THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 25, 2022 *3 hours*

Instructions: There will be little or no partial credit, so you should aim to solve at least 5 problems completely and correctly rather than attempting every problem. Skim the questions at the start, so you can focus on the ones you feel most confident about. You should attempt at least two complex analysis questions and at least two real analysis questions.

Complex Analysis: Questions 1–4.

1. Find the number of zeros of

$$f(z) = i \sin z - 2z^2$$

in the rectangle

$$D = \{z = x + iy \in \mathbb{C} \mid |x| \leq \frac{\pi}{2}; |y| \leq 1\}.$$

2. Let f be an entire function such that

$$u(r) = \int_0^{2\pi} |f(re^{i\Theta})| d\Theta$$

is bounded on $(0, \infty)$. Prove that f is a constant function.

3. Evaluate

$$\int_C \frac{z}{e^{6z} + 1} dz,$$

where C is the circle $|z| = 1$, positively oriented.

4. Find the principal part of f at i for

$$f(z) = \frac{e^z}{(z - i)^3 (z + 2)^2}.$$

Real Analysis: Questions 5–8.

5. Let (X, d) be a complete metric space and let $X^{\mathbb{N}}$ be the set of all sequences of elements of X. For $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in $X^{\mathbb{N}}$, define

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

Then ρ is a metric on $X^{\mathbb{N}}$. (There is no need to prove this.) Prove that $(X^{\mathbb{N}}, \rho)$ is complete.

6. Let $R = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1\}$ and let R' be the closed convex hull of $\{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$. With $u = y^2 + z^2$, $v = z^2 + x^2$ and $w = x^2 + y^2$, show that $(x, y, z) \mapsto (u, v, w)$ is a bijection from R to R'. Evaluate

$$\iiint_{R'} \frac{du dv dw}{\sqrt{(v + w - u)(w + u - v)(u + v - w)}}.$$

7. Let (t_n) be a sequence of real numbers, set $t_0 = 0$, and let

$$V_n = \sum_{k=1}^{n} |t_k - t_{k-1}|$$

for $n = 1, 2, \dots$. Prove that (V_n) is a bounded sequence if and only if there exist bounded increasing sequences (a_n) and (b_n) such that $t_n = a_n - b_n$ for all n.

8. Suppose f is a continuous, real-valued function on $[0, 1]$ such that

$$\int_0^1 t^n f(t)\, dt = 0$$

for $n = 0, 1, 2 \dots$. Prove that f is the zero function on $[0, 1]$.