THE WEIGHTED HARDY INEQUALITY: NEW PROOFS AND THE CASE P=1

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Abstract

An elementary proof is given of the weight characterisation for the Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p u\right)^{1/p}, \quad f \ge 0, \tag{1.1}$$

in the case $0 < q < p$, $1 < p < \infty$. It is also shown that certain weighted inequalities with monotone kernels are equivalent to inequalities in which one of the weights is monotone. Using this, a characterisation of those weights for which (1.1) holds with $0 < q < 1 = p$ is given. Results for (1.1), considered as an inequality over monotone functions f are presented.

1. *Introduction*

To ask whether or not the inequality (1.1) holds is a question about a vast class of objects—every non-negative, measurable function f on $(0, \infty)$—and the inequality must be true for each one. Moreover, there is a vast class of such questions, one for every choice of p, q, u and v. It is somewhat surprising that simple answers are available. Muckenhoupt [**6**] and Bradley [**1**] showed that if $1 < p \le q < \infty$ then each such question may be answered by calculating a one parameter supremum and Maz'ja [**5**] and Sinnamon [**10**] showed that if $0 < q < p$, $1 < p < \infty$ then the answer reduces to the finiteness of a single integral. The techniques used to prove these weighted Hardy inequalities have opened up the study of inequalities for operators with positive kernels and the simple answers to the

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above questions have provided a standard against which to measure the results obtained. Extensions have been made in many directions. There are results for more general kernels, higher dimensions, more general spaces, and different classes of functions. References may be found in [**7**].

In this paper we return to the weighted Hardy inequality with only brief excursions to results for more general kernels. In Section 2, we give a simple, elementary proof of the known characterisation in the case $0 < q < p$, $1 < p < \infty$. Then we restrict our attention to the case $p = 1$. When $p = 1 < q < \infty$, standard techniques may be used but the standard approach fails when $0 < q < 1 = p$. We are able to characterise the weights for which the Hardy inequality holds in this case by exploiting the monotonicity of the kernel. This is done in Section 3. In Section 4 we constrain the functions f in (1.1) to be non-increasing and give a weight characterisation. Some results are also given in the case that f is constrained to be non-decreasing.

The notation used is standard. χ_E will be used to denote the function whose value on the set E is 1 and whose value off the set E is 0. If $0 < q \leq \infty$, q' will denote the conjugate exponent of q defined by $1/q + 1/q' = 1$. Note that 1 and ∞ are conjugate exponents and that $q' < 0$ when $q < 1$. Similarly, p' will denote the conjugate exponent of p. Any expression of the form $0/0$, ∞/∞, or $0 \cdot \infty$ is taken to be 0. Non-negative functions are permitted to take the value ∞.

2. *An Elementary Proof*

The purpose of this section is to give a new proof of the weight characterisation for the Hardy inequality (1.1) in the case $0 < q < p$, $1 < p < \infty$. The result was originally proved by Mazja [**5**] in the case $1 < q < p < \infty$ and by Sinnamon [**10**] in the case $0 < q < 1 < p < \infty$. Sinnamon's proof of sufficiency involved the level function of I. Halperin [**2**], an object which is complicated to construct. Proofs of necessity given by Mazja and Sinnamon were also complicated and relied on the division of the index range into two cases. The proof given here does not split the index range $0 < q < p$ into two cases, sufficiency does not rely on the level function and necessity is much simplified.

To emphasize that our argument is elementary, we have made this section largely self-contained at the minor expense of including two arguments that already appear in the literature. They appear here as Propositions 2.2 and 2.3. The main result is contained in Theorem 2.4. We begin with a simple lemma.

LEMMA 2.1. *Suppose α, β and γ are non-negative functions and γ is non-decreasing. If $\int_x^\infty \alpha \le \int_x^\infty \beta$ for all x, then $\int_0^\infty \gamma\alpha \le \int_0^\infty \gamma\beta$.*

Since γ is non-decreasing there exists a non-negative constant c and a non-negative, Borel measure μ such that $\gamma(t) = c + \int_0^t d\mu$ almost everywhere. (See, for example [**8**, p262].) Now $\int_0^\infty \gamma\alpha = c\int_0^\infty \alpha + \int_0^\infty \left(\int_x^\infty \alpha\right) d\mu(x) \le c\int_0^\infty \beta + \int_0^\infty \left(\int_x^\infty \beta\right) d\mu(x) = \int_0^\infty \gamma\beta$ as stated.

The following proposition is essentially a special case of Proposition 1(b) in [**12**].

PROPOSITION 2.2. *Suppose that v, b, and F are non-negative functions with F non-decreasing such that*

$$\int_x^\infty b < \infty \quad \textit{for all } x > 0 \textit{ and } \quad \int_0^\infty b = \infty. \tag{2.1}$$

If $0 < q < p < \infty$ *and* $1/r = 1/q - 1/p$ *then*

$$\left(\int_0^\infty F^q v\right)^{1/q} \le (r/p)^{1/r} \left(\int_0^\infty \left(\int_x^\infty v\right)^{r/q} \left(\int_x^\infty b\right)^{-r/q} b(x)\,dx\right)^{1/r} \left(\int_0^\infty F^p b\right)^{1/p}. \tag{2.2}$$

Let $V(x) = \int_x^\infty v$ and $B(x) = \int_x^\infty b$. An application of Hölder's inequality with indices r/q and p/q splits the left hand side of (2.2) into two factors.

