INTERPOLATION OF SPACES DEFINED BY THE LEVEL FUNCTION

Gord Sinnamon

Appeared in *Harmonic Analysis*, The Proceedings of the ICM-90 Satellite Conference in Harmonic Analysis held in Sendai, Japan, August 14-18, 1990. Satoru Igari (Ed.) Springer-Verlag Tokyo. 190-193.

INTERPOLATION OF SPACES DEFINED BY THE LEVEL FUNCTION†

Gord Sinnamon‡
University of Western Ontario
London, Ontario, CANADA

Section 1: Introduction.

Suppose λ is a regular, Borel measure on $\mathbf{R}$ and suppose that $\lambda(-\infty, x) < \infty$ for all $x \in \mathbf{R}$. The Lebesgue spaces, L^p_λ, for $1 \leq p \leq \infty$ will then contain non-trivial, non-increasing functions. Define

$$\|f\|_{p\downarrow\lambda} = \sup\left\{\int_{-\infty}^{\infty} |f|g\,d\lambda : g \geq 0, g \text{ non-increasing}, \|g\|_{p',\lambda} \leq 1\right\}$$

where p' is defined by $1/p + 1/p' = 1$.

The norm $\|\cdot\|_{p\downarrow\lambda}$ was utilized in the early 1950's by Halperin [**2**] and Lorentz [**3**] and more recently, in proving weighted norm inequalities, by Sinnamon [**5**], Neugebauer [**4**], Stepanov [**7**] and others. Halperin showed, in the case of λ absolutely continuous, that to each non-negative function f there corresponds a function f^o which is non-increasing and satisfies $\|f\|_{p\downarrow\lambda} = \|f^o\|_{p,\lambda}$. He called (a variant of) this function f^o, the level function of f. In [**6**] this contruction is extended to regular, Borel measures and the dual of the Banach space $L^{p\downarrow}_\lambda$ is characterised for $1 \leq p < \infty$. The dual space is the space $L^{p'*}_\lambda$, with norm $\|f\|_{p'*\lambda} = \|\bar{f}\|_{p',\lambda}$ where

$$\bar{f}(x) = \operatorname{ess\,sup}_{y \geq x} |f(y)|.$$

In this paper the above duality result is used to show that interpolation between the spaces $L^{p\downarrow}_\lambda$, $1 \leq p < \infty$, again yields the spaces $L^{p\downarrow}_\lambda$.

We conclude this section with some definitions and notation. The main results of the paper are contained in Section 2.

Let f be a λ-measurable function. The distribution function of f with respect to the measure λ is $m_f(\alpha) = \lambda\{x : |f(x)| > \alpha\}$. The non-increasing rearrangement of f is $f^*(t) = \inf\{\alpha : m_f(\alpha) \leq t\}$, $t > 0$. The following simple facts follow directly

---

† This paper is in final form and no version of it has been or will be submitted elsewhere. Revised June 5, 1991.

‡ Research supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

from the definitions. 1. $m_f(f^*(t)) \le t$. 2. $f^*(m_f(\alpha)) \le \alpha$. 3. $f^*(0) = \|f\|_{\infty,\lambda}$. 4. If $|f| \le |f_0| + |f_1|$ and $0 < \varepsilon < 1$ then $f^*(s) \le f_0^*((1-\varepsilon)s) + f_1^*(\varepsilon s)$.

If $1 \le p < \infty$, L^p_λ is the space of all λ-measurable functions for which the norm $\|f\|_{p,\lambda} = (\int_{-\infty}^{\infty} |f|^p \, d\lambda)^{1/p}$ is finite. L^∞_λ is the space of all essentially bounded functions. The norm on the space is $\|f\|_{\infty,\lambda} = \operatorname{ess\,sup}|f|$.

Reference is made to several specific theorems in [**1**]. That text will also serve as our basic reference in interpolation theory and in particular we will follow the notation used there.

Section 2: Main Results.

The interpolation results are first proved in the dual spaces $L^{p'\,*}_\lambda$. To do this, a formula for the K-functional, $K(t, f; L^{1\,*}_\lambda, L^{\infty\,*}_\lambda)$, is derived and then the reiteration theorem and the results of ordinary L^p_λ interpolation are applied. The Duality Theorem for Real Interpolation provides the link to the spaces $L^{p\downarrow}_\lambda$.

We begin with a simple fact.

Lemma 1. $L^{\infty\,*}_\lambda = L^\infty_\lambda$ *with identical norms.*

Proof. Suppose f is λ-measurable. Since $f \le \bar{f}$ almost everywhere we certainly have $\|f\|_{\infty,\lambda} \le \|\bar{f}\|_{\infty,\lambda} = \|f\|_{\infty\,*\,\lambda}$. Conversely, for each x, $\bar{f}(x) = \operatorname{ess\,sup}_{y \ge x}|f(y)| \le \|f\|_{\infty,\lambda}$ so $\|f\|_{\infty\,*\,\lambda} = \|\bar{f}\|_{\infty,\lambda} \le \|f\|_{\infty,\lambda}$.

The following theorem gives a formula for the K-functional for the pair $(L^{1\,*}_\lambda, L^{\infty\,*}_\lambda)$.

