ONE-DIMENSIONAL HARDY-TYPE INEQUALITIES IN MANY DIMENSIONS

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Synopsis

Weighted inequalities for certain Hardy-type averaging operators in $\mathbf{R}^n$ are shown to be equivalent to weighted inequalities for one-dimensional operators. Known results for the one-dimensional operators are applied to give weight characterisations, with best constants in some cases, in the higher dimensional setting. Operators considered include averages over all dilations of very general starshaped regions as well as averages over all balls touching the origin. As a consequence, simple weight conditions are given which imply weighted norm inequalities for a class of integral operators with monotone kernels.

1. Introduction

The one-dimensional theory of Hardy-type inequalities is progressing well. More and more often, new problems that arise can be solved using known results or by established techniques. The theory in higher dimensions is perceived to be much more difficult and indeed there are significant problems in higher dimensions for which the one-dimensional techniques are not adequate. It is important to recognize, however, that many higher dimensional problems are really one-dimensional in

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nature and may be successfully analysed using the one-dimensional theory. We give weight characterisations for two such problems, and apply the results to establish best constants in inequalities involving weighted averages over starshaped domains. The higher dimensional results also provide an elegant approach to studying one-dimensional integral operators with monotone kernels.

For the first class of problems we look at the operator that averages functions over regions which are dilations of a fixed region, starshaped with respect to the origin. (A subset of $\mathbf{R}^n$ is starshaped with respect to the origin provided that for each point in the set, the closed segment joining that point to the origin lies entirely within the set.) This is a one-parameter family of regions so it is not surprising that weighted inequalities for the associated averaging operator can be reduced to one-dimensional inequalities. It is somewhat surprising that averages over such families are as easily treated as averages over balls centered at the origin. (The result for averages over balls was given in [**4**].) The main result is in Theorem 2.1 and Theorem 2.4 gives the best constants in these inequalities for a certain class of weight functions.

The second class of weighted inequalities involves averaging a function over a genuinely n-dimensional collection of regions—all the balls with the origin on their boundary. This time, the reduction to one-dimension depends on showing that it is sufficient to test the inequality over the class of radial functions. This is why we restrict ourselves in Theorem 3.2 to radial weight functions. Christ and Grafakos in [**3**] have given unweighted inequalities for this operator.

In the last section we look at inequalities for more general integral operators and derive sufficient conditions for weighted inequalities in one-dimension by considering them as restrictions of higher-dimensional ones.

If u is a radial function on $\mathbf{R}^n$, ie., if $u(x) = u(y)$ whenever $|x| = |y|$, then we will abuse notation by sometimes writing $u(s)$ instead of $u(x)$ when $s = |x|$.

We denote the unit sphere by $\Omega = \{x \in \mathbf{R}^n : |x| = 1\}$.

The notation $A \approx B$ means that there exist positive constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

2. Averages over starshaped regions

We will call a region $S \in \mathbf{R}^n$ *smoothly starshaped* provided there exists a piecewise-C^1 function ψ defined on the unit sphere in $\mathbf{R}^n$ and having real non-negative values, with $S = \{x \in \mathbf{R}^n \setminus \{0\} : |x| \leq \psi(x/|x|)\}$.

If S is smoothly starshaped, let $B = \{x \in \mathbf{R}^n \setminus \{0\} : |x| = \psi(x/|x|)\}$ and note that B is contained in the boundary of S. Since ψ is not assumed to be continuous, B may not be the whole boundary of S. The family of regions we average over is the collection of dilations of S.

Let E be the union of all dilations of S, $E = \cup_{\alpha>0} \alpha S$, and note that $E = \mathbf{R}^n$ whenever 0 is in the interior of S. For non-zero $x \in E$, since S is starshaped, there is a least positive dilation $\alpha_x S$ which contains x. We write $S_x = \alpha_x S$ and note that $x/\alpha_x \in B$ so that x is on the boundary of S.

The n-dimensional weighted inequality that we characterise in this section involves averages over the regions S_x.

THEOREM 2.1. *Suppose $0 < q < \infty$, $1 < p < \infty$, and u and v are non-negative weight functions on E. Then there exists a constant $C > 0$ such that*

$$\left(\int_E \left| \int_{S_x} f(y)\, dy \right|^q v(x)\, dx \right)^{1/q} \leq C \left(\int_E |f(y)|^p u(y)\, dy \right)^{1/p} \tag{2.1}$$

for all locally integrable functions f if and only if either

$p \le q$ and

$$K \equiv \sup_{z \in E\setminus\{0\}} \left(\int_{S_z} u(y)^{1-p'}\, dy \right)^{1/p'} \left(\int_{E\setminus S_z} v(x)\, dx \right)^{1/q} < \infty,$$

or $q < p$, $1/r = 1/q - 1/p$, and

$$D \equiv \left(\int_E \left(\int_{S_z} u(y)^{1-p'}\, dy \right)^{r/p'} \left(\int_{E\setminus S_z} v(x)\, dx \right)^{r/p} v(z)\, dz \right)^{1/r} < \infty.$$

Moreover, the smallest constant C for which (2.1) holds satisfies

$$K \le C \le p^{1/q}(p')^{1/p'} K \quad \text{if } p \le q, \text{ and}$$

$$(p')^{1/p'} q^{1/p}(1 - q/p) D \le C \le (r/q)^{1/r} p^{1/p} (p')^{1/p'} D \quad \text{if } q < p.$$

We will establish Theorem 2.1 as a corollary of our reduction of the n-dimensional inequality to a one-dimensional one. To prove the reduction we make the changes of variable

$$x = s\sigma \quad \text{and} \quad y = t\tau \tag{2.2}$$

where $x, y \in E \setminus \{0\}$, $s, t \in (0, \infty)$, and $\sigma, \tau \in B$. Since B is piecewise smooth we can integrate over it and we have, for any $x \in E$ and any measurable f

$$\int_{S_x} f(y)\, dy = \int_0^{\alpha_x} \int_B f(t\tau) t^{n-1}\, d\tau\, dt.$$