$$\left(\int_0^\infty F^q v\right)^{1/q} = \left(\int_0^\infty \left(\int_0^t V^{r/p} B^{-r/q} b\right)^{q/r} F(t)^q \left(\int_0^t V^{r/p} B^{-r/q} b\right)^{-q/r} v(t)\,dt\right)^{1/q}$$

$$\le \left(\int_0^\infty \left(\int_0^t V^{r/p} B^{-r/q} b\right) v(t)\,dt\right)^{1/r} \left(\int_0^\infty F(t)^p \left(\int_0^t V^{r/p} B^{-r/q} b\right)^{-p/r} v(t)\,dt\right)^{1/p}.$$

On interchanging the order of integration, the first factor becomes $\left(\int_0^\infty V^{r/q} B^{-r/q} b\right)^{1/r}$. To complete the proof we apply Lemma 2.1 to the second factor, showing that it is dominated by $(r/p)^{1/r} \left(\int_0^\infty F^p b\right)^{1/p}$. Specifically, we take $\alpha(t) = \left(\int_0^t V^{r/p} B^{-r/q} b\right)^{-p/r} v(t)$, $\beta = (r/p)^{p/r} b$, and $\gamma = F^p$ in Lemma 2.1. γ is clearly non-decreasing so it remains to check that $\int_x^\infty \alpha \le \int_x^\infty \beta$ for all x. Since $\left(\int_0^t V^{r/p} B^{-r/q} b\right)^{-p/r}$ is non-increasing,

$$\int_x^\infty \alpha = \int_x^\infty \left(\int_0^t V^{r/p} B^{-r/q} b\right)^{-p/r} v(t)\,dt \le \left(\int_0^x V^{r/p} B^{-r/q} b\right)^{-p/r} \int_x^\infty v$$

and, since V is also non-increasing,

$$\left(\int_0^x V^{r/p} B^{-r/q} b\right)^{-p/r} \int_x^\infty v \le \left(\int_0^x B^{-r/q} b\right)^{-p/r} V(x)^{-1} \int_x^\infty v = \left(\int_0^x B^{-r/q} b\right)^{-p/r}.$$

Finally, integration and (2.1) yield

$$\left(\int_0^x B^{-r/q} b\right)^{-p/r} = \left((p/r) B(x)^{-r/p}\right)^{-p/r} = \int_x^\infty \beta$$

which completes the proof.

The next result is a consequence of Muckenhoupt's weighted Hardy inequality [**6**] but we prefer to deduce it from the classical Hardy inequality.

Proposition 2.3. *Suppose $1 < p < \infty$ and w is non-negative and satisfies*

$$0 < \int_0^x w < \infty \quad \textit{for all } x > 0 \textit{ and } \quad \int_0^\infty w = \infty. \tag{2.3}$$

Then

$$\left(\int_0^\infty \left(\int_0^x f\right)^p \left(\int_0^x w\right)^{-p} w(x)\,dx\right)^{1/p} \le p' \left(\int_0^\infty f^p w^{1-p}\right)^{1/p}.$$

Let $W(x) = \int_0^x w$ and make the substitutions $t = W(x)$, $y = W(s)$, $f(s) = g(W(s))w(s)$ in Hardy's inequality [**3**, p240]

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t g(y)\,dy\right)^p dt\right)^{1/p} \le p' \left(\int_0^\infty g(y)^p\,dy\right)^{1/p}.$$

Theorem 2.4. *Suppose $0 < q < p$, $1 < p < \infty$ and $1/r = 1/q - 1/p$. Also suppose that u and v are non-negative functions and set $w = u^{1-p'}$. Then there exists a positive constant C such that the weighted Hardy inequality*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C \left(\int_0^\infty f^p u\right)^{1/p} \tag{2.4}$$

holds for all non-negative f if and only if

$$D \equiv \left(\int_0^\infty \left(\int_0^x w\right)^{r/p'} \left(\int_x^\infty v\right)^{r/p} v(x)\,dx\right)^{1/r} < \infty.$$

Moreover, the smallest constant C for which (2.4) *holds satisfies*

$$(p')^{1/p'} q^{1/p} (1 - q/p) D \le C \le (r/q)^{1/r} p^{1/p} (p')^{1/p'} D.$$

(Necessity) Suppose that (2.4) holds for all $f \geq 0$ and let v_0 and w_0 be L^1 functions such that $0 \leq v_0 \leq v$ and $0 \leq w_0 \leq w$. Set $f(t) = \left(\int_t^\infty v_0\right)^{r/pq} \left(\int_0^t w_0\right)^{r/pq'} w_0(t)$ and note that

$$\int_0^x f(t)\,dt \geq \left(\int_x^\infty v_0\right)^{r/pq} \int_0^x \left(\int_0^t w_0\right)^{r/pq'} w_0(t)\,dt$$
$$= (p'q/r) \left(\int_x^\infty v_0\right)^{r/pq} \left(\int_0^x w_0\right)^{r/p'q}.$$

