PhD Comprehensive Exam (Algebra)
Department of Mathematics
October 2022

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.
5. All questions are worth 10 marks each.

Linear Algebra

1. Let A be an $n \times n$ matrix with complex entries. Suppose that m is a positive integer such that A^m is diagonalizable. Prove that A^{m+1} is diagonalizable.

2. Let A be a $n \times n$-matrix over some field F, where $n \geq 2$, such that: $A^n = 0$ but $A^{n-1} \neq 0$.
Show that A is similar to the $n \times n$-matrix where all entries are 0, except 1 on line over the diagonal, i.e.,
$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$
Note: The field F need not be algebraically closed so you may not assume that A has a Jordan canonical form.

Groups

3. Let G be a group. Suppose that $\mathrm{Aut}(G)$ is abelian. Show that G has a normal subgroup N such that both N and G/N are both abelian. In this question $\mathrm{Aut}(G)$ is the group of group automorphisms of G. It is a group under composition.

4. Let G be a group with 30 elements. Assume additionally that G admits a unique Sylow-3 subgroup. Show that G has a normal subgroup N of order 15. Show that each group of order 15 is cyclic, and thus conclude: G has a normal subgroup N of order 15, which is necessarily cyclic.

   (Hint: Call H the unique 3-Sylow subgroup of G. Observe that H is normal in G and consider a Sylow-5 subgroup D in G/H. Use this D to construct a subgroup N of G of order 15. Show that N is a normal subgroup of G and finally conclude that N must be necessarily a cyclic group.)

Rings and Modules

5. Suppose that $\phi : \mathbb{Z}^4 \to \mathbb{Z}^2$ is a $\mathbb{Z}$-module homomorphism. Suppose that $\phi(x) = Ax$ where A is 4×2 matrix with integer entries. Prove or disprove: If ϕ is surjective then there is a 2×2 minor of A that is a unit in $\mathbb{Z}$.

6. Suppose that S is a noncommutative ring and $a \in S$. Suppose further a has a right inverse but no left inverse. Show that a has infinitely many right inverses.

   (Hint: If b is a right inverse of a, consider $c = 1 - ba$.)

Fields

7. Consider the polynomial $f = X^3 - 2tX + t \in \mathbb{C}(t)[X]$, where $\mathbb{C}(t)$ is the field of fractions of $\mathbb{C}[t]$. What is the Galois group of f over $\mathbb{C}(t)$?

8. Let $K := F(\sqrt{a})$ be an extension of degree 2 over F, where $a \in F^\times$, and $\mathrm{Char}(F) \neq 2$. Show that -1 is a norm in K/F from K down to F if and only if a is a sum of two squares in F. Reminder: let σ be the generator of $\mathrm{Gal}(K/F)$, then we define the norm from K down to F by

$$N_{K/F} \colon K \to F; \quad k \mapsto k\sigma(k).$$