THEOREM 2.2. *Suppose $0 < q < \infty$, $1 < p < \infty$, u and v are non-negative weight functions on E and $C > 0$. Set*

$$V(s) = \int_B v(s\sigma) s^{n-1}\, d\sigma, \quad \text{and} \quad U(t) = \left(\int_B u(t\tau)^{1-p'} t^{n-1}\, d\tau \right)^{1-p}.$$

Then (2.1) holds for all locally integrable functions $f : E \to \mathbf{R}$ if and only if

$$\left(\int_0^\infty \left| \int_0^s F(t)\, dt \right|^q V(s)\, ds \right)^{1/q} \le C \left(\int_0^\infty |F(t)|^p U(t)\, dt \right)^{1/p} \tag{2.3}$$

holds for all locally integrable functions $F : (0,\infty) \to \mathbf{R}$*. In particular, the best constants in inequalities* (2.1) *and* (2.3) *coincide.*

Proof. Suppose (2.3) holds and fix a locally integrable function $f : E \to \mathbf{R}$. Set

$$F(t) = \int_B f(t\tau)t^{n-1}\,d\tau. \tag{2.4}$$

Make the changes of variable (2.2) in the left hand side of (2.1) and notice that for $\sigma \in B$, $\alpha_{s\sigma} = s$.

$$\begin{aligned}
&\left(\int_E \left|\int_{S_x} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q}\\
=&\left(\int_E \left|\int_0^{\alpha_x}\int_B f(t\tau)t^{n-1}\,d\tau\,dt\right|^q v(x)\,dx\right)^{1/q}\\
=&\left(\int_0^\infty\int_B \left|\int_0^{s}\int_B f(t\tau)t^{n-1}\,d\tau\,dt\right|^q v(s\sigma)s^{n-1}\,d\sigma\,ds\right)^{1/q}\\
=&\left(\int_0^\infty \left|\int_0^{s} F(t)\,dt\right|^q V(s)\,ds\right)^{1/q}\\
\le& C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}.
\end{aligned}$$

The last inequality is the hypothesis (2.3). Use Hölder's inequality in the integral defining F to estimate the last line above as follows.

$$\begin{aligned}
&C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty \left|\int_B f(t\tau)t^{n-1}\,d\tau\right|^p U(t)\,dt\right)^{1/p}\\
\le&C\left(\int_0^\infty \left(\int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\right)\times\right.\\
&\qquad\left.\left(\int_B u(t\tau)^{1-p'}t^{n-1}\,d\tau\right)^{p/p'} U(t)\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty\int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}.
\end{aligned}$$

Thus (2.1) holds.

To prove the converse, suppose that (2.1) holds and fix a locally integrable function $F:(0,\infty)\to\mathbf{R}$. Define $f:E\to\mathbf{R}$ by

$$f(t\tau)=F(t)U(t)^{p'-1}u(t\tau)^{1-p'}$$

and use the definition of U to see that the relationship (2.4) is still valid. As in the first part of the proof we have

$$\left(\int_0^\infty \left|\int_0^s F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} = \left(\int_E \left|\int_{S_x} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q}.$$

Now the inequality (2.1) becomes

$$\left(\int_0^\infty \left|\int_0^s F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} \le C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}.$$

Using the definitions of f and U we recognize the right hand side above as the right hand side of (2.3), that is

$$\begin{aligned}
&C\left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}\\
=&C\left(\int_0^\infty \int_B |f(t\tau)|^p u(t\tau)t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty |F(t)|^p U(t)^{p'}\int_B u(t\tau)^{(1-p')p+1}t^{n-1}\,d\tau\,dt\right)^{1/p}\\
=&C\left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p}.
\end{aligned}$$

This establishes (2.3) and completes the proof.

Necessary and sufficient conditions on indices p and q and weights U and V for the weighted Hardy inequality (2.3) to hold are well known. They can now be applied to characterise the equivalent inequality (2.1) and complete the proof of Theorem 2.1.

Proof of Theorem 2.1. In the case $p \le q$, (2.3) holds if and only if, [**2**] or [**8**],

$$\sup_{\rho>0}\left(\int_0^\rho U(t)^{1-p'}\,dt\right)^{1/p'}\left(\int_\rho^\infty V(s)\,ds\right)^{1/q}<\infty.$$

Use the definitions of U and V from Theorem 2.2 to get the equivalent statement

$$\sup_{\rho>0}\left(\int_0^\rho\int_B u(t\tau)^{1-p'}t^{n-1}\,d\tau\,dt\right)^{1/p'}\left(\int_\rho^\infty\int_B v(s\sigma)s^{n-1}\,d\sigma\,ds\right)^{1/q}<\infty.$$

Now replace ρ by α_z, make the changes of variable (2.2), and notice that $y\in S_z$ if and only if $t\le\rho$ and $x\notin S_z$ if and only if $s>\rho$. The condition becomes

$$\sup_{z\in E\setminus\{0\}}\left(\int_{S_z}u(y)^{1-p'}\,dy\right)^{1/p'}\left(\int_{E\setminus S_z}v(x)\,dx\right)^{1/q}<\infty.$$

This condition on p, q, u and v is necessary and sufficient for the inequality (2.3) and, in view of Theorem 2.2, also necessary and sufficient for the inequality (2.1) as required.