This estimate, combined with (2.4), yields

$$\left(\int_0^\infty \left(\frac{p'q}{r}\right)^q \left(\int_x^\infty v_0\right)^{r/p} \left(\int_0^x w_0\right)^{r/p'} v_0(x)\,dx\right)^{1/q} \leq \left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q}$$
$$\leq C\left(\int_0^\infty f^p w^{1-p}\right)^{1/p} = C\left(\int_0^\infty \left(\int_t^\infty v_0\right)^{r/q} \left(\int_0^t w_0\right)^{r/q'} w_0(t)^p w(t)^{1-p}\,dt\right)^{1/p}$$
$$\leq C\left(\int_0^\infty \left(\int_t^\infty v_0\right)^{r/q} \left(\int_0^t w_0\right)^{r/q'} w_0(t)\,dt\right)^{1/p}$$
$$= C(p'/q)^{1/p}\left(\int_0^\infty \left(\int_x^\infty v_0\right)^{r/p} \left(\int_0^x w_0\right)^{r/p'} v_0(x)\,dx\right)^{1/p}$$

where the last inequality is integration by parts. Since v_0 and w_0 are in L^1, the integral on the right hand side is finite. Dividing by it, we have

$$(p'q/r)(q/p')^{1/p}\left(\int_0^\infty \left(\int_x^\infty v_0\right)^{r/p} \left(\int_0^x w_0\right)^{r/p'} v_0(x)\,dx\right)^{1/r} \leq C.$$

Approximating v and w from below by increasing sequences of L^1 functions and applying the Monotone Convergence Theorem we conclude that $(p')^{1/p'}q^{1/p}(1-q/p)D \leq C$ as required.

(Sufficiency) Suppose that $D < \infty$ and, for the moment, that (2.3) holds for w. Set $W(x) = \int_0^x w$ and apply Proposition 2.2 with $b = W^{-p}w$ and $F(x) = \int_0^x f$. Note that

(2.1) holds since $\int_x^\infty b = \int_x^\infty W^{-p}w = (p'/p)W(x)^{1-p}$. The conclusion of Proposition 2.2 becomes

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \\ \leq (r/p)^{1/r}\left(\int_0^\infty \left(\int_x^\infty v\right)^{r/q}\left(\int_x^\infty W^{-p}w\right)^{-r/q} W(x)^{-p}w(x)\,dx\right)^{1/r} \\ \times\left(\int_0^\infty \left(\int_0^x f\right)^p W(x)^{-p}w(x)\right)^{1/p}.$$

Performing the inner integration in the first factor, and applying Proposition 2.3 to the second factor, we reach the inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \\ \leq (r/p)^{1/r}(p/p')^{1/q}p'\left(\int_0^\infty \left(\int_x^\infty v\right)^{r/q}\left(\int_0^x w\right)^{r/q'} w(x)\,dx\right)^{1/r}\left(\int_0^\infty f^p u\right)^{1/p} \\ = (r/p)^{1/r}(p/p')^{1/q}p'(p'/q)^{1/r}D\left(\int_0^\infty f^p u\right)^{1/p} = (r/q)^{1/r}p^{1/p}(p')^{1/p'}D\left(\int_0^\infty f^p u\right)^{1/p}$$

where the next to last equality is integration by parts. (The last equality only changes the form of the constant.)

To establish sufficiency for general w, we fix non-negative functions v and w. If $w = 0$ almost everywhere on some interval $(0, x)$ then translating v and w to the left will reduce the problem to one in which this does not occur. (If $w = 0$ almost everywhere on $(0, \infty)$ sufficiency holds trivially.) We therefore assume that $0 < \int_0^x w$ for all $x > 0$. For each $n > 0$ set $v_n = v\chi_{(0,n)}$ and $w_n = \min(w, n) + \chi_{(n,\infty)}$. w_n clearly satisfies (2.3) so our previous argument applies and we have

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v_n(x)\,dx\right)^{1/q} \\ \leq c\left(\int_0^\infty \left(\int_x^\infty v_n\right)^{r/p}\left(\int_0^x w_n\right)^{r/p'} v_n(x)\,dx\right)^{1/r}\left(\int_0^\infty f^p w_n^{1-p}\right)^{1/p}$$

for all $f \geq 0$. Here $c = (r/q)^{1/r} p^{1/p} (p')^{1/p'}$. If we take f to be $g \min(w,n)^{1/p'} \chi_{(0,n)}$ and use the definitions of v_n and w_n the inequality becomes

$$\left(\int_0^n \left(\int_0^x g \min(w,n)^{1/p'}\right)^q v(x)\,dx\right)^{1/q}$$
$$\leq c \left(\int_0^n \left(\int_x^n v\right)^{r/p} \left(\int_0^x \min(w,n)\right)^{r/p'} v(x)\,dx\right)^{1/r} \left(\int_0^n g^p\right)^{1/p}$$

for all non-negative g. We let $n \to \infty$, apply the Monotone Convergence Theorem and substitute $fw^{-1/p'}$ for g to get the desired inequality and complete the proof.

Remark. Integration by parts shows that if $0 < q < p$, $1 < p < \infty$ and w is locally integrable or $q > 1$ then

$$D = (q/p')^{1/r} \left(\int_0^\infty \left(\int_x^\infty v\right)^{r/q} \left(\int_0^x w\right)^{r/q'} w(x)\,dx\right)^{1/r}. \tag{2.5}$$

Unfortunately, this alternate expression for D may not be equal to D if $q < 1$ and w is not locally integrable. For example, if $w(x) = (1/x)\chi_{(0,1)}(x)$ and $v = \chi_{(0,1)}$ we have $D = \infty$ and the integral on the right in (2.5) equal to 0.

The monotonicity of the kernel makes it possible to carry Hardy inequalities from Lebesgue spaces to Lorentz spaces. The Lorentz space $L^{s,q}(v(x)\,dx)$ is the set of measurable functions g for which

$$\|g\|_{L^{s,q}(v(x)\,dx)} \equiv \left(\int_0^\infty g^*(t)^q \, d(t^{q/s})\right)^{1/q} < \infty$$

where $g_*(\lambda) = \int_{\{x:|g(x)|>\lambda\}} v(x)\,dx$ and $g^*(t) = \inf\{\lambda : g_*(\lambda) \leq t\}$. The following result may be compared with [**9**, Theorem 3].

