nu} = \|T\|.$$

It follows that $\|T_0\| = 0$ and hence $Tf = 0$ whenever f is supported on Y_0. The remark following (2.2) shows that Y_0 has ν measure zero.

Our final result is a corollary of Lemma 4.2 and gives an iterative procedure which converges to $\|T\|$ to within any preset tolerance without an a priori bound on $\|T\|$. It provides upper and lower bounds for the norm of T at each stage of the iteration.

Corollary 4.4. *Suppose* $1 < q < p < \infty$, $T : L^+_\nu \to L^+_\mu$ *has a formal adjoint* $T : L^+_\mu \to L^+_\nu$, $S\varphi = (T^*((T\varphi)^{q-1}))^{p'-1}$, *and* $\alpha = (q-1)/(p-1)$. *Fix* $\lambda > 1$ *and a positive function* $\varphi_0 \in L^p_\nu$. *Define* ψ_n *recursively by*

$$\psi_0 = (\lambda - \lambda^\alpha)\|T\varphi_0\|^{q/(p-q)}_{L^q_\mu}\|\varphi_0\|^{p/(q-p)}_{L^p_\nu}\varphi_0 \quad \text{and} \quad \psi_{n+1} = \psi_0 + S\psi_n.$$

Then ψ_n is a non-decreasing sequence,

$$[(1/\lambda)\|\psi_n\|_{L^p_\nu}]^{(p-q)/q} \le \|T\| \le [\sup_{y\in Y} S\psi_n(y)/\psi_n(y)]^{(p-1)/q}\|\psi_n\|_{L^p_\nu}^{(p-q)/q} \tag{4.2}$$

and

$$[(1/\lambda)\lim_{n\to\infty}\|\psi_n\|_{L^p_\nu}]^{(p-q)/q} \le \|T\| \le [\lim_{n\to\infty}\|\psi_n\|_{L^p_\nu}]^{(p-q)/q} \tag{4.3}$$

for each n.

Proof. The right hand inequality in (4.2) follows from Theorem 2.2 with $A = \sup_{y\in Y} S\psi_n(y)/\psi_n(y)$. To prove the other inequalities we first suppose that $\|T\| < \infty$. Set $\psi = (\lambda^2/(\lambda-\lambda^\alpha))\psi_0$ to get

$$\|\psi\|_{L^p_\nu} = \lambda^2\|T\varphi_0\|_{L^q_\mu}^{q/(p-q)}\|\varphi_0\|_{L^p_\nu}^{q/(q-p)} \le \lambda^2\|T\|^{q/(p-q)}.$$

Lemma 4.2 shows that ψ_n is a non-decreasing sequence whose pointwise limit φ is positive and satisfies both $S\varphi \le \varphi$ and $\|\varphi\|_{L^p_\nu} \le \lambda\|T\|^{q/(p-q)}$. The latter gives the left hand inequality in (4.2) and the former, together with Theorem 2.2, provides the right hand inequality in (4.3).

If $\|T\| = \infty$ then it is enough to show that $\lim_{n\to\infty}\|\psi_n\|_{L^p_\nu}$ is infinite. The proof of Lemma 4.2 still shows that the sequence ψ_n is non-decreasing and we again define φ to be its pointwise limit. It is an exercise to show that $S\psi_n$ converges pointwise to $S\varphi$ even without assuming continuity of S. We obtain $S\varphi \le \varphi$ and apply Theorem 2.2 again to see that $\|\varphi\|_{L^p_\nu} = \infty$. The Monotone Convergence Theorem shows that $\lim_{n\to\infty}\|\psi_n\|_{L^p_\nu} = \infty$ as required.

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Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B7, CANADA

E-mail address: `sinnamon@uwo.ca`