In the case $q<p$ we define r by $1/r=1/q-1/p$. The inequality (2.3) holds if and only if, [**9**] or [**8**],

$$\left(\int_0^\infty\left(\int_0^\rho U(t)^{1-p'}\,dt\right)^{r/p'}\left(\int_\rho^\infty V(s)\,ds\right)^{r/p}V(\rho)\,d\rho\right)^{1/r}<\infty.$$

Proceeding as in the first case we use the definitions of U and V from Theorem 2.2 to get the equivalent condition

$$\left(\int_0^\infty\left(\int_0^\rho\int_B u(t\tau)^{1-p'}t^{n-1}\,d\tau\,dt\right)^{r/p'}\times\right.$$
$$\left.\left(\int_\rho^\infty\int_B v(s\sigma)t^{n-1}\,d\sigma\,ds\right)^{r/p}\int_B v(\rho\omega)\rho^{n-1}\,d\omega\,d\rho\right)^{1/r}<\infty.$$

The changes of variable (2.2) as well as $z=\rho\omega$ reduce this to

$$\left(\int_E\left(\int_{S_z}u(y)^{1-p'}\,dy\right)^{r/p'}\left(\int_{E\setminus S_z}v(x)\,dx\right)^{r/p}v(z)\,dz\right)^{1/r}<\infty$$

as required. The estimates for the best constant C are exactly the estimates for the one-dimensional case established in [**2**] and [**9**].

It is important to notice that Theorem 2.2 shows that the n-dimensional inequality is equivalent to a one-dimensional one with the identical constant. Thus the problem of finding the best constant in (2.1) is also reduced to a one-dimensional problem.

Suppose now that $1 < p \le q < \infty$ and define $B_{p,q}$ by

$$B_{p,q} = \begin{cases} p' & p = q \\ \left(\frac{p'}{q}\right)^{1/q} \left(\frac{q}{r}\right)^{1/r} \left(\frac{\Gamma(r)}{\Gamma(r/q')\Gamma(r/q)}\right)^{1/r} & p < q. \end{cases}$$

Here $1/r = |1/p - 1/q|$.

In 1930, Hardy and Littlewood [**5**] and Bliss [**1**] proved that $B_{p,q}$ is the smallest constant C for which the inequality

$$\left(\int_0^\infty \left|\int_0^s g(t)\,dt\right|^q s^{-1-q/p'}\,ds\right)^{1/q} \le C\left(\int_0^\infty |g(t)|^p\,dt\right)^{1/p} \tag{2.5}$$

holds for all g. We require a slight extension of this result suggested by the work of Manakov [**6**].

Proposition 2.3. *Suppose* $1 < p \le q < \infty$ *and let* $a > 0$. *The smallest constant* C *for which*

$$\left(\int_0^\infty \left|\int_0^x f(y)\,dy\right|^q x^{-1-aq/p'}\,dx\right)^{1/q} \le a^{-1/q-1/p'}C\left(\int_0^\infty |f(y)|^p y^{(a-1)(1-p)}\,dy\right)^{1/p} \tag{2.6}$$

holds for all f *is* $C = B_{p,q}$.

Proof. Make the substitutions $t = y^a$, $s = x^a$, and $g(t) = f(y)/(ay^{a-1})$ in (2.5) to get the equivalent inequality (2.6).

As a corollary to Theorem 2.2 we can extend the above Proposition to higher dimensions.

Theorem 2.4. *Suppose $1 < p \le q < \infty$ and let $a > 0$. The smallest constant C for which*

$$\left(\int_E \left|\int_{S_x} f(y)\,dy\right|^q |S_x|^{-1-aq/p'}\,dx\right)^{1/q} \le a^{-1/q-1/p'} C \left(\int_E |f(y)|^p |S_y|^{(a-1)(1-p)}\,dy\right)^{1/p} \tag{2.7}$$

holds for all f is $C = B_{p,q}$.

Proof. Inequality (2.7) is a special case of (2.1) with $v(x) = |S_x|^{-1-aq/p'}$, $u(y) = |S_y|^{(a-1)(1-p)}$, and C replaced by $a^{-1/q-1/p'}C$. According to Theorem 2.2, therefore, (2.7) is equivalent, with identical constants C to

$$\left(\int_0^\infty \left|\int_0^s F(t)\,dt\right|^q V(s)\,ds\right)^{1/q} \le a^{-1/q-1/p'} C \left(\int_0^\infty |F(t)|^p U(t)\,dt\right)^{1/p} \tag{2.8}$$

with U and V depending on u and v as in Theorem 2.2. To calculate U and V recall that $\alpha_{s\sigma} = s$ so $|S_{s\sigma}| = s^n|S|$.

$$\begin{aligned}
\int_0^t V(s) s^{n(1+aq/p')}\,ds &= \int_0^t \int_B v(s\sigma) s^{n-1}\,d\sigma s^{n(1+aq/p')}\,ds \\
&= \int_0^t \int_B (s^n|S|)^{-1-aq/p'} s^{n-1}\,d\sigma s^{n(1+aq/p')}\,ds \\
&= |S|^{-1-aq/p'} \int_0^t \int_B s^{n-1}\,d\sigma\,ds \\
&= |S|^{-1-aq/p'} t^n |S| = |S|^{-aq/p'} t^n.
\end{aligned}$$

Differentiate to get $V(t)t^{n(1+aq/p')} = |S|^{-aq/p'} n t^{n-1}$ and conclude that

$$V(t) = n|S|^{-aq/p'} t^{-1-naq/p'}.$$

A similar argument shows that

$$U(t)^{1-p'} = n|S|^a t^{(na-1)}.$$

With these expressions for U and V, inequality (2.8) becomes

$$\left(\int_0^\infty \left|\int_0^s F(t)\,dt\right|^q s^{-1-naq/p'}\,ds\right)^{1/q} \le (na)^{-1/q-1/p'} C \left(\int_0^\infty |F(t)|^p t^{(na-1)(1-p)}\,dt\right)^{1/p}.$$

By Proposition 2.3, with a replaced by na, the smallest constant C for which this holds is $C = B_{p,q}$. This completes the proof.

Remarks.