Corollary 2.5. *Suppose that p, q, r, u, v, and w are as in Theorem 2.4 and that $0 < s < \infty$. The inequality*

$$\left\|\int_0^x f\right\|_{L^{s,q}(v(x)\,dx)} \leq C\|f(x)\|_{L^{p,p}(u(x)\,dx)} \tag{2.6}$$

holds if and only if

$$\left(\int_0^\infty \left(\int_0^x w\right)^{r/p'} \left(\int_x^\infty v\right)^{(r/s)-1} v(x)\,dx\right)^{1/r} < \infty. \tag{2.7}$$

We may restrict our attention to non-negative f in (2.6). In this case $\int_0^x f$ is non-decreasing so we easily calculate that

$$\left\|\int_0^x f\right\|_{L^{s,q}(v(x)\,dx)} = \left(\int_0^\infty \left(\int_0^x f\right)^q \frac{q}{s}\left(\int_x^\infty v\right)^{(q/s)-1} v(x)\,dx\right)^{1/q}$$

and

$$\|f\|_{L^{p,p}(u(x)\,dx)} = \left(\int_0^\infty f^p u\right)^{1/p}.$$

Now Theorem 2.4 shows that (2.6) holds if and only if

$$\left(\int_0^\infty \left(\int_0^x w\right)^{r/p'} \left(\int_x^\infty \frac{q}{s}\left(\int_t^\infty v\right)^{(q/s)-1} v(t)\,dt\right)^{r/p} \frac{q}{s}\left(\int_x^\infty v\right)^{(q/s)-1} v(x)\,dx\right)^{1/r}$$

is finite. Straightforward integration shows that this is the condition (2.7).

3. *The Case* $p = 1$

The first result of this section, Theorem 3.2, shows that when the domain of a positive, integral operator is weighted L^1, the monotonicity of the kernel can be transferred to the weight. We apply this principle to give a characterisation of those weights for which the Hardy inequality (2.4) holds in the case $0 < q < 1 = p$.

Definition 3.1. *For a non-negative function u define $\underline{u}$ by $\underline{u}(x) = \operatorname{ess\,inf}_{0<t<x} u(t)$.*

It is easy to see that $\underline{u}$ is non-increasing and standard arguments show that $\underline{u} \le u$ almost everywhere.

Theorem 3.2. *Suppose that $0 < q < \infty$ and that $k(x,t)$ is a non-negative kernel which is non-increasing in t for each x. The best constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^\infty k(x,t)f(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu, \quad f \ge 0, \tag{3.1}$$

in unchanged when u is replaced by $\underline{u}$.

The proof is presented in Section 5.

Since $k(x,t) = \chi_{(0,x)}(t)$ is non-negative and non-increasing in t we can apply Theorem 3.2 to Hardy inequalities.

Theorem 3.3. *Suppose that $0 < q < 1$. If C is the best constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu, \quad f \ge 0, \tag{3.2}$$

then

$$(1-q)^{(1-q)/q}C \le \left(\int_0^\infty \underline{u}(x)^{q/(q-1)}\left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \le \frac{1}{q(1-q)}C. \tag{3.3}$$

Theorem 3.2 shows that C is the best constant in the inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty f\underline{u}, \quad f \ge 0. \tag{3.4}$$

We first consider the case $\underline{u}(x) = \int_x^\infty b$ for some b satisfying (2.1). The right hand side of (3.4) becomes

$$C\int_0^\infty f(t)\left(\int_t^\infty b\right)dt = C\int_0^\infty \left(\int_0^x f\right) b(x)\,dx.$$

Since any non-negative, non-decreasing function F is the limit of an increasing sequence of functions of the form $\int_0^x f$ with $f \ge 0$, we see that C is also the best constant in the inequality

$$\left(\int_0^\infty F^q v\right)^{1/q} \le C\int_0^\infty Fb, \quad F \ge 0, F\uparrow.$$

The proof of [**12**, Proposition 1(b)] yields (3.3).

Next we consider the case of general $\underline{u}$. If $\underline{u} = \infty$ on some interval $(0, x)$ then translating u and v to the left will reduce the problem to one in which this does not occur. (If $\underline{u} \equiv \infty$ then the problem is trivial.) We therefore assume that $\underline{u}(x) < \infty$ for all $x > 0$. For each $n > 0$, $\underline{u}\chi_{(0,n)}$ is finite, non-increasing and tends to 0 at ∞ so we can approximate it from above by functions of the form $\int_x^\infty b$ with b satisfying (2.1). Let $\{u_m\}$ be a non-increasing sequence of such functions that converges to $\underline{u}\chi_{(0,n)}$ pointwise almost everywhere. Set $v_n = v\chi_{(0,n)}$. The first part of the proof shows that for all non-negative functions f, the inequality

