THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 17, 2022. Duration: 3 hours

Instructions: Answer as many questions as you can. Give complete justifications. You should aim to solve 5 problems completely and correctly rather than attempting every problem. Skim the questions at the start, so you can focus on the ones you feel most confident about.

1. Let $C^1[a,b]$ denote the set of continuously differentiable functions f on the interval $[a,b]$ such that $f(a)=y_1,\ f(b)=y_2$. Consider the function $S: C^1[a,b] \to \mathbb{R}$ defined as
$$S(f) = \int_a^b |f'(x)|^2 dx.$$
Find the minimum value of S. (Hint: Solve the probem first for $y_1 = y_2 = 0$.)

2. Show that the integral $\int_0^\infty \frac{\sin x}{x} dx$ is convergent, but $\int_0^\infty |\frac{\sin x}{x}| dx$ is divergent.

3. Compute the line integral $\int_C F \cdot dr$, where C is any smooth clockwise oriented simple closed curve in the plane containing the origin in its interior, and the vector field F is given by
$$F(x,y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2+y^2)^2}.$$
(Hint: Green's theorem might be useful here).

4. It is known that for smooth real valued functions f on the interval $[0,1]$ such that $\int_0^1 f(x)dx = 0$, there is a constant C such that
$$\int_0^1 f(x)^2 dx \le C \int_0^1 f'(x)^2 dx.$$
Assuming this fact, show that $C \ge \frac{1}{4\pi^2}$.

5. Find the maximum of $|z^2 - 3z + 2|$ on the closed unit disc.

6. Evaluate
$$\int_R \frac{dz}{\sin(z)}$$
where R is the rectangle with corners $(-1,-1)$, $(7,-1)$, $(7,1)$, and $(-1,1)$, oriented counterclockwise.

7. Prove that if $\{u_n\}$ is a sequence of functions analytic in a domain Ω and such that $\sum_n u_n(z)$ converges uniformly on Ω, then $f(z) := \sum_n u_n(z)$ is analytic in Ω.

8. Let f be a holomorphic function defined in an open set containing 0. Suppose $f(0)=0$ and $f'(0)=0$. The inverse function theorem states that f has an inverse g defined near 0. Show that there is an open set U of 0 and a constant $\epsilon > 0$ such that, for all $w \in U$, we can express g as
$$g(w) = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{zf'(z)}{z-w}\, dz$$