Theorem 1. $K(t, f; L^{1\,*}_\lambda, L^{\infty\,*}_\lambda) = \int_0^t \bar{f}^{\,*}(s)\, ds = K(t, \bar{f}; L^1_\lambda, L^\infty_\lambda)$.

Proof. Fix a λ-measurable function f and fix $t > 0$. The second equality above is from [**1**, Theorem 5.2.1]. To establish the first we prove inequalities in both directions beginning with $K(t, f, L^{1\,*}_\lambda, L^{\infty\,*}_\lambda) \ge \int_0^t \bar{f}^{\,*}(s)\, ds$.

Suppose $f = f_0 + f_1$. Clearly $\bar{f} \le \bar{f}_0 + \bar{f}_1$ so for each $\varepsilon > 0$ we have $\bar{f}^{\,*}(s) \le \bar{f}_0^{\,*}((1-\varepsilon)s) + \bar{f}_1^{\,*}(\varepsilon s)$ and hence

$$\int_0^t \bar{f}^{\,*}(s)\, ds \le \int_0^t \bar{f}_0^{\,*}((1-\varepsilon)s)\, ds + \int_0^t \bar{f}_1^{\,*}(\varepsilon s)\, ds.$$

The second integral on the right hand side is dominated by

$$t\bar{f}_1^{\,*}(0) = t\|\bar{f}_1\|_{\infty,\lambda} = t\|f_1\|_{\infty\,*\,\lambda}.$$

The first is dominated by

$$\int_0^\infty \bar{f}_0^{\,*}((1-\varepsilon)s)\, ds = \frac{1}{1+\varepsilon}\int_0^\infty \bar{f}_0^{\,*}(s)\, ds = \frac{1}{1+\varepsilon}\int_{-\infty}^\infty \bar{f}_0(x)\, dx = \frac{1}{1+\varepsilon}\|f_0\|_{1\,*\,\lambda}.$$

Letting $\varepsilon \to 0$ we have

$$\int_0^t \bar{f}^{\,*}(s)\,ds \leq \|f_0\|_{1\,*\,\lambda} + t\|f_1\|_{\infty\,*\,\lambda}.$$

Since $K(t,f;L^{1\,*}_\lambda, L^{\infty\,*}_\lambda) = \inf_{f_0+f_1=f}(\|f_0\|_{1\,*\,\lambda} + t\|f_1\|_{\infty\,*\,\lambda})$ we have the inequality.

To prove the other inequality, that $K(t,f,L^{1\,*}_\lambda, L^{\infty\,*}_\lambda) \leq \int_0^t \bar{f}^{\,*}(s)\,ds$, we set $a = \bar{f}^{\,*}(t)$ and $E = \{x : |\bar{f}(x)| > a\}$. Note that $\lambda(E) = m_{\bar{f}}(a) = m_{\bar{f}}(\bar{f}^{\,*}(t)) \leq t$. Also, if $\lambda(E) \leq s \leq t$ then $\bar{f}^{\,*}(t) \leq \bar{f}^{\,*}(s) \leq \bar{f}^{\,*}(\lambda(E)) = \bar{f}^{\,*}(m_{\bar{f}}(a)) \leq a = \bar{f}^{\,*}(t)$ so $\bar{f}^{\,*}(s) = \bar{f}^{\,*}(t)$. Thus $\bar{f}^{\,*}$ is constant on the interval $[\lambda(E), t]$.

Set $f_0(x) = \max(0, |f(x)| - a)\mathrm{sgn}(f(x))$. Here $\mathrm{sgn}(z) = z/|z|$ if $z \neq 0$ and $\mathrm{sgn}(z) = 0$ otherwise. $\bar{f}_0$ is related to $\bar{f}$ in a simple way.

$$\bar{f}_0(x) = \operatorname{ess\,sup}_{y \geq x}|f_0(y)| = \operatorname{ess\,sup}_{y\geq x}\max(0, |f(y)| - a)$$

$$= \max(0, \operatorname{ess\,sup}_{y\geq x}|f(y)| - a) = \max(0, \bar{f}(x) - a).$$

The remaining portion of f is bounded by a. Let $f_1 = f - f_0$ and we have

$$|f_1(x)| = |f(x) - f_0(x)| = |f(x)| - \max(0, |f(x)| - a) \leq a.$$

Thus $\|f_1\|_{\infty,\lambda} \leq a$. We can now estimate the K-functional.

$$K(t,f;L^{1\,*}_\lambda, L^{\infty\,*}_\lambda) \leq \|f_0\|_{1\,*\,\lambda} + t\|f_1\|_{\infty\,*\,\lambda}.$$

The set E, defined earlier, contains the support of $\bar{f}_0$. $\bar{f}_0$ takes the value $\bar{f}(x) - a$ on E. This fact, together with Lemma 1 shows that the K-functional is dominated by

$$\int_{-\infty}^{\infty} \bar{f}_0(x)\,dx + t\|f_1\|_{\infty,\lambda} \leq \int_E \bar{f}(x) - a\,dx + ta \leq \int_0^{\lambda(E)} \bar{f}^{\,*}(s) - a\,ds + ta.$$

We now recall that $\bar{f}^{\,*}$ takes the value a on the interval $[\lambda(E), t]$ so the last expression is just

$$\int_0^t \bar{f}^{\,*}(s) - a\,ds + ta = \int_0^t \bar{f}^{\,*}(s)\,ds.$$

This completes the proof.