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v_n(x)\,dx\right)^{1/q} \leq (1-q)^{(q-1)/q} \left(\int_0^\infty u_m(x)^{q/(q-1)} \left(\int_x^\infty v_n\right)^{q/(1-q)} v_n(x)\,dx\right)^{(1-q)/q} \int_0^\infty f u_m$$

holds. Thus, for all non-negative functions g, the inequality

$$\left(\int_0^n \left(\int_0^x g u_m^{-1}\right)^q v(x)\,dx\right)^{1/q} \leq (1-q)^{(q-1)/q} \left(\int_0^n u_m(x)^{q/(q-1)} \left(\int_x^n v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \int_0^\infty g$$

holds. As $m \to \infty$ both u_m^{-1} and $u_m^{q/(q-1)}$ are non-decreasing sequences so the Monotone Convergence Theorem implies that for all non-negative g we have

$$\left(\int_0^n \left(\int_0^x g \underline{u}^{-1}\right)^q v(x)\,dx\right)^{1/q} \leq (1-q)^{(q-1)/q} \left(\int_0^n \underline{u}(x)^{q/(q-1)} \left(\int_x^n v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \int_0^\infty g.$$

Letting $n \to \infty$ and applying the Monotone Convergence Theorem again we obtain, with

$f = g/\underline{u}$,

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le (1-q)^{(q-1)/q} \left(\int_0^\infty \underline{u}(x)^{q/(q-1)} \left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \int_0^\infty f\underline{u}.$$

This gives the left hand inequality in (3.3). To establish the right hand inequality we suppose that (3.2) and hence (3.4) holds. Since for any n and m, $v_n \le v$ and $\underline{u} \le u_m$ we have

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v_n(x)\,dx\right)^{1/q} \le C\int_0^\infty f u_m, \quad f \ge 0.$$

Hence

$$\left(\int_0^n u_m(x)^{q/(q-1)} \left(\int_x^n v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \le \frac{1}{q(1-q)}C.$$

Using the Monotone Convergence Theorem twice in succession as above we have the right hand inequality in (3.3). This completes the proof.

Remark. If we denote the best constant in (2.4) by C_p then one can show, at least for well behaved u and v, that $\lim_{p\to 1^+} C_p = C_1$ and that

$$\begin{aligned}\lim_{p\to 1^+} D &= \lim_{p\to 1^+} \left(\int_0^\infty \left(\int_0^x \left(\frac{1}{u}\right)^{p'} u\right)^{r/p'} \left(\int_x^\infty v\right)^{r/p} v(x)\,dx\right)^{1/r} \\ &= \left(\int_0^\infty \left\|\frac{1}{u}\chi_{(0,x)}\right\|_\infty^{q/(1-q)} \left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \\ &= \left(\int_0^\infty \underline{u}(x)^{q/(q-1)} \left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q},\end{aligned}$$

the same integral that appears in Theorem 3.3. Moreover, the constants obtained in Theorem 2.4 are bounded as $p \to 1^+$, unlike the constants given previously for this problem in [**10**] and [**7**, p130]. Specifically, the conclusion

$$(p')^{1/p'} q^{1/p} (1-q/p) D \le C_p \le (r/q)^{1/r} p^{1/p} (p')^{1/p'} D.$$

of Theorem 2.4 becomes

$$q(1-q)\lim_{p\to 1^+} D \le C_1 \le (1-q)^{(q-1)/q}\lim_{p\to 1^+} D$$

when $p \to 1^+$, giving precisely the same estimates as Theorem 3.3.

When $1 < q < \infty$ there is a formula in [**4**, p316] for the best constant in (3.1). The formula, together with Theorem 3.2, yields two expressions for the best constant when $k(x,t)$ is non-increasing in t. Of course they are equal so we have

$$\operatorname*{ess\,sup}_{t>0} u(t)^{-1}\left(\int_0^\infty k(x,t)^q v(x)\,dx\right)^{1/q} = \operatorname*{ess\,sup}_{t>0} \underline{u}(t)^{-1}\left(\int_0^\infty k(x,t)^q v(x)\,dx\right)^{1/q},$$

a fact which is not difficult to establish directly. It suggests, however, that perhaps we may replace $\underline{u}$ by u in the condition of Theorem 3.3. That is, perhaps the finiteness of

$$\left(\int_0^\infty u(x)^{q/(q-1)}\left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} \tag{3.5}$$

might imply the inequality (3.2). The next example shows that this is not the case.

Example. For $n = 0, 1, 2\ldots$ let A_n be a dense, open subset of $(\frac{n}{n+1}, \frac{n+1}{n+2})$ with (Lebesgue) measure less than $2^{-n}/(n+2)$. Set $A = \cup_{n=0}^\infty A_n$, $u(x) = \chi_{A^C}(x) + (1-x)^{1/q}\chi_A(x)$ and $v = \chi_{(0,1)}$. Since A is open and dense in $(0,1)$ we have $\underline{u}(x) = (1-x)^{1/q}\chi_{(0,1)}(x)$. The inequality (3.2) does not hold because the condition of Theorem 3.3 is not finite:

$$\int_0^\infty \underline{u}(x)^{q/(q-1)}\left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx$$
$$= \int_0^1 (1-x)^{1/(q-1)}(1-x)^{q/(1-q)}\,dx = \int_0^1 (1-x)^{-1}\,dx = \infty.$$

On the other hand (3.5) is finite:

$$\int_0^\infty u(x)^{q/(q-1)}\left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx$$
$$= \int_{A^C} (1-x)^{q/(1-q)}\,dx + \sum_{n=0}^\infty \int_{A_n} (1-x)^{-1}\,dx \le 1 + \sum_{n=0}^\infty (n+2)(2^{-n}/(n+2)) \le 3.$$

4. *Inequalities for Monotone Functions*

Here we consider the weighted Hardy inequality with $0 < q < 1 = p$ where the function f is constrained to be monotone. The case in which f is non-increasing reduces to our previous results (Theorem 4.1) but the case in which f is non-decreasing is more difficult. An analogue of Theorem 3.1 is given which is of independent interest but it does not lead to a complete resolution of the problem.