1. By these methods it is easy to see that Theorem 2.4 gives all the inequalities with "power" weights. More precisely, if inequality (2.1) holds for $v(x) = |S_x|^\beta$ and $u(y) = |S_y|^\gamma$ for some β and γ then $p \le q$, $\beta = -1 - aq/p'$, and $\gamma = (a-1)(1-p)$ for some $a > 0$.
2. If S_x is taken to be the ball of radius $|x|$ centred at the origin then the case $p \le q$ of Theorem 2.1 reduces to Theorem 2.1 of [**4**]. The case $q < p$ of Theorem 2.1 essentially extends Theorem 2.2 of [**4**] by allowing q to be in $(0, 1)$. Although the weight condition (2.9) of [**4**] is different in form than the one given here, integration by parts shows that they coincide (up to a constant) in the case $1 < q < p$.
3. If S_x is taken to be the ball of radius $|x|$ centred at the origin, $p = q$, and $a = 1$ then Theorem 2.4 reduces to Theorem 1 of [**3**].
4. The condition that S be smoothly starshaped can be considerably weakened. All that is required of a starshaped region S for these arguments to be valid is that integration on $\mathbf{R}^n$ can be transformed to a kind of "polar" coordinates where instead of spherical shells we use the boundaries of dilations of S.

3. Averages over balls

In this section we consider averages over the family of all balls in $\mathbf{R}^n$ having the origin on their boundary.

DEFINITION 3.1. *For each $x \in \mathbf{R}^n$ set*

$$B_x = \{y \in \mathbf{R}^n : |y - x| < |x|\}.$$

The inequality that we will concern ourselves with is the following.

$$\left(\int_{\mathbf{R}^n} \left|\int_{B_x} f(y)\,dy\right|^q v(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbf{R}^n} |f(y)|^p u(y)\,dy\right)^{1/p}, \tag{3.1}$$

for radial weights u and v.

THEOREM 3.2. *Suppose $1 < q \le p$, $1 < p < \infty$, and u and v are radial weight functions on $\mathbf{R}^n$. The inequality (3.1) holds for all locally integrable functions f if and only either*

$p = q$ and

$$\sup_{\rho>0}\left(\int_{|y|<\rho} (\rho - |y|)^{p'(n-1)/2} u(y)^{1-p'}\,dy\right)^{1/p'} \left(\int_{|2x|>\rho} |x|^{-p(n-1)/2} v(x)\,dx\right)^{1/p},$$

$$\sup_{\rho>0}\left(\int_{|y|<\rho} u(y)^{1-p'}\,dy\right)^{1/p'} \left(\int_{|2x|>\rho} (1 - \rho/|2x|)^{p(n-1)/2} v(x)\,dx\right)^{1/p}$$

are both finite, or

$q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_{\mathbf{R}^n}\left(\int_{|y|<|z|} u(y)^{1-p'}\,dy\right)^{r/q'} \times \left(\int_{|2x|>|z|} (1 - |z|/|2x|)^{q(n-1)/2} v(x)\,dx\right)^{r/q} u(z)^{1-p'}\,dz\right)^{1/r},$$

$$\left(\int_{\mathbf{R}^n}\left(\int_{|y|<|z|} (|z| - |y|)^{p'(n-1)/2} u(y)^{1-p'}\,dy\right)^{r/p'} \times \left(\int_{|2x|>|z|} |x|^{-q(n-1)/2} v(x)\,dx\right)^{r/p} |z|^{-q(n-1)/2} v(z)\,dz\right)^{1/r}$$

are both finite.

The proof relies on the reduction of this problem to a known one-dimensional weighted norm inequality, although this time the one-dimensional inequality is not simply a Hardy inequality. To identify the one-dimensional operator that arises we must investigate the following integral which represents the $n-1$ dimensional surface area of a generalized arc, the intersection of a sphere and ball in $\mathbf{R}^n$.

DEFINITION 3.3. *Let* $\sigma \in \Omega$.

$$A(t) = \int_\Omega \chi_{B_\sigma}(2t\tau)\, d\tau.$$

Note that A does not depend on the choice of $\sigma \in \Omega$ and that $A(t) = 0$ for $t \geq 1$.

LEMMA 3.4. *For any* $\sigma, \tau \in \Omega$,

$$\int_\Omega \chi_{B_{s\sigma}}(t\tau)\, d\tau = A(t/2s) = \int_\Omega \chi_{B_{s\sigma}}(t\tau)\, d\sigma.$$

Proof. By Definition 3.1, $t\tau \in B_{s\sigma}$ means $|t\tau - s\sigma| < s$. Dividing by s this becomes $|(t/s)\tau - \sigma| < 1$ which is just $(t/s)\tau \in B_\sigma$. Thus

$$\int_\Omega \chi_{B_{s\sigma}}(t\tau)\, d\tau = \int_\Omega \chi_{B_\sigma}((t/s)\tau)\, d\tau = A(t/2s),$$

the first part of the lemma.

Using Definition 3.1 again we have $t\tau \in B_{s\sigma}$ if and only if $t^2 - 2ts\tau\cdot\sigma + s^2 < s^2$. Interchanging τ and σ has no effect on this condition so it is also equivalent to $t\sigma \in B_{s\tau}$. Therefore

$$\int_\Omega \chi_{B_{s\sigma}}(t\tau)\, d\sigma = \int_\Omega \chi_{B_{s\tau}}(t\sigma)\, d\sigma = A(t/2s).$$

where the last equality is just the first part of the lemma with σ and τ interchanged.

The first step in showing that an n-dimensional inequality is essentially one-dimensional is to identify a small class of functions over which it is sufficient to test the inequality. In the next theorem we reduce the inequality (3.1) to a one-dimensional inequality and the key observation is that it holds for all functions f if and only if it holds when f is radial.

We will regularly pass to and from polar coordinates with the changes of variable

$$x = s\sigma, \quad y = t\tau, \quad \text{and} \quad z = \rho\omega, \tag{3.2}$$

$x, y, z \in \mathbf{R}^n \setminus \{0\}$, $s, t, \rho \in (0, \infty)$, and $\sigma, \tau, \omega \in \Omega$.