With the above relation between the K-functionals for $(L^{1\,*}_\lambda, L^{\infty\,*}_\lambda)$ and $(L^1_\lambda, L^\infty_\lambda)$ we can prove the following.

Corollary 1. *If $1 \leq p_0 < p_1 \leq \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$ then $(L^{p_0\,*}_\lambda, L^{p_1\,*}_\lambda)_{\theta,p} = L^{p\,*}_\lambda$ with equivalent norms.*

Proof.

$$\|f\|_{(L^{1*}_\lambda, L^{\infty *}_\lambda)_{1/p',p}} = \left(\int_0^\infty \left(t^{-1/p'} K(t, f; L^{1*}_\lambda, L^{\infty *}_\lambda)\right)^p t^{-1}\, dt\right)^{1/p}$$

$$= \left(\int_0^\infty \left(t^{-1/p'} K(t, \bar{f}; L^1_\lambda, L^\infty_\lambda\right)^p t^{-1}\, dt\right)^{1/p} = \|\bar{f}\|_{(L^1_\lambda, L^\infty_\lambda)_{1/p',p}}.$$

By [**1**, Theorem 5.2.1] we have $(L^1_\lambda, L^\infty_\lambda)_{1/p',p} = L^p_\lambda$ with equivalent norms. Therefore the norm on $(L^{1*}_\lambda, L^{\infty *}_\lambda)_{1/p',p}$ is equivalent to $\|\bar{f}\|_{p,\lambda} = \|f\|_{p*\lambda}$. Using this fact and the reiteration theorem ([**1**, Theorem 3.5.3]) we obtain

$$(L^{p_0 *}_\lambda, L^{p_1 *}_\lambda)_{\theta,p} = ((L^{1*}_\lambda, L^{\infty *}_\lambda)_{\frac{1}{(p_0)'},p_0}, (L^{1*}_\lambda, L^{\infty *}_\lambda)_{\frac{1}{(p_1)'},p_1})_{\theta,p} = (L^{1*}_\lambda, L^{\infty *}_\lambda)_{1/p',p} = L^{p*}_\lambda.$$

with equivalent norms. Here $(L^{1*}_\lambda, L^{\infty *}_\lambda)_{0,1}$ is taken to be L^{1*}_λ and $(L^{1*}_\lambda, L^{\infty *}_\lambda)_{1,\infty}$ is taken to be $L^{\infty *}_\lambda$.

Corollary 2. *If* $1 \le p_0 < p_1 < \infty$, $0 < \theta < 1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$ *then* $(L^{p_0 \downarrow}_\lambda, L^{p_1 \downarrow}_\lambda)_{\theta,p} = L^{p\downarrow}_\lambda$ *with equivalent norms.*

Proof. As mentioned, the dual space of $L^{p\downarrow}_\lambda$ is $L^{p' *}_\lambda$ for $1 \le p < \infty$. Thus, by Corollary 1 and the Duality Theorem for Real Interpolation [**1**, Theorem 3.7.1],

$$(L^{p\downarrow}_\lambda)' \equiv L^{p'*}_\lambda = (L^{p_0' *}_\lambda, L^{p_1' *}_\lambda)_{\theta,p'} \equiv ((L^{p_0\downarrow}_\lambda)', (L^{p_1\downarrow}_\lambda)')_{\theta,p'} = (L^{p_0\downarrow}_\lambda, L^{p_1\downarrow}_\lambda)'_{\theta,p}$$

with equivalent norms. Here "$\equiv$" indicates the isomorphism $f \leftrightarrow L_f$ where $L_f(g) = \int_{\mathbf{R}} fg\, d\lambda$. Since all the $L^{p\downarrow}_\lambda$ spaces for $1 \le p < \infty$ have a common dense subset, ($\{f : f$ is bounded and supported on $(-\infty, x]$ for some $x\}$ will do,) it follows that $L^{p\downarrow}_\lambda = (L^{p_0\downarrow}_\lambda, L^{p_1\downarrow}_\lambda)_{\theta,p}$. This completes the proof.

References

1. J. Bergh and J. Löfström. *Interpolation Spaces, An Introduction.* Berlin, Heidelberg 1976, Springer-Verlag.
2. I. Halperin. Function spaces. *Canad. J. Math.* **5** (1953), 278-288.
3. G. G. Lorentz. *Bernstein Polynomials.* Toronto 1953, University of Toronto Press.
4. C. J. Neugebauer. Weighted norm inequalities for averaging operators of monotone functions. Preprint.
5. G. Sinnamon. Weighted Hardy and Opial-type inequalities. *J. Math. Anal. Appl.* To appear.
6. G. Sinnamon. Spaces defined by the level function and their duals. Preprint.
7. V. Stepanov. The weighted Hardy's inequality for nonincreasing functions. Preprint.