THEOREM 4.1. *Suppose* $0 < q < 1$. *There exists a* $C > 0$ *such that the inequality*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu \quad f \ge 0,\ f\downarrow, \tag{4.1}$$

holds if and only if both

$$\left(\int_0^\infty \left(\operatorname*{ess\,inf}_{0<t<x} \frac{1}{t}\int_0^t u\right)^{q/(q-1)} \left(\int_x^\infty v\right)^{q/(1-q)} v(x)\,dx\right)^{(1-q)/q} < \infty \tag{4.2}$$

and

$$\left(\int_0^\infty \left(\int_0^x u\right)^{q/(q-1)} \left(\int_0^x t^q v(t)\,dt\right)^{q/(1-q)} x^q v(x)\,dx\right)^{(1-q)/q} < \infty. \tag{4.3}$$

Every non-negative, non-increasing function f is the pointwise limit of an increasing sequence of functions of the form $\int_y^\infty h$, $h \ge 0$. (Set $f_n = n\int_x^{x+(1/n)} f\chi_{(0,n)}$ and note that f_n is an increasing sequence which converges pointwise almost everywhere to f even if $f(\infty) > 0$. Also, $f_n(x) = \int_x^\infty n\left[f(t)\chi_{(0,n)}(t) - f(t+(1/n))\chi_{(0,n)}(t+(1/n))\right]dt$ which is of the required form.) Thus, (4.1) is equivalent to

$$\left(\int_0^\infty \left(\int_0^x \left(\int_y^\infty h\right) dy\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty \left(\int_y^\infty h\right) u(y)\,dy, \quad h \ge 0,$$

which is, interchanging the order of integration,

$$\left(\int_0^\infty \left(\int_0^x th(t)\,dt + x\int_x^\infty h(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty h(t)\left(\int_0^t u\right) dt, \quad h \ge 0.$$

This inequality holds if and only if both

$$\left(\int_0^\infty \left(\int_0^x th(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty h(t)\left(\int_0^t u\right)\,dt, \quad h \ge 0, \tag{4.4}$$

and

$$\left(\int_0^\infty \left(x\int_x^\infty h(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty h(t)\left(\int_0^t u\right)\,dt, \quad h \ge 0, \tag{4.5}$$

hold. Theorem 3.3 shows that (4.2) and (4.3) are necessary and sufficient for (4.4) and (4.5) respectively.

Remark. For purposes of comparison we note that (4.1) holds for $1 < q < \infty$ if and only if

$$\operatorname*{ess\,sup}_{x>0}\left(\frac{1}{x}\int_0^x u\right)^{-1}\left(\int_x^\infty v\right)^{1/q} < \infty$$

and

$$\operatorname*{ess\,sup}_{x>0}\left(\int_0^x u\right)^{-1}\left(\int_0^x t^q v(t)\,dt\right)^{1/q} < \infty.$$

Here we have applied [**4**, p316] to (4.4) and (4.5), which are also jointly equivalent to (4.1) when $1 < q < \infty$.

Our analogue of Theorem 3.1 for non-decreasing f requires the notion of the level function.

Proposition 4.2. [**2, 11**] *If $0 \le u \in L^1 \cap L^\infty$, then there exists a function u^o, called the level function of u (with respect to Lebesgue measure), with the following properties:*

(1) *u^o is non-increasing on $(0,\infty)$,*

(2) *$\int_0^x u \le \int_0^x u^o$ for almost every x,*

(3) *there are disjoint intervals I_i such that u^o is constant on I_i with value $(1/|I_i|)\int_{I_i} u$ and $u^o = u$ almost everywhere off the union of the I_i.*

The level function is defined for more general u in [**11**] but we content ourselves with this special case to avoid technical difficulties. (The level functions defined in [**2**] and in [**11**] are slightly different but when taken with respect to Lebesgue measure they coincide.)

THEOREM 4.3. *Suppose that* $0 < q < \infty$ *and that* $k(x,t)$ *is a non-negative kernel which is non-increasing in* t *for each* x. *If* $0 \le u \in L^1 \cap L^\infty$ *then the best constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^\infty k(x,t)f(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu, \quad f \ge 0,\ f\uparrow, \tag{4.6}$$

is unchanged when u *is replaced by* u^o.

The proof is presented in Section 5.

THEOREM 4.4. *Suppose* $0 < q < 1$ *and* $0 \le u \in L^1 \cap L^\infty$. *The best constants in each of the following inequalities coincide.*

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu, \quad f \ge 0,\ f\uparrow. \tag{4.7}$$

$$\left(\int_0^\infty \left(\int_0^x f\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty fu^o, \quad f \ge 0,\ f\uparrow. \tag{4.8}$$

$$\left(\int_0^\infty \left(\int_0^x (x-t)h(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty h(x)\left(\int_x^\infty u\right)dx, \quad h \ge 0. \tag{4.9}$$

$$\left(\int_0^\infty \left(\int_0^x (x-t)h(t)\,dt\right)^q v(x)\,dx\right)^{1/q} \le C\int_0^\infty h(x)\left(\int_x^\infty u^o\right)dx, \quad h \ge 0. \tag{4.10}$$

$$\left(\int_0^\infty F^q v\right)^{1/q} \le C\int_0^\infty F\,d\mu, \quad F \ge 0,\ F(0) = 0,\ F \text{ convex}. \tag{4.11}$$

Here μ *is the Borel measure defined by* $u^o(x) = \int_x^\infty d\mu$ *for almost every* x.