THEOREM 3.5. *Suppose that* $1 < q \le p$, $1 < p < \infty$, *and* u *and* v *are radial weight functions. Then inequality* (3.1) *holds for all locally integrable functions* $f : \mathbf{R}^n \to \mathbf{R}$ *if and only if the inequality*

$$\left(\int_0^\infty \left| \int_0^{2s} A(t/2s) F(t) t^{n-1}\, dt \right|^q v(s) s^{n-1}\, ds \right)^{1/q} \\ \le C |\Omega|^{1/p - 1/q} \left(\int_0^\infty |F(t)|^p u(t) t^{n-1}\, dt \right)^{1/p} \tag{3.3}$$

holds for all locally integrable functions $F : (0, \infty) \to \mathbf{R}$. *In particular the best constants* C *in inequalities* (3.1) *and* (3.3) *coincide.*

Proof. Suppose that (3.1) holds and fix $F : (0, \infty) \to \mathbf{R}$. Introduce an integral over Ω into the left hand side of (3.3) and use the fact that v is radial to get

$$\begin{aligned} &\left(\int_0^\infty \left| \int_0^{2s} A(t/2s) F(t) t^{n-1}\, dt \right|^q v(s) s^{n-1}\, ds \right)^{1/q} \\ &= \left(\int_0^\infty |\Omega|^{-1} \int_\Omega \left| \int_0^{2s} A(t/2s) F(t) t^{n-1}\, dt \right|^q v(s\sigma)\, d\sigma s^{n-1}\, ds \right)^{1/q} \\ &= \left(\int_0^\infty |\Omega|^{-1} \int_\Omega \left| \int_0^{2s} \int_\Omega \chi_{B_{s\sigma}}(t\tau)\, d\tau F(t) t^{n-1}\, dt \right|^q v(s\sigma)\, d\sigma s^{n-1}\, ds \right)^{1/q} \end{aligned}$$

where the second equality uses Lemma 3.4. Continue by changing variables via (3.2) to get

$$|\Omega|^{-1/q}\left(\int_{\mathbf{R}^n}\left|\int_{B_x}F(|y|)\,dy\right|^q v(x)\,dx\right)^{1/q}.$$

Now (3.1) with $f(y)$ replaced by $F(|y|)$ shows that the last expression is no greater than

$$C|\Omega|^{-1/q}\left(\int_{\mathbf{R}^n}|F(|y|)|^p u(y)\,dy\right)^{1/p}=C|\Omega|^{1/p-1/q}\left(\int_0^\infty|F(t)|^p u(t)t^{n-1}\,dt\right)^{1/p}$$

where the last equality is a change of variable again. This establishes (3.3)

Conversely, suppose that (3.3) holds. Fix a function f and set

$$F(t)=\left(\int_\Omega|f(t\tau)|^q\,d\tau\right)^{1/q}.$$

Make the substitutions (3.2) in the left hand side of (3.1) and use Minkowski's integral inequality to get

$$\begin{aligned}&\left(\int_{\mathbf{R}^n}\left|\int_{B_x}f(y)\,dy\right|^q v(x)\,dx\right)^{1/q}\\
&=\left(\int_0^\infty\int_\Omega\left|\int_0^{2s}\int_\Omega f(t\tau)\chi_{B_{s\sigma}}(t\tau)\,d\tau t^{n-1}\,dt\right|^q d\sigma v(s)s^{n-1}\,ds\right)^{1/q}\\
&\le\left(\int_0^\infty\left|\int_0^{2s}\left(\int_\Omega\left|\int_\Omega f(t\tau)\chi_{B_{s\sigma}}(t\tau)\,d\tau\right|^q d\sigma\right)^{1/q}t^{n-1}\,dt\right|^q v(s)s^{n-1}\,ds\right)^{1/q}.\end{aligned}$$

Use Hölder's inequality and Lemma 3.4 to estimate the innermost integral.

$$\begin{aligned}\left|\int_\Omega f(t\tau)\chi_{B_{s\sigma}}(t\tau)\,d\tau\right|^q&\le\left(\int_\Omega\chi_{B_{s\sigma}}(t\tau)\,d\tau\right)^{q/q'}\int_\Omega|f(t\tau)|^q\chi_{B_{s\sigma}}(t\tau)\,d\tau\\
&=A(t/2s)^{q/q'}\int_\Omega|f(t\tau)|^q\chi_{B_{s\sigma}}(t\tau)\,d\tau.\end{aligned}$$

This estimate, and another application of Lemma 3.4, yield

$$\begin{aligned}\int_\Omega\left|\int_\Omega f(t\tau)\chi_{B_{s\sigma}}(t\tau)\,d\tau\right|^q d\sigma&\le\int_\Omega A(t/2s)^{q/q'}\int_\Omega|f(t\tau)|^q\chi_{B_{s\sigma}}(t\tau)\,d\tau\,d\sigma\\
&=A(t/2s)^{q/q'}\int_\Omega|f(t\tau)|^q\int_\Omega\chi_{B_{s\sigma}}(t\tau)\,d\sigma\,d\tau\\
&=A(t/2s)^q\int_\Omega|f(t\tau)|^q\,d\tau=A(t/2s)^qF(t)^q.\end{aligned}$$

Using this inequality and the hypothesis (3.3), deduce the inequality (3.1) to complete the proof.

$$\begin{aligned}
&\left(\int_{\mathbf{R}^n}\left|\int_{B_x} f(t)\,dt\right|^q v(x)\,dx\right)^{1/q}\\
\le&\left(\int_0^\infty\left|\int_0^{2s} A(t/2s)F(t)t^{n-1}\,dt\right|^q v(s)s^{n-1}\,ds\right)^{1/q}\\
\le& C|\Omega|^{1/p-1/q}\left(\int_0^\infty |F(t)|^p u(t)t^{n-1}\,dt\right)^{1/p}\\
=& C|\Omega|^{1/p-1/q}\left(\int_0^\infty\left(\int_\Omega |f(t\tau)|^q\,d\tau\right)^{p/q} u(t)t^{n-1}\,dt\right)^{1/p}\\
\le& C|\Omega|^{1/p-1/q}\left(\int_0^\infty\left(\int_\Omega |f(t\tau)|^p\,d\tau\right)\left(\int_\Omega d\tau\right)^{(p-q)/q} u(t)t^{n-1}\,dt\right)^{1/p}\\
=& C\left(\int_{\mathbf{R}^n} |f(y)|^p u(y)\,dy\right)^{1/p}.
\end{aligned}$$

The last inequality here is Hölder's inequality with indices p/q and $p/(p-q)$

The reduction of inequality (3.1) to a one-dimensional inequality is complete. We now use known results to give a weight characterisation for (3.3) which will finish the proof of Theorem 3.2. The following lemma examines the kernel of the integral operator in (3.3), showing that it is bounded above and below by a certain power function. This will enable us to reduce (3.3) to a weighted inequality for a Riemann-Liouville fractional integral operator.

