UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 2022 3 hours

Instructions: More credit may be given for a complete solution than for several partial solutions, so you should aim to solve some problems completely and correctly rather than attempting every problem. You should attempt at least one question from each topic. Justify all of your answers. Start each solution on a new page.

Linear Algebra.

(1) (a) Let $n \geq 1$ and let A be in $M_n(\mathbb{C})$. Assume that $A^k = I$ (the identity matrix) for some $k \geq 1$. Show that A is diagonalizable.
(b) Up to conjugacy, how many matrices A in $M_n(\mathbb{C})$ satisfy $A^2 = I$? Find one representative in each conjugacy class.

(2) (a) Let V be a complex vector space with Hermitian inner product $\langle\ ,\ \rangle$. Let S and T be linear operators on V. Show that if $\langle Sv, v\rangle = \langle Tv, v\rangle$ for all $v \in V$, then $S = T$. [Hint: consider $v = x + y$ and $v = x + iy$, for $x, y \in V$.]
(b) Is this true for a real inner product space? Prove or disprove.

Rings and modules.

(3) Let R be a commutative ring and M a Noetherian R-module. Show that any surjective module homomorphism $f\colon M \to M$ is also injective.

(4) Let R be a principal ideal domain and M a finitely generated R-module. We write $M^* = \mathrm{Hom}_R(M, R)$ for the dual R-module.
(a) Show that the assignment $m \mapsto \gamma(m)$ with

$$\gamma(m)(\alpha) = \alpha(m) \quad \text{for } \alpha \in M^*$$

gives a well-defined module homomorphism $\gamma\colon M \to M^{**}$.
(b) Show that γ is an isomorphism if and only if M is free.

Group theory.

(5) Let G be a finite group of order n.
(a) Show that the number of elements in any conjugacy class of G divides n.
(b) Suppose that $n = p^m$ for p a prime and m a positive integer. Show that the centre of G is nontrivial.

(6) Let A and B be subgroups of a (not necessarily finite) group G. Assume that B has finite index in G.
   (a) By considering the natural map $A/(A \cap B) \to G/B$ of sets, prove that $A \cap B$ has finite index in A and that $[A : A \cap B] \leq [G : B]$.
   (b) Prove that $[A : A \cap B] = [G : B]$ if and only if $G = AB$. Here, AB denotes the set $\{ ab \mid a \in A \text{ and } b \in B \}$.
   (c) Assume in addition that A has finite index in G. Prove that if $[G : A]$ and $[G : B]$ are relatively prime, then $G = AB$.

Field theory.

(7) Let L_1 and L_2 be finite extensions of the field F. Show that there is a finite extension E/F containing (isomorphic copies of) both L_1 and L_2. Moreover, E can be chosen such that

$$[E : F] \leq [L_1 : F] \cdot [L_2 : F].$$

(8) Let $\alpha = \sqrt{2 + \sqrt{2}}$. Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and determine its Galois group.

Do not forget to justify your answers!