The equivalence of (4.7) and (4.8) follows from Theorem 4.3. Each f in (4.7), even if $f(0) > 0$, may be approximated pointwise from below by a function of the form $\int_0^x h$ with

$h \geq 0$. Substituting $\int_0^x h$ for f in (4.7) and interchanging the order of integration on both sides produces (4.9). In the same way (4.10) is equivalent to (4.8).

To see that (4.11) also reduces to (4.8) we note that each F from (4.11) may be written in the form $\int_0^x f$ with $f \geq 0$ and $f \uparrow$. Interchanging the order of integration completes the equivalence and the proof.

Perhaps the most interesting consequence of this theorem is that the best constant for an embedding of the cone of convex functions (4.11) is also the best constant for a weighted Riemann-Liouville inequality (4.9).

5. *Proofs of Two Theorems*

Proof of Theorem 3.2. Let C be the best constant in (3.1) and let $\underline{C}$ be the best constant in (3.1) with u replaced by $\underline{u}$. Since $\underline{u} \leq u$ almost everywhere, $C \leq \underline{C}$. To prove the reverse inequality it is enough to show that for all non-negative $f \in L^1(\underline{x}, \infty)$

$$\left(\int_0^\infty \left(\int_{\underline{x}}^\infty k(x,t) f(t)\, dt\right)^q v(x)\, dx\right)^{1/q} \leq C \int_{\underline{x}}^\infty f \underline{u}, \tag{5.1}$$

where $\underline{x} = \inf\{x \geq 0 : \underline{u}(x) < \infty\}$.

Fix a non-negative $f \in L^1(\underline{x}, \infty)$ and an $\varepsilon \in (0,1)$ and set $A = \{x > \underline{x} : u(x) < \varepsilon + \underline{u}(x)\}$ and $B_j = \{x > \underline{x} : j\varepsilon \leq \underline{u}(x) < (j+1)\varepsilon\}$ for $j = 0, 1, \ldots$. Let $J = \{j : |B_j| > 0\}$ and note that $\cup_{j \in J} B_j$ has full measure in $(\underline{x}, \infty)$. Since $\underline{u}$ is non-increasing we can define a_j and b_j with $\underline{x} \leq a_j < b_j \leq \infty$ by $(a_j, b_j) \subset B_j \subset [a_j, b_j]$ for each $j \in J$. Moreover, if $j_1 < j_2$ with $j_1, j_2 \in J$ we have $a_{j_2} < b_{j_2} \leq a_{j_1} < b_{j_1}$.

We wish to show that each interval B_j has part of the set A, the points at which u and $\underline{u}$ are close, near its left endpoint. To do this we define $A_j = A \cap (a_j, a_j + \varepsilon(b_j - a_j)) \subset B_j$ for each $j \in J$ and demonstrate that A_j has positive measure. Since $\underline{u} \le u$ almost everywhere we have, for each $j \in J$,

$$\operatorname*{ess\,inf}_{0<t<a_j} u(t) \ge \operatorname*{ess\,inf}_{0<t<a_j} \underline{u}(t) \ge (j+1)\varepsilon > \underline{u}(a_j + \varepsilon(b_j - a_j)) = \operatorname*{ess\,inf}_{0<t<a_j+\varepsilon(b_j-a_j)} u(t).$$

It follows that on some subset of $(a_j, a_j+\varepsilon(b_j-a_j))$ of positive measure, $u < (j+1)\varepsilon \le \underline{u}+\varepsilon$. Thus $|A_j| > 0$.

We are now ready to define the functions f_ε which are closely related to f but are supported in A. We set $f_\varepsilon = \sum_{j\in J}\left(\int_{a_j}^{b_j} f\right)|A_j|^{-1}\chi_{A_j}$. The first observation to make about the functions f_ε is that their L^1_u norm is almost less than the $L^1_{\underline{u}}$ norm of f. We calculate as follows.

$$\begin{aligned}\int_{\underline{x}}^{\infty} f_\varepsilon u &= \sum_{j\in J}\left(\int_{a_j}^{b_j} f\right)|A_j|^{-1}\int_{A_j} u \le \sum_{j\in J}\left(\int_{a_j}^{b_j} f\right)(j+2)\varepsilon \\ &\le \sum_{j\in J}\left(\int_{a_j}^{b_j} f\underline{u} + 2\varepsilon\int_{a_j}^{b_j} f\right) = \int_{\underline{x}}^{\infty} f\underline{u} + 2\varepsilon\int_{\underline{x}}^{\infty} f.\end{aligned}$$

Our second observation is that the mass of the functions f_ε is almost farther left than the mass of f so the inequality $\int_{\underline{x}}^x f \le \int_{\underline{x}}^x f_\varepsilon$ almost holds. To be precise, if $x \in \cup_{j\in J}(a_j + \varepsilon(b_j - a_j), b_j)$ then $x \in (a_{j_0} + \varepsilon(b_{j_0} - a_{j_0}), b_{j_0})$ for some $j_0 \in J$ so, recalling the reversed order of the intervals B_j, we have

$$\begin{aligned}\int_{\underline{x}}^{x} f &\le \sum_{j_0\le j\in J}\int_{a_j}^{b_j} f = \sum_{j_0\le j\in J}\left(\int_{a_j}^{b_j} f\right)|A_j|^{-1}\int_{a_j}^{a_j+\varepsilon(b_j-a_j)}\chi_{A_j} \\ &= \sum_{j_0\le j\in J}\int_{a_j}^{a_j+\varepsilon(b_j-a_j)} f_\varepsilon \le \int_{\underline{x}}^{a_{j_0}+\varepsilon(b_{j_0}-a_{j_0})} f_\varepsilon \le \int_{\underline{x}}^{x} f_\varepsilon.\end{aligned}$$

Thus $\int_{\underline{x}}^x f \le \liminf_{\varepsilon\to 0^+}\int_{\underline{x}}^x f_\varepsilon$ pointwise almost everywhere.