Lemma 3.6. *Let α denote the $n-2$ dimensional volume of the unit sphere in $\mathbf{R}^{n-1}$. Then*

$$A(t)=\alpha\int_t^1 (1-s^2)^{(n-3)/2}\,ds \approx (1-t)^{(n-1)/2}.$$

Proof. Let Ω' denote the unit sphere in $\mathbf{R}^{n-1}$ and recall the induction step in the development of spherical polar coordinates.

$$A(t)=\int_\Omega \chi_{B_\sigma}(2t\tau)\,d\tau=\int_{\Omega'}\int_0^\pi \chi_{B_\sigma}(2t(\cos\theta,\omega\sin\theta))\sin^{n-2}\theta\,d\theta\,d\omega.$$

Because $A(t)$ is independent of σ in Ω, we may take $\sigma = (1, 0, \ldots, 0) \in \Omega$. The characteristic function above is 1 provided $|2t(\cos\theta, \omega\sin\theta) - \sigma| < 1$, that is, when $(2t\cos\theta - 1)^2 + 4t^2\sin^2\theta|\omega|^2 < 1$. Since $\omega \in \Omega'$, $|\omega| = 1$ so this condition simplifies to $\theta < \cos^{-1} t$. Therefore

$$A(t) = \int_{\Omega'} \int_0^{\cos^{-1} t} \sin^{n-2}\theta \, d\theta \, d\omega = \alpha \int_0^{\cos^{-1} t} \sin^{n-2}\theta \, d\theta.$$

Making the substitution $s = \cos\theta$ we have

$$A(t) = \alpha \int_t^1 (1 - s^2)^{(n-3)/2} \, ds,$$

giving the first part of the lemma. Since $0 \le s \le 1$, $1 \le 1 + s \le 2$ so $1 - s^2 = (1 + s)(1 - s) \approx 1 - s$. Therefore

$$A(t) \approx \alpha \int_t^1 (1 - s)^{(n-3)/2} \, ds = (2\alpha/(n-1))(1 - t)^{(n-1)/2},$$

which yields the second part of the lemma.

Proof of Theorem 3.2. Our object, in view of Theorem 3.5, is to show that the weight conditions of Theorem 3.2 are necessary and sufficient for the inequality (3.3). In order to cast (3.3) in the form of a Riemann-Liouville operator we first replace s by $s/2$ and then apply Lemma 3.6 to show that (3.3) is equivalent to

$$\left(\int_0^\infty \left| \int_0^s (1 - t/s)^{(n-1)/2} F(t) t^{n-1} \, dt \right|^q v(s/2)(s/2)^{n-1} \, \frac{ds}{2} \right)^{1/q} \le C|\Omega|^{1/p - 1/q} \left(\int_0^\infty |F(t)|^p u(t) t^{n-1} \, dt \right)^{1/p}.$$

Now, multiply and divide by $s^{(n-1)/2}$ on the left hand side and replace $F(t)$ by $G(t)/t^{n-1}$. The inequality becomes the weighted Riemann-Liouville inequality

$$\left(\int_0^\infty \left| \int_0^s (s - t)^{(n-1)/2} G(t) \, dt \right|^q s^{-q(n-1)/2} v(s/2)(s/2)^{n-1} \, \frac{ds}{2} \right)^{1/q} \le C|\Omega|^{1/p - 1/q} \left(\int_0^\infty |G(t)|^p u(t) (t^{n-1})^{1-p} \, dt \right)^{1/p}.$$

Apply the results of Stepanov [**10**] to show that this last inequality holds for all locally integrable G if and only if

$p = q$, and

$$\sup_{\rho>0}\left(\int_0^\rho (\rho-t)^{p'(n-1)/2}u(t)^{1-p'}t^{n-1}\,dt\right)^{1/p'}\left(\int_\rho^\infty s^{(1-p/2)(n-1)}v(s/2)2^{-n}\,ds\right)^{1/p},$$

$$\sup_{\rho>0}\left(\int_0^\rho u(t)^{1-p'}t^{n-1}\,dt\right)^{1/p'}\left(\int_\rho^\infty (s-\rho)^{p(n-1)/2}s^{(1-p/2)(n-1)}v(s/2)2^{-n}\,ds\right)^{1/p}$$

are both finite, or, $q < p$ and

$$\left(\int_0^\infty \left(\int_0^\rho t^{n-1}u(t)^{1-p'}\,dt\right)^{r/q'}\times\right.$$
$$\left.\left(\int_\rho^\infty (s-\rho)^{q(n-1)/2}s^{-q(n-1)/2}v(s/2)(s/2)^{n-1}\,\frac{ds}{2}\right)^{r/q}\rho^{n-1}u(\rho)^{1-p'}\,d\rho\right)^{1/r},$$

$$\left(\int_0^\infty \left(\int_0^\rho (\rho-t)^{p'(n-1)/2}t^{n-1}u(t)^{1-p'}\,dt\right)^{r/p'}\times\right.$$
$$\left.\left(\int_\rho^\infty s^{-q(n-1)/2}v(s/2)(s/2)^{n-1}\,\frac{ds}{2}\right)^{r/p}\rho^{-q(n-1)/2}v(\rho/2)(\rho/2)^{n-1}\,\frac{d\rho}{2}\right)^{1/r}$$

are both finite.