The monotonicity of the kernel $k(x,t)$ for each x enables us to write $k(x,t) = k_x + \int_t^\infty d\sigma_x$ for almost every t, where k_x is a non-negative constant and σ_x is a non-negative, Borel measure. This fact, together with our second observation and Fatou's Lemma allow us to estimate the operator as follows.

$$\int_{\underline{x}}^{\infty} k(x,t)f(t)\,dt = \int_{\underline{x}}^{\infty}\left(k_x + \int_t^{\infty} d\sigma_x\right) f(t)\,dt$$
$$= k_x \int_{\underline{x}}^{\infty} f(t)\,dt + \int_{\underline{x}}^{\infty}\int_{\underline{x}}^{s} f(t)\,dt\,d\sigma_x(s)$$
$$\leq \liminf_{\varepsilon\to 0^+}\left(k_x\int_{\underline{x}}^{\infty} f_\varepsilon(t)\,dt + \int_{\underline{x}}^{\infty}\int_{\underline{x}}^{s} f_\varepsilon(t)\,dt\,d\sigma_x(s)\right) = \liminf_{\varepsilon\to 0^+}\int_{\underline{x}}^{\infty} k(x,t)f_\varepsilon(t)\,dt.$$

We use this estimate and Fatou's Lemma again as well as the definition of C and our first observation to obtain

$$\left(\int_0^{\infty}\left(\int_{\underline{x}}^{\infty} k(x,t)f(t)\,dt\right)^q v(x)\,dx\right)^{1/q}$$
$$\leq \liminf_{\varepsilon\to 0^+}\left(\int_0^{\infty}\left(\int_{\underline{x}}^{\infty} k(x,t)f_\varepsilon(t)\,dt\right)^q v(x)\,dx\right)^{1/q}$$
$$\leq \liminf_{\varepsilon\to 0^+} C\int_{\underline{x}}^{\infty} f_\varepsilon u \leq C\liminf_{\varepsilon\to 0^+}\left(\int_{\underline{x}}^{\infty} f\underline{u} + 2\varepsilon\int_{\underline{x}}^{\infty} f\right) = C\int_{\underline{x}}^{\infty} f\underline{u}.$$

We have established (5.1) which completes the proof.

Proof of Theorem 4.3. Let C be the best constant in (4.6) and let C^o be the best constant in (4.6) with u replaced by u^o. Proposition 4.2, part (3) shows that $\int_0^\infty u^o = \int_0^\infty u$. Combining this with part (2) we have $\int_x^\infty u^o \leq \int_x^\infty u$ for all $x > 0$. Lemma 2.1 shows that if f is non-negative and non-decreasing then $\int_0^\infty f u^0 \leq \int_0^\infty fu$. It follows that $C \leq C^o$.

To prove that $C^o \leq C$ we fix a non-negative, non-decreasing function f and construct a related function g. On each interval I_i from Proposition 4.2, part (3) let g be constant with value $|I_i|^{-1}\int_{I_i} f$. Otherwise let $g = f$. Clearly $\int_{I_i} g = \int_{I_i} f$ for each i. It follows that if t is not interior to any of the intervals I_i then $\int_0^t g = \int_0^t f$. If, on the other hand, t is interior to one of the intervals I_i, call it (a,b), then we have

$$\int_0^t g = \int_0^a g + \int_a^t g = \int_0^a f + \frac{t-a}{b-a}\int_a^b f \geq \int_0^t f$$

where the inequality is a consequence of the monotonicity of f. We have shown that $\int_0^t f \le \int_0^t g$ for all t and we conclude, just as in Lemma 2.1, that for each x, $\int_0^\infty k(x,t)f(t)\,dt \le \int_0^\infty k(x,t)g(t)\,dt$ because $k(x,t)$ is non-increasing in t.

The definition of g was made specifically for the following calculation.

$$\int_0^\infty gu = \int_{(\cup I_i)^C} fu + \sum_i \int_{I_i} \left(|I_i|^{-1} \int_{I_i} f \right) u$$
$$= \int_{(\cup I_i)^C} fu + \sum_i \int_{I_i} \left(|I_i|^{-1} \int_{I_i} u \right) f = \int_0^\infty fu^o.$$

Now we can complete the proof.

$$\left(\int_0^\infty \left(\int_0^\infty k(x,t)f(t)\,dt \right)^q v(x)\,dx \right)^{1/q}$$
$$\le \left(\int_0^\infty \left(\int_0^\infty k(x,t)g(t)\,dt \right)^q v(x)\,dx \right)^{1/q} \le C \int_0^\infty gu = C \int_0^\infty fu^o.$$

Since the above inequality holds for all non-negative, non-decreasing functions f we have shown that $C^o \le C$.

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