To complete the proof we replace s by $2s$ in the two conditions and introduce integrals over Ω in all integrals to return the variables x, y, and z. It is a straightforward matter to reduce the expressions above to (multiples of) those in the statement of Theorem 3.2.

Remark. The analogue of Theorem 3.2 in the case $q > p$ remains an open question.

4. Back to One Dimension

We have reduced n-dimensional inequalities to one-dimensional ones in order to give weight characterisations for them. In this section we look at special cases of our

n-dimensional inequalities and interpret them as new one-dimensional inequalities. In this process something is gained and something is lost. We gain inequalities for integral operators with a wide range of kernels but because we are specializing, the weight conditions we give are sufficient but probably not necessary.

The inequalities are deduced from Theorem 2.1 with $n = 2$ and specially chosen regions S_x by restricting our attention to functions on R^2 which depend only on the first variable. The regions S_x will depend on the function φ as in the following definition.

Definition 4.1. *Suppose* $\varphi : [0, 1] \to \mathbf{R}$ *is a decreasing, continuously differentiable function with* $\varphi(0) = 1$ *and* $\varphi(1) = 0$*. Define*

$$S = \{(x_1, x_2) : 0 < x_1 \le 1, 0 \le x_2 \le x_1\varphi(x_1)\}$$

Lemma 4.2. *S is smoothly starshaped and E, the union of all dilations of S, is equal to*

$$\{(x_1, x_2) : 0 < x_1, 0 \le x_2 < x_1\}. \tag{4.1}$$

For $x = (x_1, x_2) \in E$, S_x, the least dilation of S that contains x, is $\alpha_x S$, where α_x satisfies $x_2 = x_1\varphi(x_1/\alpha_x)$.

Proof. Let $\psi(x_1, x_2) = \varphi^{-1}(x_2/x_1)(1 + (x_2/x_1)^2)^{1/2}$ when x_1 and x_2 are positive and let $\psi(x_1, x_2) = 0$ otherwise. It is easily checked that $S = \{x \in \mathbf{R}^2 \setminus \{0\} : |x| \le \psi(x/|x|)\}$ so S is smoothly starshaped.

If $(x_1, x_2) \in \alpha S$ with $\alpha > 0$ then $0 \le x_1/\alpha$ and hence $0 \le x_1$. Also $0 \le x_2/\alpha \le (x_1/\alpha)\varphi(x_1/\alpha)$ so $0 \le x_2 \le x_1\varphi(x_1/\alpha) < x_1\varphi(0) = x_1$. Thus (x_1, x_2) is in (4.1). Conversely, if (x_1, x_2) is in (4.1) then for sufficiently large α we have both $0 \le x_1/\alpha \le 1$ and $0 \le x_2/\alpha \le (x_1/\alpha)\varphi(x_1/\alpha)$ so $(x_1, x_2) \in E$.

The least dilation of S that contains the point (x_1, x_2) is the unique α for which $(x_1/\alpha, x_2/\alpha)$ is on the graph of $x_1\varphi(x_1)$. Thus $x_2 = x_1\varphi(x_1/\alpha_x)$. This completes the proof.

THEOREM 4.3. *Suppose $0 < q < \infty$, $1 < p < \infty$ and let φ be as in Definition 4.1. There exists a constant C such that the inequality*

$$\left(\int_0^\infty \left(\int_0^x \varphi(y/x)f(y)\,dy\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty |f(y)|^p u(y)\,dy\right)^{1/p}$$

holds for all f, provided either

$p \le q$ and

$$\sup_{t>0}\left(\int_0^t \varphi(y/t)u(y)^{1-p'}\,dy\right)^{1/p'}\left(\int_t^\infty v(x)\,dx\right)^{1/q} < \infty,$$

or $q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_0^\infty \left(\int_0^t \varphi(y/t)u(y)^{1-p'}\,dy\right)^{r/p'}\left(\int_t^\infty v(x)\,dx\right)^{r/p} v(t)\,dt\right)^{1/r} < \infty.$$

Proof. We wish to apply Theorem 2.1 with S as in Definition 4.1 and weights U and V defined by $U(y_1, y_2) = y_1^{p-1}u(y_1)$ and $V(x_1, x_2) = v(x_1)\delta_0(x_2)$ where δ_0 is the measure on $\mathbf{R}$ consisting of a single atom of weight 1 at 0. Because the constant C in Theorem 2.1 is controlled by the value of K or D it is straightforward to show that the theorem holds for this V by applying it to weights $V_n(x_1, x_2) = v(x_1)\xi_n(x_2)$ for some approximate identity $\{\xi_n\}$. We omit the details.

The weight condition of Theorem 2.1 in the case $p \le q$ is

$$K \equiv \sup_{z\in E\setminus\{0\}}\left(\int_{S_z} U(y)^{1-p'}\,dy\right)^{1/p'}\left(\int_{E\setminus S_z} V(x)\,dx\right)^{1/q} < \infty.$$

With $\alpha = \alpha_z$, and applying Lemma 4.2, we have

$$\begin{aligned} K &= \sup_{\alpha>0}\left(\int_0^\alpha \int_0^{y_1\varphi(y_1/\alpha)} y_1^{-1}u(y_1)^{1-p'}\,dy_2dy_1\right)^{1/p'}\left(\int_\alpha^\infty v(x_1)\,dx_1\right)^{1/q} \\ &= \sup_{\alpha>0}\left(\int_0^\alpha \varphi(y_1/\alpha)u(y_1)^{1-p'}\,dy_1\right)^{1/p'}\left(\int_\alpha^\infty v(x_1)\,dx_1\right)^{1/q}. \end{aligned}$$

The weight condition in the case $q < p$ is

$$D \equiv \left(\int_E \left(\int_{S_z} U(y)^{1-p'} \, dy \right)^{r/p'} \left(\int_{E \setminus S_z} V(x) \, dx \right)^{r/p} V(z) \, dz \right)^{1/r} < \infty.$$

D becomes

$$\left(\int_0^\infty \left(\int_0^{z_1} \int_0^{y_1 \varphi(y_1/z_1)} y_1^{-1} u(y_1)^{1-p'} \, dy_2 dy_1 \right)^{r/p'} \left(\int_{z_1}^\infty v(x_1) \, dx_1 \right)^{r/p} v(z_1) \, dz_1 \right)^{1/r}$$

$$= \left(\int_0^\infty \left(\int_0^{z_1} \varphi(y_1/z_1) u(y_1)^{1-p'} \, dy_1 \right)^{r/p'} \left(\int_{z_1}^\infty v(x_1) \, dx_1 \right)^{r/p} v(z_1) \, dz_1 \right)^{1/r}.$$

Thus the conditions of Theorem 2.1 reduce to the hypotheses of the present theorem.

The inequality guaranteed by Theorem 2.1 is

$$\left(\int_E \left| \int_{S_x} f(y) \, dy \right|^q V(x) \, dx \right)^{1/q} \leq C \left(\int_E |f(y)|^p U(y) \, dy \right)^{1/p}$$

which, if we restrict f to depend only on the first variable, becomes

$$\left(\int_0^\infty \left(\int_0^{x_1} \varphi(y_1/x_1) y_1 f(y_1) \, dy_1 \right)^q v(x_1) \, dx_1 \right)^{1/q}$$
$$\leq C \left(\int_0^\infty \int_0^{y_1} |f(y_1)|^p y_1^{p-1} u(y_1) \, dy_1 \right)^{1/p}$$
$$= C \left(\int_0^\infty |y_1 f(y_1)|^p u(y_1) \, dy_1 \right)^{1/p}.$$

Replacing $y_1 f(y_1)$ by $f(y_1)$ completes the proof.

Remark. A weight characterisation in the case $p \leq q$ has been given in [**7**] for the inequality of Theorem 4.3 without the endpoint conditions on φ but under the additional assumption that for some $D > 0$, $\varphi(ab) \leq D(\varphi(a) + \varphi(b))$ for all $a, b \in (0, 1)$.

Applying the above theorem with $\varphi(s) = (1 - s)^{k-1}$ gives a sufficient condition for the Riemann-Liouville fractional integral operators since for this φ,

$$\int_0^x \varphi(y/x) f(y) \, dy = x^{1-k} \int_0^x (x - y)^{k-1} f(y) \, dy.$$

COROLLARY 4.4. *Suppose* $0 < q < \infty$, $1 < p < \infty$ *and let* k *be a real number greater than 1. There exists a constant* C *such that the weighted Riemann-Liouville inequality*

$$\left(\int_0^\infty \left(\int_0^x (x-y)^{k-1} f(y)\,dy\right)^q v(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty |f(y)|^p u(y)\,dy\right)^{1/p}$$

holds for all f *provided either*

$p \le q$ *and*

$$\sup_{t>0}\left(\int_0^t (1-y/t)^{k-1} u(y)^{1-p'}\,dy\right)^{1/p'}\left(\int_t^\infty x^{q(k-1)} v(x)\,dx\right)^{1/q}$$

is finite, or

$q < p$, $1/r = 1/q - 1/p$, *and*

$$\left(\int_0^\infty \left(\int_0^t (1-y/t)^{k-1} u(y)^{1-p'}\,dy\right)^{r/p'}\left(\int_t^\infty x^{q(k-1)} v(x)\,dx\right)^{r/p} t^{q(k-1)} v(t)\,dt\right)^{1/r}$$

is finite.

Examples. Example 10.13 in [**8**] is easily adapted to show that the sufficient condition given in Corollary 4.4 is not necessary.

A simple estimate shows that the condition of Corollary 4.4 is easier to satisfy than the similar condition given in Theorem 10.11 of [**8**]. Also, one can check that Corollary 4.4 implies the inequality

$$\int_0^\infty \left|\int_0^x (x-y) f(y)\,dy\right|^2 e^{-x}\,\frac{dx}{x} \le C\int_0^\infty |f(y)|^2 e^{-y}\,dy,$$

but Theorem 10.11 of [**8**] does not.

Remark. In the case $0 < q < 1$ necessary and sufficient conditions are not available for the Riemann-Liouville operators but our sufficient conditions have a different form than those in [**8**, Theorem 10.11] or those that follow from [**10**, Theorem 3].

If we imitate the proof of Theorem 4.3, replacing the interval $(0,1)$ with the smoothly starshaped subset S of $\mathbf{R}^n$ we get the following result. The details of proof are omitted.

Theorem 4.5. *Let S be a smoothly starshaped subset of $\mathbf{R}^n$ and let B and α_x be as in Section 2. Suppose $0 < q < \infty$, $1 < p < \infty$ and let $\varphi : S \to [0,1]$ be radially decreasing, continuously differentiable, and satisfy $\varphi(0) = 1$, $\varphi(\sigma) = 0$ for $\sigma \in B$. Then there exists a constant C such that the inequality*

$$\left(\int_E \left(\int_{S_x} \varphi(y/\alpha_x) f(y)\,dy\right)^q v(x)\,dx\right)^{1/q} \le C \left(\int_E |f(y)|^p u(y)\,dy\right)^{1/p}$$

holds for all f provided either

$p \le q$ and

$$\sup_{z\in E\setminus\{0\}} \left(\int_{S_z} \varphi(y/\alpha_z) u(y)^{1-p'}\,dy\right)^{1/p'} \left(\int_{E\setminus S_z} v(x)\,dx\right)^{1/q} < \infty,$$

or $q < p$, $1/r = 1/q - 1/p$, and

$$\left(\int_E \left(\int_{S_z} \varphi(y/\alpha_z) u(y)^{1-p'}\,dy\right)^{r/p'} \left(\int_{E\setminus S_z} v(x)\,dx\right)^{r/p} v(z)\,dz\right)^{1/r} < \infty.$$

Remark. The appearance of α_x in the above theorem is unfortunate since it may be difficult to calculate for some regions S. On the other hand, it is often a simple function of x. If S is the unit ball, for example, then $\alpha_x = |x|$.

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